THE RANK OF A UNIMODAL SEQUENCE AND A PARTIAL THETA IDENTITY OF RAMANUJAN

BYUNGCHAN KIM AND JEREMY LOVEJOY

Abstract. We study the number of unimodal sequences of weight n and rank m using a partial theta identity discovered by Ramanujan. We obtain rank difference identities as well as a congruence for the second rank moment.

1. Introduction

Let $U(n)$ denote the number of unimodal sequences of the form

$$a_1 \leq a_2 \leq \cdots \leq a_r \leq \overline{c} \geq b_1 \geq b_2 \geq \cdots \geq b_s \tag{1.1}$$

with weight $n = c + \sum_{i=1}^{r} a_i + \sum_{i=1}^{s} b_i$. For example, $U(4) = 12$, the relevant sequences being

$$(\overline{4}), (1, \overline{3}), (\overline{3}, 1), (1, \overline{2}, 1), (\overline{2}, 2), (2, \overline{2}),$$
$$(1, 1, \overline{2}), (\overline{2}, 1, 1), (\overline{1}, 1, 1, 1), (1, \overline{1}, 1, 1), (1, 1, \overline{1}, 1), (1, 1, 1, \overline{1}).$$

The rank of a unimodal sequence is $s - r$. Let $U(m, n)$ be the number of unimodal sequences of weight n and rank m and let $U(t, m, n)$ be the number of unimodal sequences of weight n and rank congruent to t modulo m. We note the symmetries $U(m, n) = U(-m, n)$ and $U(m - t, m, n) = U(t, m, n)$, and we assume that the empty sequence has rank 0.

Define the rank difference $U_{t_1 t_2}(x)$ by

$$U_{t_1 t_2}(x) := \sum_{n \geq 0} \Big(U(t_1, m, mn + x) - U(t_2, m, mn + x) \Big) q^{mn+x}. \tag{1.2}$$

With our first result we consider the case $m = 5$ and find formulas for all of the rank differences in terms of partial theta functions and modular forms. Recall the usual q-series notation,

$$(a_1, a_2, \ldots, a_k)_n := (a_1, a_2, \ldots, a_k; q)_n := \prod_{i=1}^{k} (1 - a_i)(1 - a_i q) \cdots (1 - a_i q^{n-1}). \tag{1.3}$$

Date: December 9, 2013.

2010 *Mathematics Subject Classification.* Primary 05A15, 33D15, Secondary : 05A30, 33D90.

Key words and phrases. unimodal sequences, rank, partial theta functions.

The first author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Science, ICT & Future Planning(NRF2011-0009199), and the TJ Park Science Fellowship from the POSCO TJ Park Foundation.

Theorem 1.1. *Let $m = 5$. We have*

$$U_{02}(0) = \left(\sum_{n\geq 0} - \sum_{n\leq -1}\right)(-1)^n q^{5n(15n+1)/2} + (-1)^n q^{(5n+3)(15n+10)/2}, \tag{1.4}$$

$$U_{12}(0) = \frac{\sum_{n\geq 0}(-1)^n q^{(5n)(5n+1)/2}}{(q^5;q^{25})_\infty (q^{20};q^{25})_\infty} + \sum_{n\leq -1}(-1)^n q^{(5n+3)(15n+10)/2} - \sum_{n\geq 0}(-1)^n q^{5n(15n+1)/2}, \tag{1.5}$$

$$U_{02}(1) = \frac{\sum_{n\geq 0}(-1)^n q^{(5n+3)(5n+4)/2}}{(q^5;q^{25})_\infty (q^{20};q^{25})_\infty} + \frac{q\sum_{n\geq 0}(-1)^n q^{(5n+4)(5n+5)/2}}{(q^{10};q^{25})_\infty (q^{15};q^{25})_\infty} + q(q^{25},q^{50},q^{75};q^{75})_\infty, \tag{1.6}$$

$$U_{12}(1) = \frac{\sum_{n\geq 0}(-1)^n q^{(5n+3)(5n+4)/2}}{(q^5;q^{25})_\infty (q^{20};q^{25})_\infty} - \frac{q\sum_{n\geq 0}(-1)^n q^{(5n)(5n+1)/2}}{(q^{10};q^{25})_\infty (q^{15};q^{25})_\infty} + q(q^{25},q^{50},q^{75};q^{75})_\infty, \tag{1.7}$$

$$U_{02}(2) = \frac{q\sum_{n\geq 0}(-1)^n q^{(5n+3)(5n+4)/2}}{(q^{10};q^{25})_\infty (q^{15};q^{25})_\infty} + \sum_{n\geq 0}(-1)^n q^{(5n+1)(15n+4)/2} - \sum_{n\leq -1}(-1)^n q^{(5n+2)(15n+7)/2}, \tag{1.8}$$

$$U_{12}(2) = \left(\sum_{n\geq 0} - \sum_{n\leq -1}\right)(-1)^n q^{(5n+1)(15n+4)/2} + (-1)^n q^{(5n+2)(15n+7)/2}, \tag{1.9}$$

$$U_{02}(3) = \frac{\sum_{n\geq 0}(-1)^n q^{(5n+2)(5n+3)/2}}{(q^5;q^{25})_\infty (q^{20};q^{25})_\infty}, \tag{1.10}$$

$$U_{12}(3) = 0, \tag{1.11}$$

$$U_{02}(4) = 0, \tag{1.12}$$

$$U_{12}(4) = \frac{q\sum_{n\geq 0}(-1)^n q^{(5n+2)(5n+3)/2}}{(q^{10};q^{25})_\infty (q^{15};q^{25})_\infty}. \tag{1.13}$$

Theorem 1.1 is of course reminiscent of the many rank difference identities for partitions [9, 17, 19, 24, 26, 27] and overpartitions [21, 22, 23]. However, while those rank differences are now understood in the context of modular and mock modular forms [1, 14, 16], there is apparently no such modular structure in the case of unimodal sequences. Instead Theorem 1.1 is a curious and unexpected application of a partial theta identity discovered by Ramanujan (see (2.3)).

For $m = 7$ we are unable to find simple formulas for the rank differences using the partial theta identity. However, there is a congruence for the second rank moment modulo 7 which is reminiscent of rank moment congruences for partitions and overpartitions [4, 8, 11, 12, 18]. Define the kth rank moment $\mathcal{U}_k(n)$ by

$$\mathcal{U}_k(n) := \sum_{m\in\mathbb{Z}} m^k U(m,n). \tag{1.14}$$

Our second result is the following congruence.

