OVERPARTITIONS AND GENERATING FUNCTIONS FOR GENERALIZED FROBENIUS PARTITIONS

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Abstract. Generalized Frobenius partitions, or F-partitions, have recently played an important role in several combinatorial investigations of basic hypergeometric series identities. The goal of this paper is to use the framework of these investigations to interpret families of infinite products as generating functions for F-partitions. We employ q-series identities and bijective combinatorics.

1. Introduction

Let $P_{A,B}(n)$ denote the number of generalized Frobenius partitions of n, i.e., the number of two-rowed arrays,

$$\begin{pmatrix} a_1, a_2, \dots, a_m \\ b_1, b_2, \dots, b_m \end{pmatrix}, \tag{1.1}$$

in which the top (bottom) row is a partition from a set A (B), and such that $\sum(a_i+b_i)+m=n$ [2]. The classical example is the case $P_{D,D}(n)$, where D is the set of partitions into distinct non-negative parts. Frobenius observed that these objects are in one-to-one correspondence with the ordinary partitions of n, giving

$$\sum_{n=0}^{\infty} P_{D,D}(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}. \tag{1.2}$$

Andrews [2] later made an extensive study of two infinite families of F-partitions that begin with $P_{D,D}(n)$. He replaced D by D_k or C_k, the set of partitions where parts repeat at most k times and the set of partitions into distinct parts with k colors, respectively. The generating functions are multiple theta series, which in three known cases can be written as an infinite product.

$$\sum_{n=0}^{\infty} P_{D_2,D_2}(n)q^n = \frac{(q^2;q^2)_\infty(-q^3;q^6)_\infty}{(q)_\infty^2(-q;q^2)_\infty}, \tag{1.3}$$

$$\sum_{n=0}^{\infty} P_{D_3,D_3}(n)q^n = \frac{(q^6;q^6)_\infty(q^6;q^{12})_\infty^2(q^2;q^2)_\infty(q;q^2)_\infty^2}{(q)_\infty^3(q^3;q^6)_\infty^2}, \tag{1.4}$$

$$\sum_{n=0}^{\infty} P_{C_2,C_2}(n)q^n = \frac{(-q;q^2)_\infty^2}{(q)_\infty(q;q^2)_\infty}. \tag{1.5}$$

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Here we have employed the standard notation

$$(a_1, \ldots, a_j)_\infty := (a_1, \ldots, a_j; q)_\infty := \prod_{k=0}^{\infty} (1 - a_1 q^k) \cdots (1 - a_j q^k). \tag{1.6}$$

While Andrews' families subsequently received quite a bit of attention [10, 13, 15, 16, 18, 20], other types of Frobenius partitions have recently been turning up as novel interpretations for some infinite products that figure prominently in basic hypergeometric series identities [6, 7, 8, 21]. The combinatorial setting here is that of overpartitions, which are partitions wherein the first occurrence of a part may be overlined. Let O denote the set of overpartitions into non-negative parts. Then it turns out that

$$\sum_{m,n\geq 0} P_{D,O}(m,n) b^m q^n = \frac{(-bq)_\infty}{(q)_\infty} \tag{1.7}$$

and

$$\sum_{\ell,m,n\geq 0} P_{O,O}(\ell,m,n) a^\ell b^m q^n = \frac{(-aq,-bq)_\infty}{(q,abq)_\infty}. \tag{1.8}$$

Here $P_{D,O}(m,n)$ denotes the number of F-partitions counted by $P_{D,O}(n)$ that have m non-overlined parts in the bottom row, and $P_{O,O}(\ell,m,n)$ denotes the number of objects counted by $P_{O,O}(n)$ that have ℓ non-overlined parts in the top row and m non-overlined parts in the bottom row.

Since a thorough combinatorial understanding of (1.7) and (1.8) has been so useful, we give in this paper a variety of other infinite product generating functions for F-partitions and begin to study them using bijective combinatorics. The first goal is to use restricted overpartitions and a useful property of the ${}_1\psi_1$ summation (see Lemma 2.2) to embed some of (1.2) – (1.8) in families of infinite products that generate F-partitions.

