PARITY BIAS IN PARTITIONS

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Abstract. Let $p_o(n)$ denote the number of partitions of n with more odd parts than even parts and let $p_e(n)$ denote the number of partitions of n with more even parts than odd parts. Using q-series transformations we find a generating function for $p_o(n) - p_e(n)$, which implies that $p_o(n) > p_e(n)$ for all positive integers $n \neq 2$. Using combinatorial mappings we prove a stronger result, namely that for all $n > 7$ we have $2p_e(n) < p_o(n) < 3p_e(n)$. Finally, using asymptotic methods we show that $p_o(n)/p_e(n) \to 1 + \sqrt{2}$ as $n \to \infty$. We also examine related properties for two other types of partitions.

1. Introduction

Inequalities have a long and important history in the theory of partitions. Two of the most well-known examples are related to the Rogers-Ramanujan identities, which state that

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{1.1}$$

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.2}$$

Here and in the sequel, we use the standard q-series notation:

$$(a)_n = (a;q)_n = \prod_{k=1}^{n} (1 - aq^{k-1})$$

for $n \in \mathbb{N}_0 \cup \{\infty\}$. The Rogers-Ramanujan identities imply that for $a = 1$ or 2, the number of partitions of n into parts of size at least a and differing by at least 2 is equal to the number of partitions of n into parts congruent to $\pm a$ modulo 5. See [15] for much more on these identities.

Now, subtracting (1.2) from (1.1), we have

$$\frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} - \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = \sum_{n\geq 0} \frac{q^{n^2}(1-q^n)}{(q)_n}$$

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$$= \sum_{n \geq 1} \frac{q^{n^2}}{(q)_{n-1}},$$

and the coefficients of the right-hand side are clearly non-negative (and clearly positive for $n \geq 4$). Therefore there are at least as many partitions of n into parts $\equiv 1$ or $4 \pmod 5$ as partitions of n into parts $\equiv 2$ or $3 \pmod 5$. In 1987 at the AMS Institute on Theta Functions, Leon Ehrenpreis asked whether this inequality of partition functions could be established without using the Rogers-Ramanujan identities. The *Ehrenpreis problem*, as it became known, was first solved using a combinatorial mapping by Kadell [12] (see also [6]).

Next let $q_d(n)$ denote the number of partitions of n into parts differing by at least d and let $Q_d(n)$ denote the number of partitions into parts congruent to $\pm 1 \pmod{d+3}$. The first Rogers-Ramanujan identity gives $q_2(n) = Q_2(n)$, while Euler's classical partition identity says that $q_1(n) = Q_1(n)$. This equality does not persist for higher d, however, for a partition theorem of Schur [14] says that the number of partitions of n into parts congruent to ± 1 modulo 6 is equal to the number of partitions of n where parts differ by at least 3 and multiples of 3 differ by at least 6. But this does imply that $q_3(n) \geq Q_3(n)$.

Motivated by these results Alder [1] conjectured that $q_d(n) \geq Q_d(n)$ for all positive d and n. Alder's conjecture was proved in several steps. First, Andrews [3] established the case $d = 2^k - 1$ for $k \geq 4$ using a generalization of Schur's theorem. Much later, Yee [16] constructed an injection that settled the conjecture for $d \geq 32$. The proof was completed by Alfes, Jameson and Lemke Oliver [2] using asymptotic analysis on the finitely many remaining cases.

The point to be taken from the above discussion is that the study of partition inequalities requires a variety of methods, from q-series to bijective combinatorics to asymptotic analysis. In this paper, we use all of these methods to examine *parity bias* in partitions. By parity bias we mean the tendency of partitions to have more odd parts than even parts.

Let $p_o(n)$ denote the number of partitions of n with more odd parts than even parts and let $p_e(n)$ denote the number of partitions of n with more even parts than odd parts. For example, there are 11 partitions of 6,

$$6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1,$$
$$2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1,$$

and so we have $p_o(6) = 7$ and $p_e(6) = 3$. Our first result gives a generating function for $p_o(n) - p_e(n)$, which implies that this difference is positive with one exception.

Theorem 1. *We have*

$$\sum_{n \geq 1} (p_o(n) - p_e(n)) q^n = \frac{1}{(q;q^2)_\infty} \sum_{n \geq 1} \frac{q^{2n^2-n}(1-q^n)}{(q^2;q^2)_n^2}. \tag{1.3}$$

Moreover, for all positive integers $n \neq 2$,

$$p_o(n) > p_e(n). \tag{1.4}$$

By employing a more careful q-series analysis and constructing a combinatorial map, we prove a sharper inequality.