Theorem 1.2. *We have*

$$\sum_{n\geq 0}\mathcal{U}_2(n)q^n \equiv \sum_{n\geq 0}(n+1)U(n)q^n + \left(\sum_{n\geq 0} - \sum_{n\leq -1}\right)(-1)^n(n-1)q^{n(3n+1)/2} \pmod 7. \tag{1.15}$$

In particular, for all $n \geq 0$ we have

$$\mathcal{U}_2(7n+6) \equiv 0 \pmod 7. \tag{1.16}$$

The paper is organized as follows. In the next section we establish some useful generating functions and in Section 3 we prove the main theorems. We close in Section 4 with some remarks on the moduli 3 and 4.

Before continuing, we note that in prior studies the unimodal sequences in (1.1) have been viewed as stacks, two-quadrant Ferrers graphs or convex compositions [5, 10, 29, 31, 32]. The perspective of unimodal sequences is in line with recent work on asymptotic formulas [13] and mixed mock and quantum modular forms [15, 25].

2. Generating functions

We begin by establishing four generating functions for $U(m,n)$. Define $F(x,q)$ by

$$F(x,q) := \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} U(m,n)x^m q^n. \tag{2.1}$$

Proposition 2.1. *We have*

$$\begin{aligned} F(x,q) &= \sum_{n\geq 0}\frac{q^n}{(xq)_n(q/x)_n} &(2.2)\\ &= \frac{\sum_{n\geq 0}(-1)^n x^{2n+1}q^{\binom{n+1}{2}}}{(xq)_\infty(q/x)_\infty} + (1-x)\sum_{n\geq 0}(-1)^n x^{3n}q^{n(3n+1)/2}(1-x^2q^{2n+1}) &(2.3)\\ &= \frac{\sum_{n\geq 0}(-1)^n x^{2n+1}q^{\binom{n+1}{2}}}{(xq)_\infty(q/x)_\infty} + (1-x)\sum_{n\geq 0}\frac{(-1)^n x^{2n}q^{\binom{n+1}{2}}}{(xq)_n} &(2.4)\\ &= \frac{(1-x)}{(q)_\infty^2}\left(\sum_{n,r\geq 0} - \sum_{n,r<0}\right)\frac{(-1)^{n+r}q^{n(n+1)/2+(2n+1)r+r(r+1)/2}}{(1-xq^r)}. &(2.5)\end{aligned}$$

Proof. Equation (2.2) follows immediately from the fact that $\sum_{i=1}^r a_i$ and $\sum_{i=1}^s b_i$ in (1.1) are partitions into r and s parts, respectively. Equation (2.3) is an identity in Ramanujan's lost notebook [6, Entry 6.3.2]. Equation (2.4) follows from another identity in Ramanujan's lost notebook. It is the case $a = -1/x$ and $b = -x$ of [6, Entry 6.3.1]. We remark in passing that the equivalence of (2.3) and (2.4) follows from Franklin's involution on partitions into distinct parts [2].

For (2.5) we use Bailey pairs. It is not necessary to go into detail on these (the interested reader may consult [3] or [30]), only to note that if (α_n, β_n) is a Bailey pair relative to a, then

[20, Eq. (1.5)]

$$\sum_{n\geq 0} q^n \beta_n = \frac{1}{(aq,q)_\infty} \sum_{r,n\geq 0} (-a)^n q^{\binom{n+1}{2}+(2n+1)r} \alpha_r, \tag{2.6}$$

and that the sequences

$$\alpha_n = \begin{cases} 1, & \text{if } n=0, \\ \frac{q^{\binom{n+1}{2}}(-1)^n(1+q^n)(1-x)(1-1/x)}{(1-xq^n)(1-q^n/x)}, & \text{otherwise} \end{cases}$$

and

$$\beta_n = \frac{1}{(xq)_n (q/x)_n}$$

form a Bailey pair relative to 1 (see [28, Eq. (4.1)] with $(a,c,d) = (1,x,1/x)$). Substituting this Bailey pair into (2.6) and using the fact that for $r \geq 1$

$$\frac{(1+q^r)(1-x)(1-1/x)}{(1-xq^r)(1-q^r/x)} = \frac{1-x}{1-xq^r} + \frac{1-1/x}{1-q^r/x},$$

we have

$$\begin{aligned} F(x,q) &= \frac{1}{(q)_\infty^2}\left(\sum_{n\geq 0}(-1)^n q^{\binom{n+1}{2}} + \sum_{\substack{r\geq 1\\ n\geq 0}} (-1)^{n+r} q^{\binom{n+1}{2}+(2n+1)r+\binom{r+1}{2}} \left(\frac{1-x}{1-xq^r}\right)\right. \\ &\qquad \left. + \sum_{\substack{r\geq 1\\ n\geq 0}} (-1)^{n+r} q^{\binom{n+1}{2}+(2n+1)r+\binom{r+1}{2}} \left(\frac{1-1/x}{1-q^r/x}\right)\right) \\ &= \frac{(1-x)}{(q)_\infty^2} \sum_{r,n\geq 0} \frac{(-1)^{n+r} q^{\binom{n+1}{2}+(2n+1)r+\binom{r+1}{2}}}{1-xq^r} - \frac{(1-1/x)}{(q)_\infty^2} \sum_{r,n<0} \frac{(-1)^{n+r} q^{\binom{n+1}{2}+(2n+1)r+\binom{r}{2}}}{1-q^{-r}/x} \\ &= \frac{(1-x)}{(q)_\infty^2} \left(\sum_{n,r\geq 0} - \sum_{n,r<0} \right) \frac{(-1)^{n+r} q^{n(n+1)/2+(2n+1)r+r(r+1)/2}}{(1-xq^r)}, \end{aligned} \tag{2.7}$$

as desired. □

Setting $x=1$ in (2.2) and (2.3) (or (2.4)) we have two generating functions for $U(n)$.