Theorem 1.1. *Let O_k be the set of overpartitions where the non-overlined parts occur less than k times. Let $P_{O_k,O_k}(m,n)$ (resp. $P_{O_k,O}(m,n)$) be the number of F-partitions counted by $P_{O_k,O_k}(n)$ (resp. $P_{O_k,O}(n)$) wherein the number of overlined parts on the top minus the number of overlined parts on the bottom is m. Then*

$$\sum_{m,n\geq 0} P_{O_k,O_k}(m,n) b^m q^n = \frac{(-bq)_\infty(-q/b)_\infty(q^k;q^k)_\infty}{(q)_\infty^2(-bq^k,-q^k/b;q^k)_\infty}, \tag{1.9}$$

$$\sum_{m,n\geq 0} P_{O_k,O}(m,n) b^m q^n = \frac{(-bq)_\infty(-q/b)_\infty(q^k;q^k)_\infty}{(q)_\infty^2(-q^k/b;q^k)_\infty}. \tag{1.10}$$

Notice that in both instances the case $k \to \infty$ is the case $a = 1/b$ of (1.8), while the case $k = 1$ of (1.9) is Frobenius' example (1.2), the case $b = 1, k = 2$ of (1.9) is Andrews' (1.5), and the case $k = 1$ of (1.10) is (1.7).

Our next object is to exhibit more families like those above, but where the base cases are none of (1.2) – (1.8). We use the notation AB for the set of vector partitions $(\lambda_A, \lambda_B) \in A \times B$, and D^k for the set of partitions into non-negative parts where each part occurs 0 or k times.

Theorem 1.2.

$$\sum_{n=0}^{\infty} P_{O_k,OD^k}(n)q^n = \frac{(-q)_\infty^2}{(q)_\infty^2}(q^k;q^k)_\infty(q^{2k};q^{4k})_\infty, \tag{1.11}$$

$$\sum_{n=0}^{\infty} P_{O_k,O_{2k}}(n)q^n = \frac{(-q)_\infty^2(q^k;q^k)_\infty(q^k;q^{2k})_\infty^2}{(q)_\infty^2(q^k,q^{4k};q^{5k})_\infty}. \tag{1.12}$$

Then, by employing more general q-series identities, we find generating functions with more parameters, like Theorems 1.3 – 1.6 below. The first two contain the $k=1$ case of (1.11) and the case $k=2$ of (1.10), respectively.

Theorem 1.3. *Let $P_{D,OD}(m,n)$ be the number of F-partitions counted by $P_{D,OD}(n)$ that have m parts in λ_D. Then*

$$\sum_{m,n\geq 0} P_{D,OD}(m,n)y^m q^n = \frac{(-q;q)_\infty(-yq;q^2)_\infty}{(q;q)_\infty}. \tag{1.13}$$

Theorem 1.4. *Let O^2 denote the number of overpartitions in O where the non-overlined parts repeat an even number of times. Let $P_{O,O^2D}(\ell,m,n)$ denote the number of F-symbols counted by $P_{O,O^2D}(n)$ where ℓ is the number of non-overlined parts in the top row minus the number of parts in λ_D and m is the number of non-overlined parts in the bottom row. Then*

$$\sum_{\ell,m,n\geq 0} P_{O,O^2D}(\ell,m,n)a^\ell b^m q^n = \frac{(-aq)_\infty(-q/a,-ab^2q^2;q^2)_\infty}{(q)_\infty(q,a^2b^2q^2;q^2)_\infty}. \tag{1.14}$$

The next theorem also contains the case $k=2$ of (1.10). We are concerned here with $\bar{D}$, which denotes the set of overpartitions into distinct parts such that parts have to differ by at least two if the bigger is overlined and $\overline{0}$ does not occur. These overpartitions have recently arisen in a number of works [5, 14, 17].

Theorem 1.5. *Let $P_{O\bar{D},O}(\ell,m,n)$ denote the number of overpartitions counted by $P_{O\bar{D},O}(n)$ such that ℓ is the number of non-overlined parts in λ_O plus the number of overlined parts in $\lambda_{\bar{D}}$ and such that m is the number of non-overlined parts on the bottom minus the number of parts in $\lambda_{\bar{D}}$. Then*

$$\sum_{\ell,m,n\geq 0} P_{O\bar{D},O}(\ell,m,n)a^\ell b^m q^n = \frac{(-aq,-bq)_\infty(-q/b;q^2)_\infty}{(q,abq)_\infty(q;q^2)_\infty}. \tag{1.15}$$

The last example contains (1.8) and deals with O', the set of overpartitions in O that have no $\overline{0}$.