Theorem 2. *For all positive integers $n > 7$,*

$$2p_e(n) < p_o(n) < 3p_e(n).$$

More precisely, for positive integers n we have (i) $2p_e(n) < p_o(n)$ except for $n = 2, 4$ and (ii) $3p_e(n) > p_o(n)$ except for $n = 1, 3, 5, 7$.

Finally, we obtain the exact asymptotic ratio between $p_o(n)$ and $p_e(n)$ by employing Ingham's Tauberian theorem [11].

Theorem 3. *As $n \to \infty$,*

$$\frac{p_o(n)}{p_e(n)} \to 1 + \sqrt{2} \approx 2.4142.$$

In fact, if $p(n)$ denotes the number of partitions of n, we prove that as $n \to \infty$ we have

$$p_o(n) \sim \frac{1}{\sqrt{2}} p(n) \approx 0.7071 p(n)$$
$$p_e(n) \sim \frac{\sqrt{2}-1}{\sqrt{2}} p(n) \approx 0.2929 p(n).$$

We illustrate this in the following table.

TABLE 1. Comparison of $p(n)$, $p_o(n)$, and $p_e(n)$ for $n \leq 10000$ (values rounded to four decimal places)

n	$p_o(n)/p(n)$	$p_e(n)/p(n)$	$p_o(n)/p_e(n)$
100	0.6795	0.2764	2.4588
500	0.6946	0.2854	2.4339
1500	0.6998	0.2885	2.4255
2500	0.7015	0.2895	2.4229
5000	0.7031	0.2905	2.4204
10000	0.7043	0.2912	2.4186

The rest of the paper is organized as follows. In Section 2, we give some brief background on partitions and Frobenius symbols, q-series transformations, and Ingham's Tauberian theorem. In Section 3, we prove the main results. In Section 4, we consider parity bias in partitions without repeated odd parts and in Section 5 we examine a related property for overpartitions. We close in Section 6 with two conjectures.

2. Preliminaries

Let A and B be sets of partitions. In his seminal paper [5], Andrews defined the generalized Frobenius symbol of type (A, B) and weight n to be a two-rowed array of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix}$$

where the top row $(a_1, a_2, \ldots, a_k)$ is a partition in the set A, the bottom row $(b_1, b_2, \ldots, b_k)$ is a partition in the set B, and $n = \sum_{i=1}^{k}(a_i + b_i + 1)$. Throughout the paper, we will use $\mathcal{F}_{A,B}$ to denote the set of Frobenius symbols of type (A, B) and $F_{A,B}(n)$ to denote the number of Frobenius symbols in $\mathcal{F}_{A,B}$ having weight n.

By taking various sets of partitions in the top and bottom rows, generalized Frobenius symbols can be used to represent a number of types of well-known partitions. For example, if $\mathcal{D}$ is the set of partitions into distinct non-negative parts, then $p(n) = F_{\mathcal{D},\mathcal{D}}(n)$. For another example, let $\mathcal{O}$ be the set of overpartitions into non-negative parts. The third author and Corteel [9] bijectively showed that

$$\overline{p}(n) = F_{\mathcal{D},\mathcal{O}}(n),$$

where $\overline{p}(n)$ denotes the number of overpartitions of n. The bijection can be also used to deduce that

$$pod(n) = F_{\mathcal{D}_e,\mathcal{POD}}(n),$$

where $pod(n)$ is the number of partitions of n with odd parts distinct, $\mathcal{D}_e$ is the set of partitions into distinct non-negative even parts, and $\mathcal{POD}$ is the set of partitions into non-negative parts with odd distinct parts [13]. This shows that

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} = \sum_{n\geq 0} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n^2}. \tag{2.1}$$

Much of our combinatorial work in this paper will be in terms of Frobenius symbols. In the following table we record the notation for the sets of partitions used in the sequel.

Table 2. Notation for sets of partitions

Notation	Partitions
$\mathcal{D}_e$	Partitions into distinct non-negative even parts
$\widehat{\mathcal{D}}_e$	Partitions in $\mathcal{D}_e$ where the largest part may be overlined
$\mathcal{D}_o$	Partitions into distinct odd parts
$\mathcal{POD}$	Partitions without repeated odd parts
$\mathcal{POD}_{\geq 1}$	Partitions in $\mathcal{POD}$ having at least one odd part
$\mathcal{POD}_1$	Partitions in $\mathcal{POD}$ having exactly one odd part
$\mathcal{P}'_e$	Partitions into non-negative even parts with at least one 0

Next we recall three q-series identities. We begin with the Heine transformation [10, Appendix III.1], which says that for $|q|$, $|z|$, $|b| < 1$, we have

$$\sum_{n\geq 0} \frac{(a)_n(b)_n}{(q)_n(c)_n} z^n = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} \sum_{n\geq 0} \frac{(c/b)_n(z)_n}{(q)_n(az)_n} b^n. \tag{2.2}$$

