A MOCK THETA FUNCTION IDENTITY RELATED TO THE PARTITION RANK MODULO 3 AND 9

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For Bruce Berndt on the occasion of his 80th birthday

ABSTRACT. We prove a new mock theta function identity related to the partition rank modulo 3 and 9. As a consequence, we obtain the 3-dissection of the rank generating function modulo 9. We also evaluate all of the components of the rank-crank differences modulo 9. These are analogous to conjectures of R.P. Lewis [19] on rank-crank differences modulo 8, first proved by E. Mortenson [22].

1. Introduction

In a recent paper [2], G. E. Andrews, B. C. Berndt, S. Kim, A. Malik, and the first author gave new proofs of four identities for Ramanujan's third order mock theta functions. These four identities were first proved by H. Yesilyurt [24]. The most difficult identity to prove among the four is,

$$\frac{1}{2}(1+e^{\pi i/4})\tilde{\phi}(iq)+\frac{1}{2}(1+e^{-\pi i/4})\tilde{\phi}(-iq)$$
$$=f_{\sqrt{2}}(q)+\frac{1}{\sqrt{2}}\psi(-q)(-q^2;q^4)_\infty\prod_{n=1}^{\infty}\frac{1}{1+\sqrt{2}q^n+q^{2n}}, \tag{1.1}$$

where

$$\tilde{\phi}(q):=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(iq,q/i)_n}, \qquad \psi(q):=\sum_{n=0}^{\infty}q^{n(n+1)/2}=(-q;q)_\infty^2(q;q)_\infty$$

and for $a\in\mathbb{R}$

$$f_a(q):=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1+aq+q^2)(1+aq^2+q^4)\cdots(1+aq^n+q^{2n})}. \tag{1.2}$$

Here and in the rest of the paper, we use the standard notations

$$(x)_0:=(x;q)_0:=1,$$
$$(x)_n:=(x;q)_n:=\prod_{k=0}^{n-1}(1-xq^k), \quad (x)_\infty:=(x;q)_\infty:=\prod_{k=0}^{\infty}(1-xq^k),$$
$$(x_1,\dots,x_m)_\infty:=(x_1,\dots,x_m;q)_\infty:=(x_1;q)_\infty\cdots(x_m;q)_\infty,$$

$$J_a := (q^a;q^a)_\infty, \quad J_{a,b} := (q^a, q^{b-a}, q^b; q^b)_\infty, \quad j(x;q) := (x, q/x, q; q)_\infty,$$

and we assume $|q| < 1$ for convergence.

The q-series occurring in (1.1) and (1.2) are related to the generating function for partition ranks. Recall that F. J. Dyson [10] introduced and defined the *rank* of a partition as the largest part minus the number of parts. Let $N(m,n)$ denote the number of partitions of n with rank m. Then Dyson showed that the generating function for $N(m,n)$ is

$$G(z,q) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N(m,n) q^n z^m = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (q/z)_n},$$

and we see that f_a is a specialization of this generating function,

$$f_a(q) = G\left(\frac{-a \pm \sqrt{a^2-4}}{2}, q \right).$$

If $|a| \leq 2$ then $(-a \pm \sqrt{a^2-4})/2$ is a root of unity, and so $f_a(q)$ is a mock theta function for $-2 < a \leq 2$ by [25, Theorem 7.1]. Several of Ramanujan's third order mock theta functions are instances of $f_a(q)$.

The new proof of (1.1) in [2] uses the 2-dissections of the rank generating function modulo 4 and 8, which are proved for the first time in [2]. A natural question that arises is whether there are other dissections of $G(z;q)$ that lead to analogous identities involving mock theta functons. This brings us to our first main result.

Theorem 1.1. *Let ζ be a primitive ninth root of unity and let $\omega = \zeta^3$. Define*

$$C(z,q) := \frac{(q)_\infty}{(zq, q/z)_\infty}, \tag{1.3}$$

and recall that by (1.2),

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega q, q/\omega)_n} \qquad \text{and} \qquad f_{-2\Re(\zeta)}(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta q, q/\zeta)_n}.$$

Then

$$\begin{aligned} &\frac{1}{3}(1-\zeta)\big(f_1(\omega q) - \omega^2 C(\omega, \omega q)\big) + \frac{1}{3}(1-\zeta^{-1})\big(f_1(\omega^2 q) - \omega C(\omega, \omega^2 q)\big) \\ &= f_{-2\Re(\zeta)}(q) + \frac{1}{3}(\zeta^2 + \zeta^{-2} - \zeta - \zeta^{-1}) \frac{J_3^3}{J_1 J_9}. \end{aligned} \tag{1.4}$$

The proof of identity (1.4) relies on properties of Appell-Lerch series [14, Eq. (16)] along with the 3-dissection of the rank modulo 3, which first appeared in [9]. (See (2.1) and (2.2)).

Observe that (1.4), unlike (1.1), contains extra infinite products on the left-hand side. These products are specializations of $C(z,q)$, which is the generating function for the partition crank. The crank is a partition statistic that was predicted by Dyson [10] and finally discovered by Andrews and Garvan [3] in 1988. To define it, let $w(\lambda)$ denote the number of ones in a partition λ and let $v(\lambda)$ denote the number of parts greater than $w(\lambda)$. Then the crank is equal to the largest part, if $w(\lambda) = 0$, and $v(\lambda) - w(\lambda)$ otherwise. Andrews and Garvan showed that if $M(m,n)$ is defined via

$$C(z,q) = \sum_{n=0}^{\infty} M(m,n) z^m q^n,$$

then for $n \geq 2$, $M(m,n)$ is the number of partitions of n with crank equal to m.