Theorem 1.6. *Let $P_{OO',O}(k,\ell,m,n)$ denote the number of F-partitions counted by $P_{OO',O}(n)$ where k is the number of non-overlined parts in λ_O plus the number of overlined parts in $\lambda_{O'}$, ℓ is the number of non-overlined parts in the bottom row, and m is the number of parts in $\lambda_{O'}$. Then*

$$\sum_{k,\ell,m,n\geq 0} P_{OO',O}(k,\ell,m,n)a^k b^\ell c^m q^n = \frac{(-aq,-bq,-cq)_\infty}{(q,abq,bcq)_\infty}. \tag{1.16}$$

Finally, we give bijective proofs for some of the generating functions above. We are able to establish (1.5), (1.13), and the case $k=2$ of (1.10) in this way.

2. Recollections and Proofs

Given a set A of partitions we denote by $P_A(n,k)$ the number of partitions of n from the set A having k parts. We recall from [2] that

Lemma 2.1. *The generating function for Frobenius partitions is given by*

$$\sum_{n=0}^{\infty} P_{A,B}(n)q^n = [z^0]\sum_{n,k} P_A(n,k)q^n(zq)^k \sum_{n,k} P_B(n,k)q^n z^{-k}, \tag{2.1}$$

where $[z^k]\sum A_n z^n = A_k$.

We assume enough familiarity with the elementary theory of partitions and overpartitions [1, 8] that we can state generating functions for simple $P_A(n,k)$ without explanation. The following is the key lemma mentioned in the introduction.

Lemma 2.2. *If*

$$[z^0]\frac{(-bzq,-1/bz)_\infty}{(zq,1/z)_\infty}G(z,q) = \frac{(-bq,-q/b)_\infty}{(q)_\infty^2}H(q), \tag{2.2}$$

then

$$[z^0]\frac{(-bzq,-1/bz)_\infty}{(zq,1/z)_\infty}G(z^k,q^k) = \frac{(-bq,-q/b)_\infty}{(q)_\infty^2}H(q^k). \tag{2.3}$$

Proof. Let $H(q) = [z^0]F(z,q)G(z,q)$, with $F(z,q) = \sum A_j(q)z^j$ and $G(z,q) = \sum B_j(q)z^j$. If $A_{kj}(q) = A_j(q^k)$, then

$$\begin{aligned} [z^0]F(z,q)G(z^k,q^k) &= [z^0]\sum A_j(q)z^j \sum B_j(q^k)z^{kj} \\ &= \sum A_{-kj}(q)B_j(q^k) \\ &= \sum A_{-j}(q^k)B_j(q^k) \\ &= H(q^k). \end{aligned}$$

The proof is finished when we apply the above observation to

$$F(z,q) = \sum \frac{(1+b)q^j}{1+bq^j}z^j. \tag{2.4}$$

Substituting $a = 1/b$ and $z = bz$ in the ${}_1\psi_1$ summation,

$$\sum_{n=-\infty}^{\infty} \frac{(-1/a)_n(azq)^n}{(-bq)_n} = \frac{(q,abq,-zq,-1/z)_\infty}{(-bq,-aq,azq,b/z)_\infty}, \tag{2.5}$$

we have

$$F(z,q) = \frac{(q,q,-bzq,-1/bz)_\infty}{(-bq,-q/b,zq,1/z)_\infty}. \tag{2.6}$$

Then

$$\frac{(-bq,-q/b)_\infty}{(q)_\infty^2}[z^0]F(z,q)G(z^k,q^k) = \frac{(-bq,-q/b)_\infty}{(q)_\infty^2}H(q^k),$$