By appropriately iterating Heine's transformation two times, we obtain another version [10, Appendix III.3], sometimes called the q-analogue of Euler's transformation. Namely, for $|z|$, $|abz/c| < 1$, we have

$$\sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n(q)_n} z^n = \frac{(abz/c)_\infty}{(z)_\infty} \sum_{n\geq 0} \frac{(c/a)_n(c/b)_n}{(c)_n(q)_n} (abz/c)^n. \tag{2.3}$$

Finally, we record an identity of Sylvester [4, Eq. (9.2.3)]. For $|q| < 1$ we have

$$(-xq)_\infty = \sum_{n\geq 0} \frac{(-xq)_n}{(q)_n} (1 + xq^{2n+1}) x^n q^{n(3n+1)/2}. \tag{2.4}$$

To conclude this section, we state Ingham's Tauberian theorem [11, Theorem 1.1], as presented in [7].

Theorem 4 (Theorem 1.1 in [7]). *Let $f(q) = \sum_{n\geq 0} a(n)q^n$ be a power series whose radius of convergence is equal to 1 and whose coefficients $a(n)$ are non-negative and weakly increasing. Suppose that for $A > 0, \lambda, \alpha \in \mathbb{R}$,*

$$f(e^{-t}) \sim \lambda t^\alpha e^{A/t} \quad \text{as } t \to 0^+, \qquad f(e^{-z}) \ll |z|^\alpha e^{A/|z|} \quad \text{as } z \to 0,$$

with $z = x + iy$ ($x > 0, y \in \mathbb{R}$) in each region of the form $|y| \leq \Delta x$ for $\Delta > 0$. Then

$$a(n) \sim \frac{\lambda}{2\sqrt{\pi}} \frac{A^{\frac{\alpha}{2}+\frac{1}{4}}}{n^{\frac{\alpha}{2}+\frac{3}{4}}} e^{2\sqrt{An}}$$

as $n \to \infty$.

3. Proofs of the main theorems

In this section we prove Theorems 1 – 3. We begin by computing some generating functions. Using standard combinatorial arguments in partition theory, we see that

$$\frac{q^{bn}}{(q^2;q^2)_n}$$

generates a partition into n odd parts when $b = 1$, a partition into at most n even parts when $b = 0$, and a partition into exactly n even parts when $b = 2$. This implies that

$$\begin{aligned} P_o(q) := \sum_{n\geq 0} p_o(n)q^n &= \sum_{n\geq 0} \frac{q^n}{(q^2;q^2)_n^2} - \sum_{n\geq 0} \frac{q^{3n}}{(q^2;q^2)_n^2} \\ &= q + q^2 + 2q^3 + 3q^4 + 4q^5 + 7q^6 + 9q^7 + 14q^8 + \cdots \end{aligned}$$

$$P_e(q) := \sum_{n\geq 0} p_e(n)q^n = \frac{1}{(q)_\infty} - \sum_{n\geq 0}\frac{q^n}{(q^2;q^2)_n^2}$$
$$= q^2 + 2q^4 + q^5 + 3q^6 + 3q^7 + 6q^8 + 7q^9 + 10q^{10} + \cdots .$$

We are now ready to prove Theorem 1.

Proof of Theorem 1. Applying (2.3) with $q \to q^2$, $a, b \to 0$, $c = q^4$, $z = q$ gives

$$P_o(q) = \frac{q}{(1-q^2)}\sum_{n\geq 0}\frac{q^n}{(q^4;q^2)_n(q^2;q^2)_n}$$
$$= \frac{1}{(q;q^2)_\infty}\sum_{n\geq 0}\frac{q^{2n^2+3n+1}}{(q^2;q^2)_{n+1}(q^2;q^2)_n}$$
$$= \frac{1}{(q;q^2)_\infty}\sum_{n\geq 1}\frac{q^{2n^2-n}}{(q^2;q^2)_n(q^2;q^2)_{n-1}}. \tag{3.1}$$

Also, by (2.3) with $q \to q^2$, $a, b \to 0$, $c = q^2$, $z = q$, we have

$$P_e(q) = \frac{1}{(q;q^2)_\infty}\frac{1}{(q^2;q^2)_\infty} - \sum_{n\geq 0}\frac{q^n}{(q^2;q^2)_n^2}$$
$$= \frac{1}{(q;q^2)_\infty}\sum_{n\geq 0}\frac{q^{2n^2}}{(q^2;q^2)_n^2} - \frac{1}{(q;q^2)_\infty}\sum_{n\geq 0}\frac{q^{2n^2+n}}{(q^2;q^2)_n^2}$$
$$= \frac{1}{(q;q^2)_\infty}\sum_{n\geq 1}\frac{q^{2n^2}(1-q^n)}{(q^2;q^2)_n^2}. \tag{3.2}$$