A consequence of (1.4) is the 3-dissection of the rank generating function modulo 9, which is presented here for the first time. This is the second objective of this paper.

Theorem 1.2. *Let ζ be a primitive ninth root of unity. Then*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta q, q/\zeta)_n} \\
&= (\zeta^2+\zeta^{-2}) \frac{q^3 J_{27}^3}{J_9 J_{12,27}} + \frac{J_{12,27} J_{27}^3}{J_9 J_{3,27} J_{6,27}} + \frac{\zeta+\zeta^{-1}-2}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1-q^{27n+21}} \\
&\quad + \frac{q J_{27}^3}{J_9 J_{3,27}} + (\zeta+\zeta^{-1}-1) \frac{q^4 J_{3,27} J_{27}^3}{J_9 J_{6,27} J_{12,27}} + \frac{\zeta+\zeta^{-1}+\zeta^2+\zeta^{-2}-1}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+7}}{1-q^{27n+12}} \\
&\quad + (\zeta+\zeta^{-1}-1) \frac{q^2 J_{27}^3}{J_9 J_{6,27}} - (\zeta^2+\zeta^{-2}) \frac{q^2 J_{6,27} J_{27}^3}{J_9 J_{3,27} J_{12,27}} + \frac{1+\zeta^2+\zeta^{-2}}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+2}}{1-q^{27n+3}}.
\end{aligned}$$

Dissections like this can be used to obtain results on $N(k,m,n)$, the number of partitions of n where the rank is congruent to k modulo m. Dyson [10] was the first to observe identities like

$$\begin{aligned}
N(1,5,5n+1) &= N(2,5,5n+1), \\
N(i,5,5n+4) &= N(j,5,5n+4), \\
N(i,7,7n+5) &= N(j,7,7n+5), \\
N(0,7,7n+6) + N(1,7,7n+6) &= N(2,7,7n+6) + N(3,7,7n+6),
\end{aligned}$$

the middle two being true for all i,j. Dyson's observations were confirmed by A.O.L. Atkin and H.P.F. Swinnerton-Dyer [5], and since then there have been many studies of identities of this type – see [4, 8, 14, 15, 18, 20, 21], for example.

By comparing Theorem 1.2 with the 3-dissection of the rank modulo 3 given in (2.1), we see that the same series and products appear in the dissections. This leads to the following rank identities, which were conjectured by R. P. Lewis [16] and first proved by Santa-Gadea in his thesis [23]. This phenomenon is also present in [2], where dissections of the rank modulo 4 and 8 imply relations on ranks modulo 8.

Corollary 1.3. *For all $n \geq 0$ we have*

$$N(3,9,3n) = N(4,9,3n) \tag{1.5}$$

$$N(1,9,3n+1)+N(2,9,3n+1)=N(3,9,3n+1)+N(4,9,3n+1) \tag{1.6}$$
$$N(0,9,3n+2)=N(4,9,3n+2). \tag{1.7}$$

There is also an extensive literature on similar identities for the crank as well as mixed rank-crank identities (e.g. [11, 12, 16, 19, 20, 22]). For example, letting $M(k,m,n)$ denote the number of partitions of n whose crank is congruent to k modulo m, work of Lewis and Santa-Gadea [17, 20] contains identities such as

$$M(4,9,3n+3)=N(4,9,3n+3), \tag{1.8}$$
$$M(0,9,3n+1)+M(1,9,3n+1)=N(1,9,3n+1)+N(2,9,3n+1), \tag{1.9}$$
$$M(1,4,2n+2)=N(2,4,2n+2), \tag{1.10}$$
$$M(0,8,4n+3)+M(1,8,4n+3)=N(2,8,4n+3)+N(3,8,4n+3), \tag{1.11}$$

all valid for $n \geq 0$.

This brings us to the third aim of this paper. In 2009, R. P. Lewis [19] proposed three conjectures involving the difference between ranks and cranks modulo 8. Define

$$R_8(i,j,k) := \sum_{n\geq 0} \big(N(i,8;4n+k)-M(j,8;4n+k)\big)q^n.$$

In his first conjecture, Lewis gave explicit formulas for $R_8(i,j,k)$ for all possible values of (i,j,k). Every $R_8(i,j,k)$ is expressed as a combination of at most one infinite sum and at most two infinite products. For several triples (i,j,k), we have $R_8(i,j,k)=0$. A proof of this conjecture was recently given by E. Mortenson [22].

Here we show that Theorem 1.2 and Mortenson's method can be used to find similar expressions for

$$R_9(i,j,k) := \sum_{n\geq 0} (N(i,9;3n+k)-M(j,9;3n+k))q^n$$

for all possible values of (i,j,k). See Theorem 3.6 for a list of all 75 identities.

The paper is organized as follows. In Section 2, we prove Theorems 1.1 and 1.2 as well as Corollary 1.3. In Section 3, we list the results on $R_9(i,j,k)$ for all combinations of (i,j,k) in Theorem 3.6 and provide a proof.