and the lemma follows. □

Above we have introduced the notation

$$(a)_n := (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{2.7}$$

Proof of Theorem 1.1. For the first part, take $G(z,q) = (zq, 1/z)_\infty$ in Lemma 2.2. By (1.2), $H(q) = (q)_\infty/(-bq)_\infty(-q/b)_\infty$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} P_{O_k,O_k}(m,n)b^m q^n &= [z^0]\frac{(-bzq,-1/bz)_\infty(z^kq^k,z^{-k};q^k)_\infty}{(zq,1/z)_\infty} \\ &= [z^0]\frac{(-bzq,-1/bz)_\infty}{(zq,1/z)_\infty}G(z^k,q^k) \\ &= \frac{(-bq,-q/b)_\infty}{(q)_\infty^2}H(q^k) \\ &= \frac{(-bq)_\infty(-q/b)_\infty(q^k;q^k)_\infty}{(q)_\infty^2(-bq^k,-q^k/b;q^k)_\infty}.\end{aligned}$$

Similarly, take $G(z,q) = (zq)_\infty$ for the second part. □

In the following we will use the ${}_1\psi_1$ summation (2.5) or one of its corollaries for the first step of each proof.

Proof of Theorem 1.2. For the first part, take $G(z,q) = (-1/z, zq)_\infty$ in the case $b = 1$ of Lemma 2.2. Then

$$\begin{aligned}H(q) &= \frac{(q)_\infty^2}{(-q,q)_\infty^2}[z^0]\frac{(-zq;q)_\infty(-z^{-1};q)_\infty(-z^{-1};q)_\infty}{(z^{-1};q)_\infty} \\ &= [z^0]\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{z^n q^{\binom{n+1}{2}}}{(-q;q)_n}\sum_{n=0}^{\infty}\frac{z^{-n}q^{\binom{n}{2}}}{(q;q)_n} \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^2;q^2)_n} \\ &= \frac{(q;q)_\infty(-q;q^2)_\infty}{(-q;q)_\infty},\end{aligned}$$

by the q-binomial theorem,

$$\sum_{n=0}^{\infty}\frac{(a)_n z^n}{(q)_n} = \frac{(az)_\infty}{(z)_\infty}. \tag{2.8}$$

For the second part we again apply the case $b = 1$ of Lemma 2.2, this time with $G(z, q) = (z^{-1}, zq)_\infty$. Then

$$\begin{aligned} H(q) &= \frac{(q)^2_\infty}{(-q,q)^2_\infty}[z^0](-zq;q)_\infty(-z^{-1};q)_\infty(-z^{-1};q)_\infty \\ &= \frac{(q)_\infty}{(-q)^2_\infty}[z^0]\sum_{n\in\mathbb{Z}} z^n q^{n(n+1)/2} \sum_{n\geq 0} \frac{z^{-n}q^{n(n-1)/2}}{(q)_n} \\ &= \frac{(q)_\infty}{(-q)^2_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n} \\ &= \frac{(q)_\infty}{(-q)^2_\infty(q,q^4;q^5)_\infty}, \end{aligned}$$

by the first Rogers-Ramanujan identity,

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n} = \frac{1}{(q,q^4;q^5)_\infty}. \tag{2.9}$$

□

Proof of Theorem 1.3.

$$\begin{aligned} \sum_{m,n\geq 0} P_{D,OD}(m,n)y^m q^n &= [z^0]\frac{(-zq;q)_\infty(-z^{-1};q)_\infty(-yz^{-1};q)_\infty}{(z^{-1};q)_\infty} \\ &= [z^0]\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{z^n q^{\binom{n+1}{2}}}{(-q;q)_n}\sum_{n=0}^{\infty}\frac{y^n z^{-n} q^{\binom{n}{2}}}{(q;q)_n} \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{y^n q^{n^2}}{(q^2;q^2)_n} \\ &= \frac{(-q;q)_\infty(-yq;q^2)_\infty}{(q;q)_\infty}, \end{aligned}$$

the final equality being the case $q = q^2, z = -q/a$, and $a \to \infty$ of (2.8). □

Proof of Theorem 1.4.