Subtracting, we arrive at

$$P_o(q) - P_e(q) = \frac{1}{(q;q^2)_\infty}\sum_{n\geq 1}\frac{q^{2n^2-n}(1-q^n)}{(q^2;q^2)_n^2},$$

as desired. Each summand on the right-hand side above clearly has non-negative coefficients since the term $(1-q^n)$ cancels with a term in $(q^2;q^2)_n$ if n is even and with a term in $(q;q^2)_\infty$ if n is odd. Moreover, the second summand is

$$\frac{q^6}{(1-q^2)(1-q^4)^2(1-q)(q^3;q^2)_\infty},$$

which implies that for $n \geq 6$ the coefficient of q^n is positive. Checking for $n < 6$ we find that $p_o(n) > p_e(n)$ for all positive $n \neq 2$, as claimed. □

We now turn to Theorem 2. We prove parts (i) and (ii) separately. For part (i) we give both a q-series proof and a combinatorial proof. For part (ii) we only give a combinatorial proof and leave finding a q-series proof to the motivated reader.

First proof of part (i) of Theorem 2. First, from (3.1) and (3.2), we find that

$$P_o(q) - 2P_e(q) = \frac{1}{(q;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2-n}(1-q^n)^2}{(q^2;q^2)_n^2}. \tag{3.3}$$

For $n \geq 2$, each summand on the right-hand side above has non-negative coefficients. When n is even, this is clear and when $n > 1$ is odd, this follows from

$$\frac{(1-q^n)^2}{(1-q)(1-q^n)} = 1 + q + \cdots + q^{n-1}.$$

For the case $n = 1$, we use (2.4) with $x = 1$ and compute

$$\begin{aligned}
&q^2 + q^4 + \frac{q(1-q)^2}{(1-q^2)^2} \frac{1}{(q;q^2)_\infty} \\
&= q^2 + q^4 + \frac{q(1-q)^2}{(1-q^2)^2} (-q)_\infty \\
&= q^2(1+q^2) + \frac{q}{(1+q)^2} \sum_{n\geq 0} \frac{(-q)_n}{(q)_n} (1+q^{2n+1}) q^{(3n^2+n)/2} \\
&= q^2(1+q^2) + \frac{q}{1+q} + \frac{q^3(1+q^3)}{1-q^2} + \frac{q}{(1+q)^2} \sum_{n\geq 2} \frac{(-q)_n}{(q)_n} (1+q^{2n+1}) q^{(3n^2+n)/2} \\
&= \frac{q(1+q^2)}{1-q^2} + \frac{q}{(1+q)^2} \sum_{n\geq 2} \frac{(-q)_n}{(q)_n} (1+q^{2n+1}) q^{(3n^2+n)/2},
\end{aligned}$$

which gives

$$\frac{q(1-q)^2}{(1-q^2)^2} \frac{1}{(q;q^2)_\infty} = -q^2 - q^4 + \frac{q(1+q^2)}{1-q^2} + \frac{q}{(1-q^2)} \sum_{n\geq 2} \frac{(-q^2)_{n-1}}{(q^2)_{n-1}} (1+q^{2n+1}) q^{(3n^2+n)/2}. \tag{3.4}$$

We conclude that $P_o(q) - 2P_e(q)$ has non-negative coefficients except for $n = 2, 4$, where the coefficient is -1.

To see that the coefficients are positive, we note that the second summand in (3.3) is

$$\frac{q^6}{(1-q^4)^2(1-q)(q^3;q^2)_\infty},$$

which implies that for $n \geq 6$ the coefficient of q^n is positive. Checking the coefficients for $n < 6$ then confirms the result. □

Second proof of part (i) of Theorem 2. First, we fix some notation. Recall that $\mathcal{D}_e$ denotes the set of partitions into distinct non-negative even parts and that $\widehat{\mathcal{D}}_e$ denotes the set of partitions into distinct non-negative even parts where the largest part may be overlined. If λ is a Frobenius symbol we write $|\lambda|$ for its weight and $\ell(\lambda)$ for the number of its columns. If π is a partition then $|\pi|$ denotes the sum of the parts.

We define two kinds of partition pairs involving generalized Frobenius symbols. First, let $\mathcal{A}$ be the set of partition pairs (π, λ) where π is a partition into distinct parts and λ is a Frobenius symbol in $\mathcal{F}_{\mathcal{D}_e,\mathcal{D}_e}$ such that 0 always occurs as a part on the top row. Next, let $\mathcal{B}$ be the set of partition pairs (π,λ) where λ is a Frobenius symbol in $\mathcal{F}_{\widehat{\mathcal{D}}_e,\mathcal{D}_e}$ such that 0 is always a part of the top row and π is a partition into distinct parts not equal to $\ell(\lambda)$.