Corollary 2.2. *We have*

$$\sum_{n\geq 0} U(n) q^n = \sum_{n\geq 0} \frac{q^n}{(q)_n^2} \tag{2.8}$$

$$= \frac{1}{(q)_\infty^2} \sum_{n\geq 0} (-1)^n q^{\binom{n+1}{2}}. \tag{2.9}$$

Next we find a generating function for the second rank moment.

Proposition 2.3. *We have*

$$\begin{aligned}\sum_{n\geq 0}\mathcal{U}_2(n)q^n = \frac{1}{(q)_\infty^2}\left(2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}} + \sum_{n=0}^{\infty}(-1)^n(4n^2+4n+1)q^{\binom{n+1}{2}}\right)\\ - \left(\sum_{n\geq 0} - \sum_{n\leq -1}\right)(-1)^n(6n+1)q^{n(3n+1)/2}.\end{aligned} \tag{2.10}$$

Proof. From the definition of rank moment we have that

$$\sum_{n\geq 0}\mathcal{U}_k(n)q^n = \partial_x^k\big|_{x=1}F(x,q), \tag{2.11}$$

where $\partial_x := x\frac{d}{dx}$. We calculate $\partial_x^2\big|_{x=1}F(x,q)$ using equation (2.4). Let $G(x,q)$ and $H(x,q)$ denote the first and second terms on the right-hand side. The fact that

$$\partial_x^2\big|_{x=1}H(x,q) = \sum_{n\geq 0}(-1)^{n+1}(6n+1)q^{n(3n+1)/2} + \sum_{n\leq -1}(-1)^n(6n+1)q^{n(3n+1)/2} \tag{2.12}$$

is a straightforward calculation.

For $G(x,q)$ we observe that

$$G(x,q) = \frac{1}{(q)_\infty}\frac{(q)_\infty}{(xq)_\infty(q/x)_\infty}\sum_{m\geq 0}(-1)^m x^{2m+1}q^{\binom{m+1}{2}},$$

and note that

$$\frac{(q)_\infty}{(xq)_\infty(q/x)_\infty} =: C_0(x,q)$$

is the two-variable generating function for the crank of a partition [7].

We compute that

$$\partial_x G(x,q) = \frac{1}{(q)_\infty}C_1(x,q)\sum_{m\geq 0}(-1)^m x^{2m+1}q^{\binom{m+1}{2}} + \frac{1}{(q)_\infty}C_0(x,q)\sum_{m\geq 0}(-1)^m(2m+1)x^{2m+1}q^{\binom{m+1}{2}}$$

and

$$\begin{aligned}\partial_x^2 G(x,q) = \frac{1}{(q)_\infty}C_2(x,q)\sum_{m=0}^{\infty}(-1)^m x^{2m+1}q^{\binom{m+1}{2}} + \frac{2}{(q)_\infty}C_1(x,q)\sum_{m=0}^{\infty}(-1)^m(2m+1)x^{2m+1}q^{\binom{m+1}{2}}\\ + \frac{1}{(q)_\infty}C_0(x,q)\sum_{m=0}^{\infty}(-1)^m(2m+1)^2 q^{\binom{m+1}{2}},\end{aligned}$$

where $C_k(x,q) = \partial_x^k C_0(x,q)$. Now we have [8]

$$\begin{aligned}C_0(1,q) &= \frac{1}{(q)_\infty},\\ C_1(1,q) &= 0,\\ C_2(1,q) &= \frac{2}{(q)_\infty}\sum_{n\geq 1}\frac{nq^n}{1-q^n},\end{aligned}$$

and so

$$\partial_x^2 G(x,q)|_{x=1} = \frac{1}{(q)_\infty^2}\left(2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}} + \sum_{n=0}^{\infty}(-1)^n(4n^2+4n+1)q^{\binom{n+1}{2}}\right). \quad (2.13)$$

Together with (2.12), this completes the proof. □

Finally we record generating functions for $U(m,n)$ and $U(t,m,n)$. These are not necessary for the sequel but are quite useful for computations.

Proposition 2.4.

(1) *For $m \in \mathbb{Z}$ we have*

$$\sum_{n\geq 0} U(m,n)q^n = \chi(m=0) + \frac{-1}{(q)_\infty^2}\sum_{r,n\geq 0}(-1)^{n+r}q^{n(n+1)/2+r(r+1)/2+2rn+|m|r}(1-q^r). \quad (2.14)$$

(2) *For $m \geq 1$ and $0 \leq t \leq m-1$ we have*

$$\sum_{n\geq 0} U(t,m,n)q^n = \chi(t=0) + \frac{-1}{(q)_\infty^2}\sum_{r,n\geq 0}(-1)^{n+r}q^{n(n+1)/2+r(r+1)/2+2rn}(1-q^r)\frac{(q^{rt}+q^{r(m-t)})}{1-q^{rm}}. \quad (2.15)$$

Proof. For $m \geq 1$ equation (2.14) follows from (2.5) after expanding

$$(1-x)/(1-xq^r) = (1-x)\sum_{m\geq 0} x^m q^{mr}$$

and picking off the coefficient of x^m. The case $m < 0$ follows from the symmetry $U(m,n) = U(-m,n)$. The case $m = 0$ is trickier. For this we need the identity

$$\left(\sum_{r,n\geq 0} - \sum_{r,n<0}\right)(-1)^{n+r}q^{\binom{n+1}{2}+(2n+1)r+\binom{r+1}{2}} = (q)_\infty^2, \quad (2.16)$$

which follows from (2.6) and the unit Bailey pair relative to 1, [3, Theorem 1],

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{\binom{n}{2}}(-1)^n(1+q^n), & \text{otherwise} \end{cases}$$

and

$$\beta_n = \chi(n=0).$$

Specifically, we obtain

$$\begin{aligned}(q)_\infty^2 &= \sum_{n\geq 0}(-1)^n q^{\binom{n+1}{2}} + \sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^{n+r}q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^r) \\ &= \sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^{n+r}q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}} + \sum_{r,n\geq 0}(-1)^{n+r}q^{\binom{n+1}{2}+(2n+1)r+\binom{r+1}{2}}.\end{aligned}$$

Replacing (r,n) by $(-r,-n-1)$ in the first sum gives (2.16).