2. Proofs of Theorems 1.1 and 1.2

First, we record some results from [9]. Let ω be a primitive third root of unity. Theorem 1.1 and Corollary 1.2 of [9] give two versions of the 3-dissections of the rank modulo 3,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega q, q/\omega)_n}$$
$$= -\frac{q^3 J_{27}^3}{J_9 J_{12,27}} + \frac{J_{12,27} J_{27}^3}{J_9 J_{3,27} J_{6,27}} - \frac{3}{J_9} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(27n^2+45n)/2+9}}{1-q^{27n+21}}$$

$$+\frac{qJ_{27}^3}{J_9J_{3,27}}+\frac{q^4J_{3,27}J_{27}^3}{J_9J_{6,27}J_{12,27}}+\frac{3}{J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(27n^2+45n)/2+7}}{1-q^{27n+12}}$$

$$+\frac{q^2J_{27}^3}{J_9J_{6,27}}+\frac{q^2J_{6,27}J_{27}^3}{J_9J_{3,27}J_{12,27}}-\frac{3}{J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(27n^2+45n)/2+2}}{1-q^{27n+3}} \tag{2.1}$$

$$=\frac{1}{J_9}\sum_{n=-\infty}^{\infty}\left\{\frac{(-1)^nq^{(27n^2+9n)/2}}{1-q^{27n+3}}-\frac{(-1)^nq^{(27n^2+27n)/2+3}}{1-q^{27n+12}}-2\frac{(-1)^nq^{(27n^2+45n)/2+9}}{1-q^{27n+21}}\right\}$$

$$+\frac{q}{J_9}\sum_{n=-\infty}^{\infty}\left\{\frac{(-1)^nq^{(27n^2+27n)/2}}{1-q^{27n+3}}+2\frac{(-1)^nq^{(27n^2+45n)/2+6}}{1-q^{27n+12}}+\frac{(-1)^nq^{(27n^2+63n)/2+15}}{1-q^{27n+21}}\right\}$$

$$+\frac{q^2}{J_9}\sum_{n=-\infty}^{\infty}\left\{-2\frac{(-1)^nq^{(27n^2+45n)/2}}{1-q^{27n+3}}-\frac{(-1)^nq^{(27n^2+63n)/2+9}}{1-q^{27n+12}}+\frac{(-1)^nq^{(27n^2+81n)/2+21}}{1-q^{27n+21}}\right\}. \tag{2.2}$$

Similarly, from Theorem 1.3 and Corollary 1.4 of [9], we have respectively,

$$\frac{(q)_\infty}{(\omega q,q/\omega)_\infty}$$

$$=2\frac{q^3J_{27}^3}{J_9J_{12,27}}+\frac{J_{12,27}J_{27}^3}{J_9J_{3,27}J_{6,27}}-2\frac{qJ_{27}^3}{J_9J_{3,27}}+\frac{q^4J_{3,27}J_{27}^3}{J_9J_{6,27}J_{12,27}}-2\frac{q^2J_{27}^3}{J_9J_{6,27}}+\frac{q^2J_{6,27}J_{27}^3}{J_9J_{3,27}J_{12,27}} \tag{2.3}$$

$$=\frac{1}{J_9}\sum_{n=-\infty}^{\infty}\left\{\frac{(-1)^nq^{(27n^2+9n)/2}}{1-q^{27n+3}}+2\frac{(-1)^nq^{(27n^2+27n)/2+3}}{1-q^{27n+12}}+\frac{(-1)^nq^{(27n^2+45n)/2+9}}{1-q^{27n+21}}\right\}$$

$$+\frac{q}{J_9}\sum_{n=-\infty}^{\infty}\left\{-2\frac{(-1)^nq^{(27n^2+27n)/2}}{1-q^{27n+3}}-\frac{(-1)^nq^{(27n^2+45n)/2+6}}{1-q^{27n+12}}+\frac{(-1)^nq^{(27n^2+63n)/2+15}}{1-q^{27n+21}}\right\}$$

$$+\frac{q^2}{J_9}\sum_{n=-\infty}^{\infty}\left\{\frac{(-1)^nq^{(27n^2+45n)/2}}{1-q^{27n+3}}-\frac{(-1)^nq^{(27n^2+63n)/2+9}}{1-q^{27n+12}}-2\frac{(-1)^nq^{(27n^2+81n)/2+21}}{1-q^{27n+21}}\right\}. \tag{2.4}$$

Next, by Lemma 2.2 of [9], we see that

$$m(q,q^3,\omega)=\frac{1}{(1-\omega)J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(27n^2+9n)/2}}{1-q^{27n+3}}\left(-\omega-\omega^2q^{9n+1}-q^{18n+2}\right)$$

$$+\frac{1}{(1-\omega)J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(27n^2+27n)/2+3}}{1-q^{27n+12}}\left(\omega^2+q^{9n+4}+\omega q^{18n+8}\right)$$

$$+\frac{1}{(1-\omega)J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(27n^2+45n)/2+9}}{1-q^{27n+21}}\left(-1-\omega q^{9n+7}-\omega^2q^{18n+14}\right) \tag{2.5}$$

and

$$\begin{aligned}\omega^2m(q,q^3,\omega^2)=&\frac{1}{(1-\omega^2)J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(27n^2+9n)/2}}{1-q^{27n+3}}\left\{-\omega-q^{9n+1}-\omega^2q^{18n+2}\right\}\\&+\frac{1}{(1-\omega^2)J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(27n^2+27n)/2+3}}{1-q^{27n+12}}\left\{1+\omega^2q^{9n+4}+\omega q^{18n+8}\right\}\\&+\frac{1}{(1-\omega^2)J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(27n^2+45n)/2+9}}{1-q^{27n+21}}\left\{-\omega^2-\omega q^{9n+7}-q^{18n+14}\right\},\end{aligned} \tag{2.6}$$

where

$$m(x,q,z):=\frac{1}{(z,q/z,q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(n-1)/2}z^n}{1-q^{n-1}xz}$$

is the Appell-Lerch series.