Using standard combinatorial arguments, we have that

$$\sum_{(\pi,\lambda)\in\mathcal{A}} q^{|\pi|+|\lambda|} = (-q)_\infty \sum_{n\geq 1} \frac{q^{2n^2-n}}{(q^2;q^2)_n(q^2;q^2)_{n-1}},$$

$$\sum_{(\pi,\lambda)\in\mathcal{B}} q^{|\pi|+|\lambda|} = 2(-q)_\infty \sum_{n\geq 1} \frac{q^{2n^2-n}(1-q^n)}{(q^2;q^2)_n^2}.$$

Using (3.1) and (3.2), we find that

$$\begin{aligned} P_o(q)-2P_e(q) &= 2\left(P_o(q)-P_e(q)\right)-P_o(q) \\ &= 2(-q)_\infty \sum_{n\geq 1} \frac{q^{2n^2-n}(1-q^n)}{(q^2;q^2)_n^2} - (-q)_\infty \sum_{n\geq 1} \frac{q^{2n^2-n}}{(q^2;q^2)_n(q^2;q^2)_{n-1}} \\ &= \sum_{(\pi,\lambda)\in\mathcal{B}} q^{|\pi|+|\lambda|} - \sum_{(\pi,\lambda)\in\mathcal{A}} q^{|\pi|+|\lambda|}. \end{aligned}$$

Now let $\mathcal{A}_n$ (resp. $\mathcal{B}_n$) denote the number of partition pairs (π,λ) in $\mathcal{A}$ (resp. $\mathcal{B}$) such that $|\pi|+|\lambda|=n$. To prove the theorem, we will describe an injection $\phi:\mathcal{A}_n\to\mathcal{B}_n$ for all $n\neq 2,4$.

Suppose that $(\pi,\lambda)\in\mathcal{A}_n$. Then we define $\phi((\pi,\lambda))=(\pi',\lambda')\in\mathcal{B}_n$ as follows:

Case 1: $\ell(\lambda)$ is not a part of π. Then $(\pi',\lambda')=(\pi,\lambda)$.

Case 2: $\ell(\lambda)$ is a part of π and π has at least two parts. Then λ' is the Frobenius symbol obtained by overlining the top leftmost part of λ and π' is constructed by deleting the part $\ell(\lambda)$ from the partition π and adding $\ell(\lambda)$ to the largest part among the remaining parts of π.

Case 3: $\pi=(\ell(\lambda))$ and $\ell(\lambda)>1$.

(1) If $\ell(\lambda)$ is even, λ' is obtained by adding $\ell(\lambda)$ to the top leftmost part of λ after overlining the top leftmost entry, and $\pi'=\emptyset$,

(2) if $\ell(\lambda)$ is odd, λ' is obtained by adding $\ell(\lambda)-1$ to the largest part of the top row of λ after overlining the top leftmost entry, and $\pi'=(1)$.

Case 4: $\pi=(\ell(\lambda))$ and $\ell(\lambda)=1$. For this case, we define

$$\phi((\pi,\lambda))=\phi\left(\left((1),\begin{pmatrix}0\\2k\end{pmatrix}\right)\right)=(\pi',\lambda')=\left(\emptyset,\begin{pmatrix}\overline{2} & 0\\ 2k-2 & 0\end{pmatrix}\right)$$

for $k>1$.

We note several easily verified facts about the mapping ϕ. First, ϕ is defined on all pairs $(\pi,\lambda)\in\mathcal{A}$ except for $k=0,1$ in **Case 4**. Second, the image (π',λ') of a pair $(\pi,\lambda)\in\mathcal{A}_n$ is indeed an element of $\mathcal{B}_n$. Third, the sets of images obtained in the four cases are disjoint. Finally, in each of the four cases, the unique preimage (π,λ) can be recovered from (π',λ').

Taken together, the above facts imply that $\phi : \mathcal{A}_n \to \mathcal{B}_n$ is an injection for $n \neq 2, 4$ and therefore we have $|\mathcal{B}_n| \geq |\mathcal{A}_n|$ for all such n. To obtain strict inequality, we observe that there are classes of partition pairs in $\mathcal{B}\backslash\phi(\mathcal{A})$. For example,

$$\left((2,1), \begin{pmatrix} \overline{2k} & 0 \\ 2 & 0 \end{pmatrix}\right), \left((2), \begin{pmatrix} \overline{2k} & 0 \\ 2 & 0 \end{pmatrix}\right) \in \mathcal{B}$$

are not in $\phi(\mathcal{A})$ for $k \geq 1$. If they were, then they would necessarily come from **Case 2**, but then the largest part of π' would be at least 3 since $\ell(\lambda') = \ell(\lambda) = 2$. This gives $|\mathcal{B}_n| > |\mathcal{A}_n|$ for $n \geq 8$. By checking the coefficients for $n < 8$, we complete the proof. □

Now we prove the upper bound in Theorem 2.