Now picking off the coefficient of x^0 in (2.5) (c.f. equation (2.7)), we have

$$\begin{aligned}\sum_{n\geq 0} U(0,n)q^n &= \frac{1}{(q)_\infty^2}\Big(\sum_{r,n\geq 0}(-1)^{n+r}q^{\binom{n+1}{2}+(2n+1)r+\binom{r+1}{2}} + \sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^{n+r}q^{\binom{n+1}{2}+(2n+1)r+\binom{r+1}{2}}\Big)\\ &= \frac{1}{(q)_\infty^2}\Big((q)_\infty^2 + \sum_{r,n<0}(-1)^{n+r}q^{\binom{n+1}{2}+(2n+1)r+\binom{r+1}{2}} + \sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^{n+r}q^{\binom{n+1}{2}+(2n+1)r+\binom{r+1}{2}}\Big)\\ &= 1 + \frac{1}{(q)_\infty^2}\Big(-\sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^{n+r}q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}} + \sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^{n+r}q^{\binom{n+1}{2}+(2n+1)r+\binom{r+1}{2}}\Big),\end{aligned}$$

which gives (2.14) when $m = 0$.

Finally, equation (2.15) follows from (2.14) after noting that

$$U(t,m,n) = \sum_{v\geq 0} U(mv+t,n) + \sum_{v\geq 1} U(mv-t,n).$$

□

3. Proofs of the main results

We are now ready to prove Theorems 1.1 – 1.2. For $0 \leq i \leq 4$ define the sums X_i and Y_i by

$$X_i := \sum_{m\in\mathbb{Z}}(-1)^m q^{(5m+i)(5m+i+1)/2} \tag{3.1}$$

and

$$Y_i := \sum_{n\geq 0}(-1)^n q^{(5n+i)(5n+i+1)/2}. \tag{3.2}$$

We will frequently use the fact that $X_0 = -X_4$, $X_1 = -X_3$, and $X_2 = 0$, which follow upon replacing m by $-m-1$ in X_i.

Proof of Theorem 1.1. We begin by observing that

$$\begin{aligned}F(\zeta_5,q) &= \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} U(m,n)\zeta_5^m q^n\\ &= \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}}\sum_{i=0}^{4} U(5m+i,n)\zeta_5^{5m+i}q^n\\ &= \sum_{n\geq 0}\sum_{i=0}^{4} U(i,5,n)\zeta_5^i q^n.\end{aligned}$$

This together with (2.3) gives

$$\begin{aligned}\sum_{n\geq 0}\sum_{i=0}^{4} U(i,5,n)\zeta_5^i q^n = \frac{1}{(\zeta_5 q, \zeta_5^{-1} q)_\infty} \sum_{n\geq 0} (-1)^n \zeta_5^{2n+1} q^{\binom{n+1}{2}} \\ + (1-\zeta_5)\sum_{n\geq 0}(-1)^n \zeta_5^{3n} q^{n(3n+1)/2}(1-\zeta_5^2 q^{2n+1}).\end{aligned} \tag{3.3}$$

Using the fact that

$$(q^5;q^5)_\infty = (\zeta_5 q, \zeta_5^{-1} q, \zeta_5^2 q, \zeta_5^{-2} q, q)_\infty,$$

together with the triple product identity,

$$\sum_{n\in\mathbb{Z}} z^n q^{\binom{n+1}{2}} = (-1/z, -zq, q)_\infty, \tag{3.4}$$

we may rewrite (3.3) as

$$\begin{aligned}\sum_{n\geq 0}\sum_{i=0}^{4} U(i,5,n)\zeta_5^i q^n = \frac{1}{(1-\zeta_5^2)(q^5;q^5)_\infty} \sum_{m\in\mathbb{Z}} (-1)^m \zeta_5^{-2m} q^{\binom{m+1}{2}} \sum_{n\geq 0} (-1)^n \zeta_5^{2n+1} q^{\binom{n+1}{2}} \\ + (1-\zeta_5)\sum_{n\geq 0}(-1)^n \zeta_5^{3n} q^{n(3n+1)/2}(1-\zeta_5^2 q^{2n+1}).\end{aligned} \tag{3.5}$$

We first treat equations (1.10) - (1.13). These are the simplest cases since the exponent of q is never of the form q^{5n+3} or q^{5n+4} in the final sum on the right-hand side of (3.5) .

To obtain an exponent of the form $5n+3$ in the product of the first two sums on the right-hand side of (3.5) we require $(m,n) \equiv (0,2)$, $(2,0)$, $(4,2)$, or $(2,4)$ modulo 5. Thus we have

$$\begin{aligned}\sum_{n\geq 0}\sum_{i=0}^{4} U(i,5,5n+3)\zeta_5^i q^{5n+3} &= \frac{1}{(1-\zeta_5^2)(q^5;q^5)_\infty}\left(X_0Y_2 + \zeta_5^2 X_2 Y_0 + \zeta_5^2 X_4 Y_2 + X_2 Y_4\right) \\ &= \frac{1}{(1-\zeta_5^2)(q^5;q^5)_\infty}\left(X_0Y_2 + \zeta_5^2 X_4 Y_2\right) \\ &= \frac{1}{(1-\zeta_5^2)(q^5;q^5)_\infty}\left(X_0Y_2 - \zeta_5^2 X_0 Y_2\right) \\ &= \frac{1}{(q^5;q^5)_\infty}\sum_{m\in\mathbb{Z}}(-1)^m q^{5m(5m+1)/2}\sum_{n\geq 0}(-1)^n q^{(5n+2)(5n+3)/2} \\ &= \frac{1}{(q^5;q^{25})_\infty (q^{20};q^{25})_\infty}\sum_{n\geq 0}(-1)^n q^{(5n+2)(5n+3)/2},\end{aligned}$$

by an application of (3.4). Thus, writing

$$U_i(x) := \sum_{n\geq 0} U(i,5,5n+x) q^{5n+x}, \tag{3.6}$$

we have

$$U_0(3) - \frac{\sum_{n\geq 0}(-1)^n q^{(5n+2)(5n+3)/2}}{(q^5;q^{25})_\infty (q^{20};q^{25})_\infty} + U_1(3)\zeta_5 + U_2(3)\zeta_5^2 + U_3(3)\zeta_5^3 + U_4(3)\zeta_5^4 = 0.$$

The fact that the minimal polynomial of ζ_5 over $\mathbb{Q}$ is $1+x+x^2+x^3+x^4$ implies that the coefficients of ζ_5^i are all identical, giving equations (1.10) and (1.11).