Using the above results, we derive the following lemma.

Lemma 2.1.

$$m(q,q^3,\omega)=\frac{1}{3}\big(f_1(q)-\omega C(\omega,q)\big), \tag{2.7}$$

$$m(q,q^3,\omega^2)=\frac{1}{3}\left(f_1(q)-\omega^2C(\omega,q)\right). \tag{2.8}$$

Proof. Identity (2.7) follows from applying (2.2) and (2.4) to the right side of (2.5). Similarly, (2.8) follows from applying (2.2) and (2.4) to (2.6). □

Next we need an expression for the rank generating function in terms of Appell-Lerch series.

Lemma 2.2.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(xq,q/x)_n}=&(1-x)m(x^3q,q^3,x^{-3}z^{-1})+(1-x^{-1})m(x^{-3}q,q^3,x^3z)\\&+\frac{x(1-x)J_1^2j(xz;q)j(z;q^3)}{j(x;q)j(z;q)j(x^3z;q^3)}.\end{aligned}$$

Proof. From [14, Eq. (16)], we have

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(xq,q/x)_n}=x(1-x)\Bigg(-\frac{1}{x}m(x^{-3}q^2,q^3,x^3z)-\frac{1}{x^2}m(x^{-3}q,q^3,x^3z)$$

$$+\frac{J_1^2 j(xz;q)j(z;q^3)}{j(x;q)j(z;q)j(x^3z;q^3)}\Bigg) + (1-x)$$
$$= (1-x)(1-m(x^{-3}q^2,q^3,x^3z)) + (1-1/x)m(x^{-3}q,q^3,x^3z)$$
$$+\frac{x(1-x)J_1^2 j(xz;q)j(z;q^3)}{j(x;q)j(z;q)j(x^3z;q^3)}. \tag{2.9}$$

Using [14, Prop. 3.1, (21b)(21c)], we get

$$\begin{aligned}1-m(x^{-3}q^2,q^3,x^3z) &= 1-x^3q^{-2}m(x^3q^{-2},q^3,x^{-3}z^{-1})\\ &= m(x^3q,q^3,x^{-3}z^{-1}).\end{aligned} \tag{2.10}$$

Substituting identity (2.10) into (2.9), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(xq,q/x)_n} =&(1-x)m(x^3q,q^3,x^{-3}z^{-1}) + (1-x^{-1})m(x^{-3}q,q^3,x^3z)\\ &+\frac{x(1-x)J_1^2 j(xz;q)j(z;q^3)}{j(x;q)j(z;q)j(x^3z;q^3)},\end{aligned} \tag{2.11}$$

and this completes the proof of the lemma. □

Set $x=\zeta$, and let z tend to 1 in Lemma 2.2. Noting that $j(\omega,q^3)=(1-\omega)J_9$, we obtain the following corollary.

Corollary 2.3.

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(\zeta q,q/\zeta)_n} = (1-\zeta)m(\omega q,q^3,\omega^2) + (1-1/\zeta)m(\omega^2 q,q^3,\omega) + \frac{\zeta(1-\zeta)}{1-\omega}\frac{J_3^3}{J_1J_9}. \tag{2.12}$$

We need one more lemma for our proof of Theorems 1.1 and 1.2.

Lemma 2.4.

$$\frac{1}{J_1} = \frac{J_9}{J_3^4}\left\{(J_{12,27}^2 - q^3J_{3,27}J_{6,27}) + (qJ_{6,27}J_{12,27} + q^4J_{3,27}^2) + (q^2J_{6,27}^2 + q^2J_{3,27}J_{12,27})\right\}. \tag{2.13}$$

Proof. First note that a simple consequence of Jacobi triple product identity [6, p. 35, Entry 19] and [6, p. 48, Entry 31] is

$$J_1 = J_{12,27} - qJ_{6,27} - q^2J_{3,27}. \tag{2.14}$$

Substituting (2.14) into (2.13), cross multiplying and simplifying, we see that it suffices to prove

$$\frac{J_9}{J_3^4}\left\{J_{12,27}^3 - q^3J_{6,27}^3 - q^6J_{3,27}^3 - 3q^3J_{3,27}J_{6,27}J_{12,27}\right\} = 1. \tag{2.15}$$

By applying the second and third identity from the bottom of page 351 of [6], we see that (2.15) is equivalent to

$$\frac{J_9}{J_3^4}\left\{J_1^3+3qJ_9^3-9q^3J_{3,27}J_{6,27}J_{12,27}\right\}=1. \tag{2.16}$$

Multiplying both sides by J_3^4 and rearranging, we see that (2.16) is equivalent to

$$J_9\left\{J_1^3+3qJ_9^3\right\}=J_3\left\{J_3^3+9q^3J_{27}^3\right\}.$$

This is in turn equivalent to

$$qJ_9^4\left\{3+\frac{J_1^3}{qJ_9^3}\right\}=J_3^4\left\{1+9q^3\frac{J_{27}^3}{J_3^3}\right\}. \tag{2.17}$$

By applying the first equality of Entry 1(iv) on page 345 of [6] on the left side of (2.17) and the last equality of the same entry on the right side, we see that equation (2.17) follows. This completes the proof of Lemma 2.4. □

Theorems 1.1 and 1.2 now follow readily from the work above.