Proof of part (ii) of Theorem 2. We begin by defining two sets of partition pairs. First, let $\mathcal{A}$ be the set of partition pairs (π, λ) where λ is a Frobenius symbol in $\mathcal{F}_{\mathcal{D}_e,\mathcal{D}_e}$ such that 0 is always a part of the top row and π is a partition into distinct parts not equal to $\ell(\lambda)$. Second, let $\mathcal{B}$ be the set of partition pairs (π, λ) where λ is a Frobenius symbol in $\mathcal{F}_{\mathcal{D}_o,\mathcal{D}_e}$ such that that 1 is always a part of the top row and the smallest part of the bottom row may be overlined, and π is a partition into distinct parts not equal to $\ell(\lambda)$. Recall that $\mathcal{D}_o$ is the set of partitions into distinct odd parts.

Using standard combinatorial arguments we have the generating functions

$$\sum_{(\pi,\lambda)\in\mathcal{A}} q^{|\pi|+|\lambda|} = (-q)_\infty \sum_{n\geq 1} \frac{q^{2n^2-n}(1-q^n)}{(q^2;q^2)_n^2},$$

$$\sum_{(\pi,\lambda)\in\mathcal{B}} q^{|\pi|+|\lambda|} = 2(-q)_\infty \sum_{n\geq 1} \frac{q^{2n^2}(1-q^n)}{(q^2;q^2)_n^2}.$$

Now by (3.1) and (3.2), we have

$$\begin{aligned} 3P_e(q) - P_o(q) &= 2P_e(q) - (P_o(q) - P_e(q)) \\ &= 2(-q)_\infty \sum_{n\geq 1} \frac{q^{2n^2}(1-q^n)}{(q^2;q^2)_n^2} - (-q)_\infty \sum_{n\geq 1} \frac{q^{2n^2-n}(1-q^n)}{(q^2;q^2)_n^2}, \end{aligned}$$

and so

$$3P_e(q) - P_o(q) = \sum_{(\pi,\lambda)\in\mathcal{B}} q^{|\pi|+|\lambda|} - \sum_{(\pi,\lambda)\in\mathcal{A}} q^{|\pi|+|\lambda|}. \tag{3.5}$$

As before, let $\mathcal{A}_n$ (resp. $\mathcal{B}_n$) denote the set of partition pairs (π, λ) in $\mathcal{A}$ (resp. $\mathcal{B}$) with weight $n = |\pi| + |\lambda|$. For all positive integers $n \geq 14$, we will construct an injection $\phi : \mathcal{A}_n \to \mathcal{B}_n$

$$(\pi, \lambda) = \left((\pi_1, \pi_2, \cdots, \pi_{\ell(\pi)}), \begin{pmatrix} a_1 & \cdots & a_{k-1} & 0 \\ b_1 & \cdots & b_{k-1} & b_k \end{pmatrix}\right) \to (\pi', \lambda') = \left(\pi', \begin{pmatrix} c_1 & \cdots & c_{k'-1} & 1 \\ d_1 & \cdots & d_{k'-1} & d_{k'} \end{pmatrix}\right),$$

where $\pi_1 > \pi_2 > \cdots > \pi_{\ell(\pi)}$ with the number of parts in π denoted by $\ell(\pi)$. This gives the non-negativity of the coefficients in (3.5) for $n \geq 14$. Later we will see that $\mathcal{B}_n \backslash \phi(\mathcal{A}_n)$ is nonempty for these values of n, giving the positivity. Checking the coefficients for $n \leq 13$ then completes the proof.

We now describe the injection. It is somewhat involved due to a number of exceptional cases. We invite the reader to verify at every step that the image is indeed in $\mathcal{B}_n$, that the images in the various cases are disjoint, and that the mapping is always invertible. We make some notes along the way concerning the disjointness, and a precise description of the inverse map is given later.

Case 1: $b_k \neq 0$.