$$\begin{aligned} \sum_{\ell,m,n\geq 0} P_{O,O^2D}(\ell,m,n)a^\ell b^m q^n &= [z^0]\frac{(-zq,-1/z,-1/az)_\infty}{(azq)_\infty(b^2/z^2;q^2)_\infty} \\ &= [z^0]\frac{(-aq,-bq)_\infty}{(q,abq)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1/a)_n(azq)^n}{(-bq)_n}\sum_{n=0}^{\infty}\frac{(1/ab)_n(-b/z)^n}{(q)_n} \\ &= \frac{(-aq,-bq)_\infty}{(q,abq)_\infty}\sum_{n=0}^{\infty}\frac{(-1/a,1/ab)_n(-abq)^n}{(-bq,q)_n} \\ &= \frac{(-aq)_\infty(-q/a,-ab^2q^2;q^2)_\infty}{(q)_\infty(q,a^2b^2q^2;q^2)_\infty}, \end{aligned}$$

by the q-Kummer identity,

$$\sum_{n=0}^{\infty} \frac{(a,b)_n(-q/b)^n}{(q,aq/b)_n} = \frac{(aq,aq^2/b^2;q^2)_\infty}{(-q/b,aq/b)_\infty(q;q^2)_\infty}. \tag{2.10}$$

Proof of Theorem 1.5.

$$\begin{aligned}
\sum_{\ell,m,n\geq 0} P_{O\bar{D},O}(\ell,m,n)a^\ell b^m q^n &= [z^0]\frac{(-zq,-1/z)_\infty}{(azq,b/z)_\infty}\sum_{n=0}^{\infty}\frac{(-aq)_n q^{n(n+1)/2}(z/b)^n}{(q)_n} \\
&= [z^0]\frac{(-aq,-bq)_\infty}{(q,abq)_\infty}\sum_{n\in Z}\frac{(-1/b)_n(b/z)^n}{(-aq)_n}\sum_{n=0}^{\infty}\frac{(-aq)_n q^{n(n+1)/2}(z/b)^n}{(q)_n} \\
&= \frac{(-aq,-bq)_\infty}{(q,abq)_\infty}\sum_{n=0}^{\infty}\frac{(-1/b)_n q^{n(n+1)/2}}{(q)_n} \\
&= \frac{(-aq,-bq)_\infty(-q/b;q^2)_\infty}{(q,abq)_\infty(q;q^2)_\infty},
\end{aligned}$$

the final equality being Lebesgue's identity which is the case $b\to\infty$ and $a=-1/b$ of (2.10). □

Proof of Theorem 1.6.

$$\begin{aligned}
\sum_{k,\ell,m,n\geq 0} P_{OO',O}(k,\ell,m,n)a^k b^\ell c^m q^n &= [z^0]\frac{(-zq,-1/z,-aczq^2)_\infty}{(azq,b/z,czq)_\infty} \\
&= [z^0]\frac{(-aq,-bq)_\infty}{(q,abq)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1/b)_n(b/z)^n}{(-aq)_n}\sum_{n=0}^{\infty}\frac{(-aq)_n(czq)^n}{(q)_n} \\
&= \frac{(-aq,-bq)_\infty}{(q,abq)_\infty}\sum_{n=0}^{\infty}\frac{(-1/b)_n(bcq)^n}{(q)_n} \\
&= \frac{(-aq,-bq,-cq)_\infty}{(q,abq,bcq)_\infty},
\end{aligned}$$

by the q-binomial theorem (2.8). □

It should be noted that Theorems 1.4 – 1.6 have k-generalizations like Theorems 1.1 and 1.2 but the combinatorial definition of the F-partitions are less palatable.

3. Bijections

In this section we establish some partitions that explains some of the first cases.

Bijection for (1.5) We will here give a combinatorial proof of

$$\sum_{n=0}^{\infty} P_{O_2,O_2}(n)q^n = \frac{(-q;q)_\infty(-q;q^2)_\infty^2}{(q;q)_\infty}.$$

We start with a F-partition and add one to each entry of the first row. Let k be the number of overlined parts in the first row minus the number of overlined parts of the second row. Suppose without loss of generality that $k\geq 0$. Then we split the F-partition into two F-partitions :