Proof of Theorem 1.1. First, using

$$1+\zeta^3+\zeta^6=0, \tag{2.18}$$

it is easy to verify that $\zeta(1-\zeta)/(1-\omega)=\frac{1}{3}(\zeta+\zeta^{-1}-\zeta^2-\zeta^{-2})$. To complete the proof, it suffices to apply Lemma 2.1 to Corollary 2.3 and rearrange terms. □

Proof of Theorem 1.2. Apply (2.1) and (2.3) on the left hand side of (1.4) and Lemma 2.4 on the right hand side and rearrange terms. □

We end this section with a proof of Corollary 1.3.

Proof of Corollary 1.3. We use the notation

$$N_a=\sum_{n\geq 0}N(a,9,n)q^n,$$
$$N_a(d)=\sum_{n\geq 0}N(a,9,3n+d)q^{3n+d},$$

and note that since the rank generating function is invariant under $z\leftrightarrow 1/z$, we have $N_a=N_{9-a}$ and $N_a(d)=N_{9-a}(d)$. Together with (2.18) and the fact that

$$\zeta+\zeta^2+\zeta^4+\zeta^5+\zeta^7+\zeta^8=0, \tag{2.19}$$

we find that

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(\zeta q,q/\zeta)_n}=N_0-N_3+(\zeta+\zeta^{-1})(N_1-N_4)+(\zeta^2+\zeta^{-2})(N_2-N_4) \tag{2.20}$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(\omega q,q/\omega)_n}=N_0-N_1-N_2+2N_3-N_4. \tag{2.21}$$

Comparing coefficients of $1, \zeta+\zeta^{-1}$, and $\zeta^2+\zeta^{-2}$ in (2.20) and the first line of Theorem 1.2, we find

$$\begin{aligned}
N_0(0)-N_3(0) &= \frac{J_{12,27}J_{27}^3}{J_9 J_{3,27}J_{6,27}} - \frac{2}{J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(27n^2+45n)/2+9}}{1-q^{27n+21}},\\
N_1(0)-N_4(0) &= \frac{1}{J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(27n^2+45n)/2+9}}{1-q^{27n+21}},\\
N_2(0)-N_4(0) &= \frac{q^3 J_{27}^3}{J_9 J_{12,27}}.
\end{aligned}$$

Comparing this with (2.21) and (2.1) we deduce that

$$N_0(0)-N_3(0)-N_1(0)+N_4(0)-N_2(0)+N_4(0) = N_0(0)+2N_3(0)-N_1(0)-N_2(0)-N_4(0).$$

Simplifying, we obtain (1.5). The proofs of equations (1.6) and (1.7) are similar. □

3. Differences of Ranks and Cranks

Recall that the 3-dissection of the crank modulo 3 is given in (2.3). We record the 3-dissection of the crank modulo 9 given in [12, (2.15)] as the following theorem.

Theorem 3.1. *Let ζ be a primitive ninth root of unity. Then*

$$\frac{(q)_\infty}{(\zeta q)_\infty(\zeta^{-1}q)_\infty} = \frac{J_3 J_{27}^2}{J_9}\left(\frac{1}{J_{3,27}} + \left(\zeta+\zeta^{-1}-1\right)\frac{q}{J_{6,27}} + \left(\zeta^2+\zeta^{-2}\right)\frac{q^2}{J_{12,27}}\right). \tag{3.1}$$

Next, we reproduce the definition of rank deviation and crank deviation as given by D. Hickerson and E. Mortenson [14, 22].

Definition 3.2. *The rank and crank deviations are given respectively by*

$$D(a,m) = D(a,m;q) = \sum_{n=0}^{\infty}\left(N(a,m;n) - \frac{p(n)}{m}\right)q^n$$

and

$$D_C(a,m) = D_C(a,m;q) = \sum_{n=0}^{\infty}\left(M(a,m;n) - \frac{p(n)}{m}\right)q^n.$$

We do not compute rank and crank deviations from the definitions. Instead, we use the next two formulas. The first is [14, Eq. (25)], given with a slight variation. The next is the analogous result for cranks which is given in [22, Eq. (2.12)].

Theorem 3.3. *Let ζ_m be a primitive m-th root of unity. Then*

$$D(a,m) = \frac{1}{m}\sum_{j=1}^{m-1}\zeta_m^{-aj}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(\zeta_m^j q)_n(\zeta_m^{-j}q)_n} \tag{3.2}$$

and

$$D_C(a,m) = \frac{1}{m}\sum_{j=1}^{m-1} \zeta_m^{-aj}\frac{(q)_\infty}{(\zeta_m^j q)_\infty(\zeta_m^{-j}q)_\infty}. \tag{3.3}$$

Since the generating functions for $N(m,n)$ and $M(m,n)$ are invariant under $z \leftrightarrow 1/z$, one easily sees that $N(a,m;n) = N(m-a,m;n)$ and $M(a,m;n) = M(m-a,m;n)$. Therefore, it suffices to examine the values of $D(i,9)$ and $D_C(j,9)$ for $i,j = 0,1,2,3,4$.