(1) If $\pi \neq \emptyset$,

$$(\pi, \lambda) \to \left((\pi_1 + k, \pi_2, \ldots, \pi_{\ell(\pi)}), \begin{pmatrix} a_1+1 & \cdots & a_{k-1}+1 & 1 \\ b_1-2 & \cdots & b_{k-1}-2 & b_k-2 \end{pmatrix} \right).$$

(2) If $\pi = \emptyset$,

$$(\emptyset, \lambda) \to \left((k-1, 1), \begin{pmatrix} a_1+1 & \cdots & a_{k-1}+1 & 1 \\ b_1-2 & \cdots & b_{k-1}-2 & b_k-2 \end{pmatrix} \right) \quad \text{if } k > 2,$$

$$\left(\emptyset, \begin{pmatrix} a_1 & 0 \\ b_1 & b_2 \end{pmatrix} \right) \to \left(\emptyset, \begin{pmatrix} a_1+3 & 1 \\ b_1-2 & b_2-2 \end{pmatrix} \right),$$

$$\left(\emptyset, \begin{pmatrix} 0 \\ 2m \end{pmatrix} \right) \to \left((1), \begin{pmatrix} 3 & 1 \\ 2m-6 & 0 \end{pmatrix} \right).$$

Note that the last mapping is well defined as $n \geq 14$ implies that $m > 6$. Clearly there is no overlap between the four sets of images here.

Case 2: $b_k = 0$ and $k \geq 2$.

(1) If $a_{k-1} > 2$,

(a) If $\pi_1 \neq 2$

$$(\pi, \lambda) \to \left((\pi_1 + (k-2), \pi_2, \ldots, \pi_{\ell(\pi)}), \begin{pmatrix} a_1-1 & \cdots & a_{k-1}-1 & 1 \\ b_1 & \cdots & b_{k-1} & \overline{0} \end{pmatrix} \right).$$

We set $\pi_1 = 0$ if $\pi = \emptyset$ and $\pi' = \emptyset$ if $\pi'_1 = 0$.

(b) If $\pi_1 = 2$ (which implies that $k \geq 3$),

$$((2), \lambda) \to \begin{cases} \left((k-2, 2), \begin{pmatrix} a_1-1 & \cdots & a_{k-1}-1 & 1 \\ b_1 & \cdots & b_{k-1} & \overline{0} \end{pmatrix} \right) & \text{if } k > 4, \\ \left((2, 1), \begin{pmatrix} a_1-1 & \cdots & a_{k-1}-1 & 1 \\ b_1 & \cdots & b_{k-1} & \overline{0} \end{pmatrix} \right) & \text{if } k = 3, \\ \left(\emptyset, \begin{pmatrix} a_1+1 & a_2+1 & a_3-1 & 1 \\ b_1 & b_2 & b_3 & \overline{0} \end{pmatrix} \right) & \text{if } k = 4, \end{cases}$$

$$((2,1),\lambda) \to \begin{cases} \left((k-1,2), \begin{pmatrix} a_1-1 & \cdots & a_{k-1}-1 & 1 \\ b_1 & \cdots & b_{k-1} & \overline{0} \end{pmatrix}\right) & \text{if } k \neq 3, \\ \left(\emptyset, \begin{pmatrix} a_1+1 & a_2+1 & 1 \\ b_1 & b_2 & \overline{0} \end{pmatrix}\right) & \text{if } k = 3. \end{cases}$$

We note that the resulting partition π' from **Case 2**-(1)-(a) cannot be of the form $(k'-2,2)$ or $(k'-1,2)$, and it can only be $\emptyset$ if $k=2$. Therefore there is no overlap between **Case 2**-(1)-(a) and **Case 2**-(1)-(b). Also, since there is an overlined part in λ' there is no overlap with **Case 1**.

(2) If $a_{k-1}=2$,

(a) If $k-1 \notin \pi$ or $\pi_1 = k-1$,

$$(\pi,\lambda) \to \left((\pi_1+k,\pi_2,\ldots,\pi_{\ell(\pi)}), \begin{pmatrix} a_1-1 & \cdots & a_{k-1}-1 \\ b_1 & \cdots & \overline{b_{k-1}} \end{pmatrix}\right).$$

Note that $k'=k-1$ and $b_{k-1}\neq 0$. As before, we take $\pi_1=0$ if $\pi=\emptyset$.

(b) If $k-1\in\pi$ and $\pi_1\neq k-1$, replace $k-1$ by k, and add $k-1$ to π_1. Recall that $k\notin\pi$.

$$(\pi,\lambda) \to \left((\pi_1+(k-1),\pi_2,\ldots,k,\ldots,\pi_{\ell(\pi)}), \begin{pmatrix} a_1-1 & \cdots & a_{k-1}-1 \\ b_1 & \cdots & \overline{b_{k-1}} \end{pmatrix}\right).$$

Again there is no overlap with **Case 1** since there is an overlined part in λ'. Since this overlined part must be nonzero, there is also no overlap with **Case 2**-(1). Finally, the subcases (a) and (b) above are disjoint because in (b) π' contains both k and at least one other part while in (a) π' only contains k if it has just one part.

Case 3: $\lambda = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

We proceed as $\pi_1-\pi_2$ decreases.