Equations (1.12) and (1.13) are similar. To obtain an exponent of the form $5n+4$ in the product of the first two sums on the right-hand side of (3.5) we require $(m,n) \equiv (2,1)$, $(2,3)$, $(1,2)$, or $(3,2)$ modulo 5. Arguing as above we find that

$$\begin{aligned}
\sum_{n\geq 0}\sum_{i=0}^{4} U(i,5,5n+4)\zeta_5^i q^{5n+4} &= \frac{1}{(1-\zeta_5^2)(q^5;q^5)_\infty}\left(-\zeta_5^3 X_1Y_2 - \zeta_5^4 X_3Y_2\right) \\
&= \frac{1}{(1-\zeta_5^2)(q^5;q^5)_\infty}\left(-\zeta_5^3 X_1Y_2 + \zeta_5^4 X_1Y_2\right) \\
&= \frac{(\zeta_5+\zeta_5^4)}{(q^5;q^5)_\infty}\sum_{m\in\mathbb{Z}}(-1)^m q^{(5m+1)(5m+2)/2}\sum_{n\geq 0}(-1)^n q^{(5n+2)(5n+3)/2} \\
&= \frac{(\zeta_5+\zeta_5^4)q}{(q^{10};q^{25})_\infty(q^{15};q^{25})_\infty}\sum_{n\geq 0}(-1)^n q^{(5n+2)(5n+3)/2}.
\end{aligned}$$

Recalling (3.6) we have

$$\begin{aligned}
U_0(4) + \left(U_1(4) - \frac{q\sum_{n\geq 0}(-1)^n q^{(5n+2)(5n+3)/2}}{(q^{10};q^{25})_\infty(q^{15};q^{25})_\infty}\right)\zeta_5 + U_2(4)\zeta_5^2 + U_3(4)\zeta_5^3 \\
+\left(U_4(4) - \frac{q\sum_{n\geq 0}(-1)^n q^{(5n+2)(5n+3)/2}}{(q^{10};q^{25})_\infty(q^{15};q^{25})_\infty}\right)\zeta_5^4 = 0.
\end{aligned}$$

As before, the coefficients of ζ_5^i are identical, giving (1.12) and (1.13).

Next we turn to equations (1.8) and (1.9). Here we will need to take into account the fact that the exponent of q in the final term on the right-hand side of (3.5) may be $5n+2$. But first, to obtain an exponent of the form $5n+2$ in the product of the first two sums on the right-hand side of (3.5) we require $(m,n) \equiv (1,1)$, $(1,3)$, $(3,1)$, or $(3,3)$ modulo 5. The contribution to

$$\sum_{n\geq 0}\sum_{i=0}^{4} U(i,5,5n+2)\zeta_5^i q^{5n+2}$$

is thus

$$\begin{aligned}
\frac{1}{(1-\zeta_5^2)(q^5;q^5)_\infty}\left(\zeta_5 X_1Y_1 + X_1Y_3 + \zeta_5^2 X_3Y_1 + \zeta_5 X_3Y_3\right) \\
= \frac{1}{(1-\zeta_5^2)(q^5;q^5)_\infty}\left((\zeta_5-\zeta_5^2)X_1Y_1 + (1-\zeta_5)X_1Y_3\right).
\end{aligned} \tag{3.7}$$

Next we turn to the final sum in (3.5). The contribution to q^{5n+2} comes from $q^{m(3m+1)/2}$ with $m \equiv 1,2 \pmod 5$ or $q^{m(3m+1)/2+2m+1}$ with $m \equiv 2,3 \pmod 5$. Thus the contribution is

$$(1-\zeta_5)\zeta_5^3 C_1 + (1-\zeta_5)\zeta_5 C_2 + (1-\zeta_5)\zeta_5^3 C_3 + (1-\zeta_5)\zeta_5 C_4, \tag{3.8}$$

where

$$\begin{aligned} C_1 &= \sum_{n\geq 0}(-1)^{n+1}q^{(5n+1)(15n+4)/2}, \\ C_2 &= \sum_{n\geq 0}(-1)^{n}q^{(5n+2)(15n+7)/2}, \\ C_3 &= \sum_{n\geq 0}(-1)^{n+1}q^{(5n+2)(15n+7)/2+10n+5} = \sum_{n\leq -1}(-1)^{n}q^{(5n+2)(15n+7)/2}, \\ C_4 &= \sum_{n\geq 0}(-1)^{n}q^{(5n+3)(15n+10)/2+10n+7} = \sum_{n\leq -1}(-1)^{n+1}q^{(5n+1)(15n+4)/2}. \end{aligned}$$

Putting equations (3.7) and (3.8) together we have

$$\sum_{n\geq 0}\sum_{i=0}^{4} U(i,5,5n+2)\zeta_5^i q^{5n+2} = \frac{\zeta_5 X_1Y_1 + X_1Y_3}{(1+\zeta_5)(q^5;q^5)_\infty} + (\zeta_5^3-\zeta_5^4)(C_1+C_3) + (\zeta_5-\zeta_5^2)(C_2+C_4).$$

Now, multiplying both sides of the above by $(1+\zeta_5)$, recalling the notation (3.6), and simplifying, we have

$$\begin{aligned} 0 &= \Big(U_0(2)+U_4(2)-X_1Y_3+C_1+C_3\Big) + \zeta_5\Big(U_1(2)+U_0(2)-X_1Y_1-(C_2+C_4)\Big) \\ &\quad + \zeta_5^2\Big(U_2(2)+U_1(2)\Big) + \zeta_5^3\Big(U_3(2)+U_2(2)+C_2+C_4-(C_1+C_3)\Big) + \zeta_5^4\Big(U_4(2)+U_3(2)\Big). \end{aligned}$$

Again since the minimal polynomial of ζ_5 over $\mathbb{Q}$ is $1+x+x^2+x^3+x^4$ we have that the coefficients of ζ_5^i must be equal. Subtracting the coefficient of ζ_5^2 from the coefficient of ζ_5^0 and applying (3.4) gives (1.8), and subtracting the coefficient of ζ_5^3 from the coefficient of ζ_5^2 gives (1.9).