Theorem 3.4. *Let us denote*

$$A := \frac{J_3 J_{27}^2}{J_9 J_{3,27}}, \qquad B := q\frac{J_3 J_{27}^2}{J_9 J_{6,27}}, \qquad C := q^2\frac{J_3 J_{27}^2}{J_9 J_{12,27}},$$

$$D := 2q^3\frac{J_{27}^3}{J_9 J_{12,27}} + \frac{J_{12,27}J_{27}^3}{J_9 J_{3,27}J_{6,27}} - 2q\frac{J_{27}^3}{J_9 J_{3,27}} + q^4\frac{J_{3,27}J_{27}^3}{J_9 J_{6,27}J_{12,27}} - 2q^2\frac{J_{27}^3}{J_9 J_{6,27}} + q^2\frac{J_{6,27}J_{27}^3}{J_9 J_{3,27}J_{12,27}}.$$

Then we have,

$$\begin{aligned}
D_C(0,9) &= \frac{1}{9}(6A - 6B + 2D),\\
D_C(1,9) &= \frac{1}{9}(6B - 3C - D),\\
D_C(2,9) &= \frac{1}{9}(-3B + 6C - D),\\
D_C(3,9) &= \frac{1}{9}(-3A + 3B + 2D),\\
D_C(4,9) &= \frac{1}{9}(-3B - 3C - D).
\end{aligned}$$

Proof. Using the above notation, (3.1) and (2.3) simplify respectively to

$$\frac{(q)_\infty}{(\zeta q)_\infty(\zeta^{-1}q)_\infty} = A + (\zeta + \zeta^{-1} - 1)B + (\zeta^2 + \zeta^{-2})C \tag{3.4}$$

and

$$\frac{(q)_\infty}{(\omega q)_\infty(\omega^{-1}q)_\infty} = D. \tag{3.5}$$

Computing directly using (3.3) and applying (3.4) and (3.5), we find that

$$\begin{aligned}
D_C(0,9) &= \frac{1}{9}\sum_{j=1}^{8}\frac{(q)_\infty}{(\zeta^j q)_\infty(\zeta^{-j}q)_\infty}\\
&= \frac{1}{9}\Big(2\frac{(q)_\infty}{(\zeta q, q/\zeta)_\infty} + 2\frac{(q)_\infty}{(\zeta^2 q, q/\zeta^2)_\infty} + 2\frac{(q)_\infty}{(\omega q, q/\omega)_\infty} + 2\frac{(q)_\infty}{(\zeta^4 q, q/\zeta^4)_\infty}\Big)\\
&= \frac{2}{9}\Big(A + (\zeta + \zeta^{-1} - 1)B + (\zeta^2 + \zeta^{-2})C + A + (\zeta^2 + \zeta^{-2} - 1)B + (\zeta^4 + \zeta^{-4})C
\end{aligned}$$

$$+ D + A + (\zeta^4 + \zeta^{-4} - 1)B + (\zeta^8 + \zeta^{-8})C\Big)$$

$$= \frac{2}{9}\Big(3A - 3B + D + (\zeta + \zeta^{-1} + \zeta^2 + \zeta^{-2} + \zeta^4 + \zeta^{-4})B$$

$$+ (\zeta^2 + \zeta^{-2} + \zeta^4 + \zeta^{-4} + \zeta^8 + \zeta^{-8})C\Big)$$

$$= \frac{1}{9}(6A - 6B + 2D),$$

by (2.19). The other cases are handled similarly and so we omit the proofs. □

Theorem 3.5. *Denote by*

$$A' = q^3\frac{J_{27}^3}{J_9J_{12,27}} - q^2\frac{J_{6,27}J_{27}^3}{J_9J_{3,27}J_{12,27}}, \qquad D' = \frac{1}{J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(27n^2+45n)/2+9}}{1-q^{27n+21}},$$

$$B' = \frac{J_{12,27}J_{27}^3}{J_9J_{3,27}J_{6,27}}, +q\frac{J_{27}^3}{J_9J_{3,27}}, \qquad E' = \frac{1}{J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(27n^2+45n)/2+7}}{1-q^{27n+12}},$$

$$C' = q^4\frac{J_{3,27}J_{27}^3}{J_9J_{6,27}J_{12,27}} + q^2\frac{J_{27}^3}{J_9J_{6,27}}, \qquad F' = \frac{1}{J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(27n^2+45n)/2+2}}{1-q^{27n+3}}.$$

Then we have

$$D(0,9) = \frac{-2A' + 8B' - 4C' - 18D'}{9},$$
$$D(1,9) = \frac{-2A' - B' + 5C' + 9D'}{9},$$
$$D(2,9) = \frac{7A' - B' - 4C' + 9F'}{9},$$
$$D(3,9) = \frac{-2A' - B' + 5C' + 9E' - 9F'}{9},$$
$$D(4,9) = \frac{-2A' - B' - 4C' - 9E'}{9}.$$

Proof. With the above notation, Theorem 1.2 and equation (2.1) can be rewritten as, respectively,

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(\zeta q, q/\zeta)_n} = (\zeta^2 + \zeta^{-2})A' + B' + (\zeta + \zeta^{-1} - 1)C' + (\zeta + \zeta^{-1} - 2)D'$$
$$+ (\zeta + \zeta^{-1} + \zeta^2 + \zeta^{-2} - 1)E' + (1 + \zeta^2 + \zeta^{-2})F' \tag{3.6}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(\omega q, q/\omega)_n} = -A' + B' + C' - 3D' + 3E' - 3F'. \tag{3.7}$$