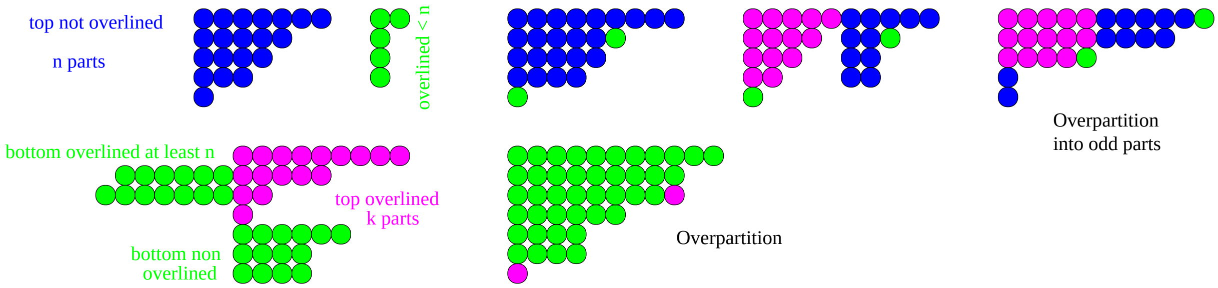

FIGURE 1. Bijection for 1.10

one that contains the overlined parts and one that contains the non-overlined parts. Apply Wright's bijection (see [20]) to both F-partitions and get two ordinary partitions and two triangles $(k, k-1, \dots, 1)$ and $(k-1, \dots, 1)$. We keep the first partition, which gives $1/(q;q)_\infty$ and the odd parts of the second partition, which gives $1/(q;q^2)_\infty$. Then we divide the even parts of the second partition by 2. To the left of these parts we put $k, k-1, \dots, 1$. By applying the reverse of Wright's bijection, we obtain two partitions into distinct parts. We multiply by two and decrease by 1 the parts of the first partition and we multiply by 2 and increase by 1 the parts of the second partition. We get two partitions into distinct odd parts, which is generated by $(-q;q^2)^2$. It is easy to check that the weight is preserved and that every step is reversible.

Bijection for (1.10). Now let us prove combinatorially that

$$\sum_{n=0}^{\infty} P_{O_2,O}(n)q^n = \frac{(-q;q)_\infty(-q;q^2)_\infty}{(q;q)_\infty(q;q^2)_\infty}$$

This proof is inspired by some ideas of [21]. We start with the Frobenius partition and add one to each entry of the first row. The top row is an overpartition in O_2. Therefore the non-overlined parts form a partition into distinct parts. Let us suppose that this top row has n non-overlined parts and k overlined parts. We separate these into partitions α and β. The bottom row is an overpartition into $n+k$ non-negative parts. Using algorithm Z [4, 22], we can decompose it into a partition δ into parts at most $n+k$ and a partition γ into non-negative distinct parts less than $n+k$.

We take the parts of γ that are less than n, conjugate them, and add them to α to create an overpartition into distinct parts (a part is overlined if the difference with the previous part is at least 2). Then we change α into an overpartition into odd parts generalizing Sylvester's bijection [19]. First we take off $2\lfloor (n-1)/2 \rfloor + 2 - i$ from the i^{th} part and change it to η a $(2\lfloor (n-1)/2 \rfloor + 1) \times \lceil n/2 \rceil$ rectangle. Then we look at the conjugate of what is left. Every odd part is inserted in η and every even part $2i$ is changed to two i parts that are inserted in the conjugate of η. The overlines follow. Note that this part is a bijective proof of Lebesgue's identity.

Now we take the parts of γ that are equal to or greater than n. If the part $n+i-1$ occurs we add it to the i^{th} part of β and take off the overline. After sorting, that creates an overpartition where the non-overlined parts are greater than $n+k$. We add then δ and get an overpartition, which is generated by $(-q;q)_\infty/(q;q)_\infty$.