Equations (1.4) and (1.5) are similar. To obtain an exponent of the form $5n$ in the product of the first two sums on the right-hand side of (3.5) we require $(m,n)\equiv(0,0)$, $(0,4)$, $(4,0)$, or $(4,4)$ modulo 5. The contribution to

$$\sum_{n\geq 0}\sum_{i=0}^{4} U(i,5,5n)\zeta_5^i q^{5n}$$

is thus

$$\begin{aligned} &\frac{1}{(1-\zeta_5^2)(q^5;q^5)_\infty}\left(\zeta_5X_0Y_0+\zeta_5^4X_0Y_4+\zeta_5^3X_4Y_0+\zeta_5X_4Y_4\right) \\ &\qquad = \frac{1}{(1-\zeta_5^2)(q^5;q^5)_\infty}\left((\zeta_5-\zeta_5^3)X_0Y_0+(\zeta_5-\zeta_5^4)X_4Y_4\right). \end{aligned} \tag{3.9}$$

Next the contribution to q^{5n} from the final sum in (3.5) comes from $q^{m(3m+1)/2}$ with $m\equiv 0,3 \pmod 5$ or $q^{m(3m+1)/2+2m+1}$ with $m\equiv 1,4 \pmod 5$. Thus the contribution is

$$(1-\zeta_5)D_1+(1-\zeta_5)\zeta_5^4D_2+(1-\zeta_5)D_3+(1-\zeta_5)\zeta_5^4D_4, \tag{3.10}$$

where

$$
\begin{aligned}
D_1 &= \sum_{n\geq 0}(-1)^n q^{(5n)(15n+1)/2},\\
D_2 &= \sum_{n\geq 0}(-1)^{n+1} q^{(5n+3)(15n+10)/2},\\
D_3 &= \sum_{n\geq 0}(-1)^n q^{(5n+1)(15n+4)/2+10n+3} = \sum_{n\leq -1}(-1)^{n+1} q^{(5n+3)(15n+10)/2},\\
D_4 &= \sum_{n\geq 0}(-1)^{n+1} q^{(5n+4)(15n+13)/2+10n+9} = \sum_{n\leq -1}(-1)^n q^{(5n)(15n+1)/2}.
\end{aligned}
$$

Putting equations (3.9) and (3.10) together we have

$$
\begin{aligned}
\sum_{n\geq 0}\sum_{i=0}^{4} U(i,5,5n)\zeta_5^i q^{5n} =& \frac{(\zeta_5+\zeta_5^2)X_0Y_0 + (\zeta_5+\zeta_5^2+\zeta_5^3)X_4Y_4}{(1+\zeta_5)(q^5;q^5)_\infty}\\
&+ (1-\zeta_5)(D_1+D_3) + (\zeta_5^4-1)(D_2+D_4).
\end{aligned} \tag{3.11}
$$

Now, multiplying both sides of the above by $(1+\zeta_5)$, recalling the notation (3.6), and simplifying, we have

$$
\begin{aligned}
0 =& \Big(U_0(0)+U_4(0)-(D_1+D_3)\Big) + \zeta_5\Big(U_1(0)+U_0(0)-X_0Y_0-X_4Y_4+D_2+D_4\Big)\\
&+ \zeta_5^2\Big(U_2(0)+U_1(0)-X_0Y_0-X_4Y_4+D_1+D_3\Big) + \zeta_5^3\Big(U_3(0)+U_2(0)-X_4Y_4\Big)\\
&+ \zeta_5^4\Big(U_4(0)+U_3(0)-(D_2+D_4)\Big).
\end{aligned} \tag{3.12}
$$

As usual since the minimal polynomial of ζ_5 over $\mathbb{Q}$ is $1+x+x^2+x^3+x^4$ we have that the coefficients of ζ_5^i must be equal. Subtracting the coefficient of ζ_5^2 from the coefficient of ζ_5 gives (1.4), and subtracting the coefficient of ζ_5^3 from the coefficient of ζ_5^2 and and applying (3.4) gives (1.5).

The final case is the progression $5n+1$. Here there are nine pairs (m,n) which give an exponent of q of the form $5n+1$ in the first term on the right-hand side of (3.5), namely $(0,1)$, $(0,3)$, $(4,1)$, $(4,3)$, $(2,2)$, $(1,0)$, $(3,0)$, $(1,4)$, and $(3,4)$ We obtain a contribution of

$$
\begin{aligned}
&\frac{\Big(-\zeta_5^3X_0Y_1-\zeta_5^2X_0Y_3-X_4Y_1-\zeta_5^4X_4Y_3+\zeta_5X_2Y_2-\zeta_5^4X_1Y_0-X_3Y_0-\zeta_5^2X_1Y_4-\zeta_5^3X_3Y_4\Big)}{(1-\zeta_5^2)(q^5;q^5)_\infty}\\
&= \frac{\Big((\zeta_5^4-\zeta_5^2)X_0Y_3+(1-\zeta_5^4)X_1Y_0+(1-\zeta_5^3)X_0Y_1+(\zeta_5^3-\zeta_5^2)X_1Y_4\Big)}{(1-\zeta_5^2)(q^5;q^5)_\infty}.
\end{aligned} \tag{3.13}
$$

The contribution from the final sum in (3.5) comes from $q^{m(3m+1)/2}$ with $m \equiv 4 \pmod 5$ or $q^{m(3m+1)/2+2m+1}$ with $m \equiv 0 \pmod 5$. We obtain

$$\begin{aligned}
&(1-\zeta_5)\zeta_5^2 \sum_{n\geq 0} q^{(5n+4)(15n+13)/2} - (1-\zeta_5)\zeta_5^2 \sum_{n\geq 0} (-1)^n q^{5n(15n+1)/2+10n+1} \\
&= (\zeta_5^2-\zeta_5^3)\left(\sum_{n\geq 0} q^{(5n+4)(15n+13)/2} + \sum_{n\leq -1} (-1)^n q^{(5n+4)(15n+13)/2}\right) \\
&= (\zeta_5^3-\zeta_5^2)\sum_{n\in\mathbb{Z}} q^{(5n+1)(15n+2)/2} \\
&= (\zeta_5^3-\zeta_5^2) q (q^{25}, q^{50}, q^{75}; q^{75})_\infty \\
&=: (\zeta_5^3-\zeta_5^2)P.
\end{aligned} \tag{3.14}$$