To prove the theorem, we compute each of $D(a,9)$ using (3.2) by applying (3.6) and (3.7). The computations are straightforward and similar to those in the proof of Theorem 3.4 and so we omit them. □

Theorems 3.4 and 3.5 provide us the 3-dissections of $D(i,9)$ and $D_C(j,9)$, which in turn leads us to $R_9(i,j,k)$. To display all the results in a neat way, here we define some notation. First, let

$$f_n(i,j,k) := \frac{J_9^2}{J_3 J_{2^n,9}}(i \cdot J_{12,27} + j \cdot qJ_{6,27} + k \cdot q^2 J_{3,27}),$$

Next, for $k = 0, 1, 2$ we define h_k as follows.

$$h_0 := \frac{1}{J_3} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(9n^2+15n)/2+3}}{1-q^{9n+7}},$$

$$h_1 := \frac{1}{J_3} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(9n^2+15n)/2+2}}{1-q^{9n+4}},$$

$$h_2 := \frac{1}{J_3} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(9n^2+15n)/2}}{1-q^{9n+1}}.$$

We are now ready to list the expressions for $R_9(i,j,k)$.

Theorem 3.6. *For $R_9(i,j,k)$, we have,*

$R_9(0,0,0) = f_0(0,0,2) - 2h_0$
$R_9(0,0,1) = f_1(2,0,0)$
$R_9(0,0,2) = 0$
$R_9(0,1,0) = f_0(1,0,1) - 2h_0$
$R_9(0,1,1) = f_1(0,1,1)$
$R_9(0,1,2) = f_2(0,0,-1)$
$R_9(0,2,0) = f_0(1,0,1) - 2h_0$
$R_9(0,2,1) = f_1(1,0,0)$
$R_9(0,2,2) = f_2(-1,1,0)$
$R_9(0,3,0) = f_0(1,-1,1) - 2h_0$
$R_9(0,3,1) = f_1(1,1,1)$
$R_9(0,3,2) = 0$
$R_9(0,4,0) = f_0(1,0,1) - 2h_0$
$R_9(0,4,1) = f_1(1,0,0)$
$R_9(0,4,2) = f_2(0,0,-1)$

$R_9(1,0,0) = f_0(-1,0,1) + h_0$
$R_9(1,0,1) = f_1(1,0,-1)$
$R_9(1,0,2) = f_2(1,0,1)$
$R_9(1,1,0) = h_0$
$R_9(1,1,1) = f_1(-1,1,0)$
$R_9(1,1,2) = f_2(1,0,0)$
$R_9(1,2,0) = h_0$
$R_9(1,2,1) = f_1(0,0,-1)$
$R_9(1,2,2) = f_2(0,1,1)$
$R_9(1,3,0) = f_0(0,-1,0) + h_0$
$R_9(1,3,1) = f_1(0,1,0)$
$R_9(1,3,2) = f_2(1,0,1)$
$R_9(1,4,0) = h_0$
$R_9(1,4,1) = f_1(0,0,-1)$
$R_9(1,4,2) = f_2(1,0,0)$

$R_9(2,0,0) = f_0(-1,1,0)$
$R_9(2,0,1) = f_1(1,-1,0)$
$R_9(2,0,2) = f_2(-1,-1,0) + h_2$
$R_9(2,1,0) = f_0(0,1,-1)$

$R_9(2,1,1) = f_1(-1,0,1)$
$R_9(2,1,2) = f_2(-1,-1,-1) + h_2$
$R_9(2,2,0) = f_0(0,1,-1)$
$R_9(2,2,1) = f_1(0,-1,0)$
$R_9(2,2,2) = f_2(-2,0,0) + h_2$
$R_9(2,3,0) = f_0(0,0,-1)$
$R_9(2,3,1) = f_1(0,0,1)$
$R_9(2,3,2) = f_2(-1,-1,0) + h_2$
$R_9(2,4,0) = f_0(0,1,-1)$
$R_9(2,4,1) = f_1(0,-1,0)$
$R_9(2,4,2) = f_2(-1,-1,-1) + h_2$

$R_9(3,0,0) = f_0(-1,0,1)$
$R_9(3,0,1) = f_1(1,0,-1) + h_1$
$R_9(3,0,2) = f_2(1,0,1) - h_2$
$R_9(3,1,0) = 0$
$R_9(3,1,1) = f_1(-1,1,0) + h_1$
$R_9(3,1,2) = f_2(1,0,0) - h_2$
$R_9(3,2,0) = 0$
$R_9(3,2,1) = f_1(0,0,-1) + h_1$
$R_9(3,2,2) = f_2(0,1,1) - h_2$
$R_9(3,3,0) = f_0(0,-1,0)$
$R_9(3,3,1) = f_1(0,1,0) + h_1$
$R_9(3,3,2) = f_2(1,0,1) - h_2$
$R_9(3,4,0) = 0$
$R_9(3,4,1) = f_1(0,0,-1) + h_1$
$R_9(3,4,2) = f_2(1,0,0) - h_2$

$R_9(4,0,0) = f_0(-1,0,1)$
$R_9(4,0,1) = f_1(1,-1,0) - h_1$
$R_9(4,0,2) = 0$
$R_9(4,1,0) = 0$
$R_9(4,1,1) = f_1(-1,0,1) - h_1$
$R_9(4,1,2) = f_2(0,0,-1)$
$R_9(4,2,0) = 0$
$R_9(4,2,1) = f_1(0,-1,0) - h_1$
$R_9(4,2,2) = f_2(-1,1,0)$
$R_9(4,3,0) = f_0(0,-1,0)$
$R_9(4,3,1) = f_1(0,0,1) - h_1$
$R_9(4,3,2) = 0$
$R_9(4,4,0) = 0$
$R_9(4,4,1) = f_1(0,-1,0) - h_1$
$R_9(4,4,2) = f_2(0,0,-1)$.