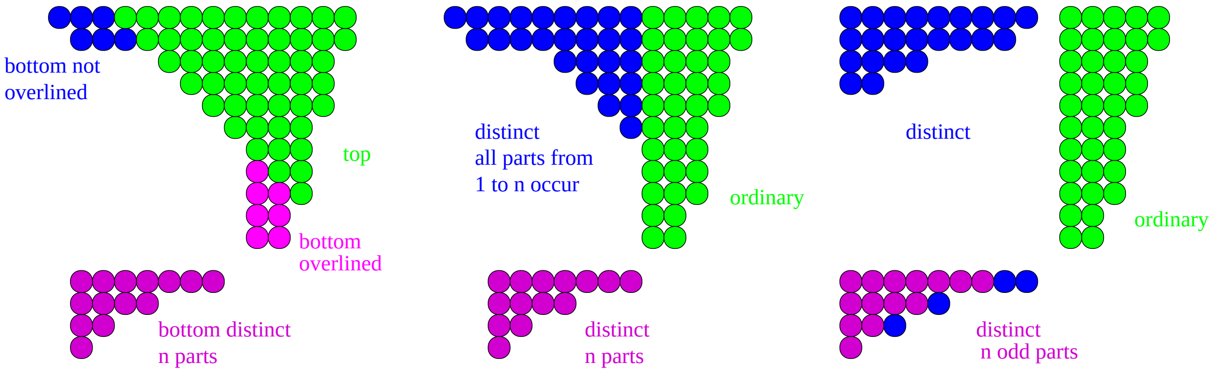

FIGURE 2. Bijection for (1.13)

For example, we start with $\begin{pmatrix} \bar{8}, & 6, \bar{4}, 4, 3, 2, \bar{1}, \bar{0}, 0 \\ \overline{10}, & \bar{9}, \bar{5}, 3, 3, \bar{1}, 1, 0, 0 \end{pmatrix}$. We get $\alpha = (7,5,4,3,1)$, $\beta = (9,5,2,1)$, $\gamma = (7,6,4,1)$ and $\delta = (6,4,4)$. Then we apply the mapping and get $\eta = (1\bar{1}, 9, \bar{5}, 1, 1)$ and $\mu = (11, \bar{9}, 9, 6, 4, 4, \bar{1})$. See Fig 1.

Bijection for (1.13) We will here give a combinatorial proof of

$$\sum_{n=0}^{\infty} P_{D,OD}(n)q^n = \frac{(-q;q)_\infty(-q;q^2)_\infty}{(q;q)_\infty}.$$

We start with a Frobenius partition and add one to each entry of the first row. The bottom row is made of an overpartition β and a partition into distinct parts α. Let n be the number of parts of α. We apply a generalization of Wright's bijection to the top row and β. We draw the Ferrers diagram of the top row and we shift the i^{th} part by $i-1$. We draw the Ferrers diagram of the non overlined parts of β and put it at the left of the diagram of the top row. We draw the Ferrers diagram of the overlined parts of β, conjugate it and put it at the bottom right of the diagram of the top row, as shown on the left of Figure 2. Then we break the diagram into two parts : the left and the right of the largest overlined part. On the right we get an ordinary partition, which gives $1/(q;q)_\infty$ and on the left we get a partition λ into distinct parts where all the parts from 1 to n occur.

Let i be the index of the smallest even part in α. Then we take off i from λ and add it to the conjugate of α. We do that until α has only odd parts. We finally get λ a partition into distinct parts, which is generated by $(-q;q)_\infty$ and α a partition into distinct odd parts, which is generated by $(-q;q^2)_\infty$. Each step is easily reversible.

For example starting with $\begin{pmatrix} 11, 10, 8, 7, 6, 4, 3, 2, 1 \\ (\bar{4}, 3, 3, \bar{3}, \bar{0}), (7, 4, 2, 1) \end{pmatrix}$ we get $\beta = (5,5,4,4,4,3,3,3,3,3,2,2)$, $\lambda = (9,8,4,2)$ and $\alpha = (9,5,3,1)$. See Fig 2.

Bijection for (1.15). A combinatorial proof of

$$\sum_{n=0}^{\infty} P_{OO_2',O}(n)q^n = \frac{(-q;q)^2_\infty(-q;q^2)_\infty}{(q;q)^2_\infty(q;q^2)_\infty},$$

can be easily done mixing the combinatorial proof of the ${}_1\psi_1$ summation of [21] and a combinatorial proof of the Lebesgue's identity for example the one used in the previous bijection.

Bijection for (1.16). A combinatorial proof of

$$\sum_{n=0}^{\infty} P_{OO',O}(n)q^n = \frac{(-q;q)_\infty^3}{(q;q)_\infty^3},$$

can be easily done mixing the combinatorial proof of the ${}_1\psi_1$ summation of [21] and a combinatorial proof of the q-binomial identity (see for example [12]).

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