Putting equations (3.13) and (3.14) together we have

$$\begin{aligned}
\sum_{n\geq 0}\sum_{i=0}^{4} U(i,5,5n+1)\zeta_5^i q^{5n+1} &= \frac{\Big((\zeta_5^4-\zeta_5^2)X_0Y_3 + (1-\zeta_5^4)X_1Y_0 + (1-\zeta_5^3)X_0Y_1 + (\zeta_5^3-\zeta_5^2)X_1Y_4\Big)}{(1-\zeta_5^2)(q^5;q^5)_\infty} \\
&\quad + (\zeta_5^3-\zeta_5^2)P.
\end{aligned} \tag{3.15}$$

Multiplying both sides of the above by $(1-\zeta_5^2)$, recalling the notation (3.6), and simplifying, we have

$$\begin{aligned}
0 &= \Big(U_0(1) - U_3(1) - X_1Y_0 - X_0Y_1 - P\Big) + \zeta_5\Big(U_1(1) - U_4(1)\Big) \\
&\quad + \zeta_5^2\Big(U_2(1) - U_0(1) + X_0Y_3 + X_1Y_4 + P\Big) + \zeta_5^3\Big(U_3(1) - U_1(1) + X_0Y_1 - X_1Y_4 - P\Big) \\
&\quad + \zeta_5^4\Big(U_4(1) - U_2(1) - X_0Y_3 + X_1Y_0 - P\Big).
\end{aligned} \tag{3.16}$$

Now the coefficients of ζ_5^i are all equal to 0 since the coefficient of ζ_5 is 0. The fact that the coefficient of ζ_5^2 is 0 gives (1.6) and the fact that the coefficient of ζ_5^4 is 0 gives (1.7).

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Define $U(q)$ by

$$U(q) := \sum_{n\geq 0} U(n)q^n = \frac{1}{(q)_\infty^2}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}.$$

Then, we calculate that

$$\begin{aligned}
q\frac{d}{dq}U(q) &= \sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}\Big(-2(q)_\infty^{-3}(q)_\infty \sum_{n=1}^{\infty}\frac{-nq^n}{1-q^n}\Big) + \frac{1}{(q)_\infty^2}\sum_{n=0}^{\infty}(-1)^n\frac{n^2+n}{2}q^{\binom{n+1}{2}} \\
&\equiv \frac{2}{(q)_\infty^2}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \frac{1}{(q)_\infty^2}\sum_{n=0}^{\infty}(-1)^n(4n^2+4n)q^{\binom{n+1}{2}} \pmod 7.
\end{aligned} \tag{3.17}$$

Comparing this with equation (2.10) gives equation (1.15). □

We remark that Theorem 1.2 implies the congruence

$$U(1,7,7n+6)+4U(2,7,7n+6)+2U(3,7,7n+6)\equiv 0 \pmod 7. \tag{3.18}$$

We also note that the proof of Theorem 1.2 only works modulo 7, as 7 is the only prime p for which $2^{-1}\equiv 4 \pmod p$.

4. Remarks on the moduli 3 and 4

We have focused on the moduli 5 and 7, but equation (2.3) can also be used to obtain results modulo 3 and modulo 4. In the latter case, we consider $F(\imath, q)$, and find that on one hand

$$\begin{aligned} F(\imath,q) &= \sum_{n\geq 0}\Big(U(0,4,n)+U(1,4,n)\imath-U(2,4,n)-U(3,4,n)\imath\Big)q^n \\ &= \sum_{n\geq 0}\Big(U(0,4,n)-U(2,4,n)\Big)q^n, \end{aligned} \tag{4.1}$$

while on the other hand using (2.3) we have (assuming that q is real)

$$F(\imath,q)=\Re\Big(\frac{\imath\sum_{n\geq 0}q^{\binom{n+1}{2}}}{(-q^2;q^2)_\infty}+(1-\imath)\sum_{n\geq 0}\imath^n q^{n(3n+1)/2}(1+q^{2n+1})\Big). \tag{4.2}$$

Thus picking off the real part of the final sum gives:

Theorem 4.1.

$$\sum_{n\geq 0}\Big(U(0,4,n)-U(2,4,n)\Big)q^n=\sum_{n\geq 0}(-1)^{\binom{n}{2}}q^{n(3n+1)/2}(1+q^{2n+1}). \tag{4.3}$$

Turning to the modulus 3, we have

$$\begin{aligned} F(\zeta_3,q) &= \sum_{n\geq 0}\Big(U(0,3,n)+(\zeta_3+\zeta_3^2)U(1,3,n)\Big)q^n \\ &= \frac{(q)_\infty}{(q^3;q^3)_\infty}\sum_{n\geq 0}(-1)^n\zeta_3^{2n+1}q^{\binom{n+1}{2}}+(1-\zeta_3)\sum_{n\geq 0}(-1)^n q^{n(3n+1)/2}(1-\zeta_3^2q^{2n+1}). \end{aligned} \tag{4.4}$$

After expanding $(q)_\infty=\sum_{m\in\mathbb{Z}}(-1)^m q^{m(3m+1)/2}$ it is a straightforward calculation to determine the coefficients of q^{3n+x} on the right-hand side of (4.4). We omit the details, but record the result.

Theorem 4.2.

$$\begin{aligned} U_{01}(0) &= \frac{q^2(q^3,q^{24},q^{27};q^{27})_\infty}{(q^3;q^3)_\infty}\sum_{n\geq 0}(-1)^n q^{(3n+1)(3n+2)/2} \\ &\quad -\frac{(q^{12},q^{15},q^{27};q^{27})_\infty}{(q^3;q^3)_\infty}\sum_{n\geq 0}(-1)^n q^{(3n+2)(3n+3)/2}+\left(\sum_{n\geq 0}-2\sum_{n\leq -1}\right)(-1)^n q^{(3n)(9n+1)/2}, \end{aligned} \tag{4.5}$$

$$U_{01}(1) = \frac{q(q^6, q^{21}, q^{27}; q^{27})_\infty}{(q^3; q^3)_\infty} \sum_{n\geq 0} (-1)^n q^{(3n+2)(3n+3)/2}$$
$$- \frac{(q^{12}, q^{15}, q^{27}; q^{27})_\infty}{(q^3; q^3)_\infty} \sum_{n\geq 0} (-1)^n q^{(3n+1)(3n+2)/2} + \left(\sum_{n\geq 0} - 2 \sum_{n\leq -1} \right) (-1)^n q^{(3n+2)(9n+7)/2}, \tag{4.6}$$

$$U_{01}(2) = \frac{q(q^6, q^{21}, q^{27}; q^{27})_\infty}{(q^3; q^3)_\infty} \sum_{n\geq 0} (-1)^n q^{(3n+1)(3n+2)/2}$$
$$+ q^2 \frac{(q^3, q^{24}, q^{27}; q^{27})_\infty}{(q^3; q^3)_\infty} \sum_{n\geq 0} (-1)^n q^{(3n+2)(3n+3)/2} - \left(\sum_{n\geq 0} - 2 \sum_{n\leq -1} \right) (-1)^n q^{(3n+1)(9n+4)/2}. \tag{4.7}$$