Proof. All of these components $R_9(i,j,k)$ can be computed directly from Theorems 3.4 and 3.5. We illustrate the method of proof for the cases $(i,j,k) = (0,0,1)$ and $(0,0,2)$. The other cases are proved similarly and we omit the details.

One can easily check that

$$\begin{aligned}D(0,9) - D_C(0,9) &= \frac{-2A' + 8B' - 4C' - 18D' - 6A + 6B - 2D}{9}\\ &= -\frac{6}{9}\frac{J_3J_{27}^2}{J_9J_{3,27}} + \frac{6}{9}q\frac{J_3J_{27}^2}{J_9J_{6,27}} - \frac{6}{9}q^3\frac{J_{27}^3}{J_9J_{12,27}} + \frac{6}{9}\frac{J_{12,27}J_{27}^3}{J_9J_{3,27}J_{6,27}} + \frac{12}{9}q\frac{J_{27}^3}{J_9J_{3,27}}\\ &\quad - \frac{6}{9}q^4\frac{J_{3,27}J_{27}^3}{J_9J_{6,27}J_{12,27}} - \frac{18}{9}\frac{1}{J_9}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(27n^2+45n)/2+9}}{1-q^{27n+21}}.\end{aligned}$$

Note that all the coefficients of q^{3n+2} are 0 for all integers n. Therefore, $R_9(0,0,2) = 0$. Collecting only terms with q^{3n+1} for all integers n, we have

$$\frac{6}{9}\frac{J_3J_{27}^2}{J_9J_{6,27}} + \frac{12}{9}\frac{J_{27}^3}{J_9J_{3,27}} - \frac{6}{9}q^3\frac{J_{3,27}J_{27}^3}{J_9J_{6,27}J_{12,27}}.$$

Therefore,

$$\begin{aligned}
R_9(0,0,1) &= \frac{6}{9}\frac{J_1J_9^2}{J_3J_{2,9}} + \frac{12}{9}\frac{J_9^3}{J_3J_{1,9}} - \frac{6}{9}q\frac{J_{1,9}J_9^3}{J_3J_{2,9}J_{4,9}} \\
&= \frac{2}{3}\frac{J_9^2}{J_3J_{2,9}}\left(J_1 + 2\frac{J_{2,9}J_9}{J_{1,9}} - q\frac{J_{1,9}J_9}{J_{4,9}}\right) \\
&= \frac{2}{3}\frac{J_9^2}{J_3J_{2,9}}\big((J_{12,27} - qJ_{6,27} - q^2J_{3,27}) + (2J_{12,27} + 2qJ_{6,27}) - (qJ_{6,27} - q^2J_{3,27})\big) \\
&= 2\frac{J_9^2J_{12,27}}{J_3J_{2,9}},
\end{aligned}$$

where in the penultimate equality, we use (2.14) and [7, Eqs. (4.2)–(4.4)] (which were reproduced in [1, Eqs. (3.4.2)–(3.4.4)]).

□

We close with a few remarks on Theorem 3.6. First, we have used the notation $f_n(i,j,k)$ for consistency, but some of these sums of infinite products can be simplified to a single infinite product. For example, using [7, Eq. (4.2)], we have

$$f_0(0,1,-1) = \frac{qJ_9^3}{J_3J_{4,9}}.$$

Similarly, by using [7, Eq. (4.3)], we have

$$f_2(1,0,1) = \frac{J_9^3}{J_3J_{2,9}}.$$

These appear in several places in Theorem 3.6.

Second, Theorem 3.6 immediately gives rank-crank identities like (1.8) and (1.9) and other results of Lewis [17]. For example, by inspection we see that $R_9(1,0,1) = -R_9(2,1,1)$, which implies (1.9). All of the rank-crank identites of Lewis in [17] follow in the same way.

Finally, the 75 results in Theorem 3.6 can also be used to deduce rank-crank inequalities. To give just one example out of many, we have

$$R_9(1,1,2) = f_2(1,0,0) = \frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty(1-q^4)(1-q^5)}\left(1 + \sum_{n=1}^{\infty} a_nq^n\right),$$

where $a_n \geq 0$. Since

$$\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} = 1 + q^3 + 2q^6 + \cdots$$

is the generating function for the number of 3-core partitions of $n/3$ [13] and every natural number at least 12 can be written as $4x+5y$ with $x,y \geq 0$, we deduce that $R_9(1,1,2) > 0$ for all $n \geq 12$. Checking the expansion for $n < 12$ we find that

$$N(1,9,3n+2) > M(1,9,3n+2)$$

except for $n = 8$. We leave the pursuit of other inequalities to the interested reader.

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