References

[1] S. Ahlgren and S. Treneer, Rank generating functions as weakly holomorphic modular forms, *Acta Arith.* **133** (2008), 267-279

[2] G.E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, MA, 1976; reissued by Cambridge University Press, Cambridge, 1984.

[3] G.E. Andrews, Bailey's transform, lemma, chains and tree, in: Special functions 2000: current perspective and future directions (Tempe, AZ), 1–22, NATO Sci. Ser. II Math. Phys. Chem., **30**, Kluwer Acad. Publ., Dordrecht, 2001.

[4] G.E. Andrews, Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks, *Invent. Math.* **169** (2007), 37–73.

[5] G.E. Andrews, Concave and convex compositions, *Ramanujan J.* **31** (2013), 67–82.

[6] G.E. Andrews and B.C. Berndt, Ramanujan's Lost Notebook Part II, Springer, New York, 2009.

[7] G.E. Andrews and F.G. Garvan, Dyson's crank of a partition, *Bull. Am. Math. Soc.* **18** (1988), 167–171.

[8] A.O.L. Atkin and F.G. Garvan, Relations between the ranks and cranks of partitions, *Ramanujan J.* **7** (2003), 343–366.

[9] A.O.L. Atkin and H.P.F. Swinnerton-Dyer, Some properties of partitions, *Proc. London Math. Soc.* **66** (1954), 84–106.

[10] F. C. Auluck, On some new types of partitions associated with generalized Ferrers graphs, *Proc. Cambridge Phil. Soc.* **47** (1951), 679–686.

[11] K. Bringmann, F. Garvan and K. Mahlburg, Partition statistics and quasiharmonic Maass forms, *Int. Math. Res. Not. IMRN*, (2009), 63–97.

[12] K. Bringmann, J. Lovejoy, and R. Osburn, Rank and crank moments for overpartitions, *J. Number Theory* **129** (2009), 1758–1772.

[13] K. Bringmann and K. Mahlburg, Asymptotic formulas for stacks and unimodal sequences, preprint.

[14] K. Bringmann, K. Ono, and R. Rhoades, Eulerian series as modular forms, *J. Amer. Math. Soc.* **21** (2008), 1085–1104

[15] J. Bryson, K. Ono, S. Pitman, and R.C. Rhoades, Unimodal sequences and quantum and mock modular forms, *Proc. Natl. Acad. Sci. USA* **109** vol. 40 (2012), 16063–16067.

[16] M. Dewar, The nonholomorphic parts of certain weak Maass forms, *J. Number Theory* **130** (2010) 559–573.

[17] F. Dyson, Some guesses in the theory of partitions, *Eureka (Cambridge)* **8** (1944), 10–15.

[18] F.G. Garvan, Congruences for Andrews' spt-function modulo powers of 5, 7, and 13, *Trans. Amer. Math. Soc.* **364** (2012), 4847–4873.

[19] R.P. Lewis, On the rank and the crank moduli 4 and 8, *Trans. Amer. Math. Soc.* **341** (1994), 449–465.

[20] J. Lovejoy, Ramanujan-type partial theta identities and conjugate Bailey pairs, *Ramanujan J.* **29** (2012), 51–67.

[21] J. Lovejoy and R. Osburn, Rank differences for overpartitions, *Quart. J. Math. (Oxford)* **59** (2008) 257–273.
[22] J. Lovejoy and R. Osburn, $M2$-rank differences for partitions without repeated odd parts, *J. Théor. Nombres Bordeaux* **21** (2009), 313–334.
[23] J. Lovejoy and R. Osburn, $M2$-rank differences for overpartitions, *Acta Arith.* **144** (2010), 193–212.
[24] R. Mao, Ranks of partitions modulo 10, *J. Number Theory* **133** (2013), 3678–3702.
[25] R.C. Rhoades, Strongly unimodal sequences and mixed mock modular forms, *Int. Math. Res. Notices IMRN*, to appear.
[26] N. Santa-Gadea, On the rank and crank moduli 8, 9, and 12, PhD thesis, Pennsylvania State University, USA, 1990.
[27] N. Santa-Gadea, On some relations for the rank moduli 9 and 12, *J. Number Theory* **40** (1992), 130–145.
[28] L.J. Slater, A new proof of Rogers's transformations of infinite series, *Proc. London Math. Soc. (2)* **53** (1951), 460–475.
[29] H.N.V. Temperley, Statistical mechanics and the partitions of numbers II. The form of crystal surfaces, *Proc. Camb. Phil. Soc.* **48** (1952), 683–697.
[30] S.O. Warnaar, 50 years of Bailey's lemma, in: Algebraic combinatorics and applications (Gößweinstein, 1999), 333–347, Springer, Berlin, 2001.
[31] E.M. Wright, Stacks, *Quart. J. Math. Oxford Ser. (2)* **19** (1968), 313–320.
[32] E.M. Wright, Stacks, II, *Quart. J. Math. Oxford Ser. (2)* **22** (1971), 107–116.

School of Liberal Arts and Institute of Convergence Fundamental Studies, Seoul National University of Science and Technology, 172 Gongreung 2 dong, Nowongu, Seoul 139–743, Republic of Korea
E-mail address: bkim4@seoultech.ac.kr

CNRS, LIAFA, Université Denis Diderot - Paris 7, Case 7014, 75205 Paris Cedex 13, FRANCE
E-mail address: lovejoy@liafa.univ-paris-diderot.fr