(1) $\pi_1-\pi_2>3$. (If $\ell(\pi)=1$, then we take $\pi_2=0$. Since $n\geq 14$, this case covers all instances where $\ell(\pi)=1$.)

$$\left(\pi, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \to \left((\pi_1-1,\pi_2,\ldots,\pi_{\ell(\pi)}), \begin{pmatrix} 1 \\ \overline{0} \end{pmatrix}\right).$$

Note that the image cannot come from **Case 2**-(1) as $k'=1$, or from **Case 2**-(2) as $\overline{0}$ is in the bottom row of λ'.

(2) $\pi_1-\pi_2=3$ and $\ell(\pi)>2$.

$$(\pi,\lambda) \to \begin{cases} \left((\pi_1-2,\pi_2,\ldots,\pi_{\ell(\pi)}-1), \begin{pmatrix} 1 \\ \overline{2} \end{pmatrix}\right) & \text{if } \pi_{\ell(\pi)}>2, \\ \left((\pi_1-1,\pi_2,\ldots,\pi_{\ell(\pi)-1}), \begin{pmatrix} 1 \\ \overline{2} \end{pmatrix}\right) & \text{if } \pi_{\ell(\pi)}=2. \end{cases}$$

Note that the condition $\pi'_1 - \pi'_2 = 1$ for the case $\pi_{\ell(\pi)} > 2$ means that it cannot come from **Case 2**-(2). For the case $\pi_{\ell(\pi)} = 2$ we have $\pi'_1 - \pi'_2 = 2$. This could happen in **Case 2**-(2)-(b), but only if $k = 2$. Since $k \in \pi'$ there, this contradicts $\pi_{\ell(\pi)-1} > 2$. Therefore the case $\pi_{\ell(\pi)} = 2$ cannot come from **Case 2**-(2).

(3) $\pi_1 - \pi_2 = 3$, $\ell(\pi) = 2$.

$$(\pi, \lambda) \to \left((\pi_1 - 1, \pi_2), \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).$$

As in **Case 3**-(1), this cannot come from **Case 2**, and there is no overlap with **Case 3**-(1) since $\pi'_1 - \pi'_2 = 2$ here and $\pi'_1 - \pi'_2 \geq 3$ in **Case 3**-(1).

(4) $\pi_1 - \pi_2 = 2$. Note that since $n \geq 14$, we have $\ell(\pi) \geq 2$.

$$(\pi, \lambda) \to \left((\pi_1 - 1, \pi_2, \ldots, \pi_{\ell(\pi)}), \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).$$

This cannot come from **Case 1**-(1) as $\pi'_1 - \pi'_2 \geq 2$ there while $\pi'_1 - \pi'_2 = 1$ here.

For the remaining cases we have $\pi_1 - \pi_2 = 1$. Since $n \geq 14$ this implies that $\ell(\pi) \geq 2$ and $\pi_1 \geq 5$. Let g denote the length of the initial run in π. That is, g is the largest integer such that $\pi_i - \pi_{i+1} = 1$ holds for all $1 \leq i < g$. Note that $2 \leq g \leq \ell(\pi)$.

(5) $\pi_1 - \pi_2 = 1$ and either $g < \ell(\pi)$ or $g = \ell(\pi) > 2$ with $\pi_{\ell(\pi)} > 2$. In other words, $\ell(\pi) > 2$ and $\pi_g > 2$.

$$(\pi, \lambda) \to \left((\pi_1, \cdots, \pi_g - 1, \cdots, \pi_{\ell(\pi)}), \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).$$

Since $\pi'_1 - \pi'_2 \leq 2$, there is no overlap with **Case 3**-(1). Moreover, the image cannot come from **Case 3**-(3) as π' has at least three parts here.

(6) $\pi_1 - \pi_2 = 1$, $g = \ell(\pi) = 2$. Since $n \geq 14$, we have $\pi_1 > 6$ and $\pi_2 > 5$ for this case.

$$(\pi, \lambda) \to \left((\pi_1 - 2, \pi_2 - 3), \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right).$$

Note that $\pi'_1 - \pi'_2 = 2$ and $\pi'_2 > 2$ together guarantee that this image cannot come from **Case 2**-(2).

(7) $\pi_1 - \pi_2 = 1$, $g = \ell(\pi) \geq 3$, and $\pi_{\ell(\pi)} = 2$. Note that since $\pi_1 \geq 5$, this actually gives $g = \ell(\pi) > 3$.

$$(\pi, \lambda) = \left((\ldots, 5, 4, 3, 2), \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \to \left((\ldots, 5, 4), \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right).$$

Since $\pi'_1 - \pi'_2 = 1$, this cannot come from **Case 2**-(2) or **Case 3**-(6).

The map ϕ is well-defined and injective. To emphasize the latter point, we explain how to determine whether a given $(\pi', \lambda') \in \mathcal{B}$ has a preimage and how to find the preimage if it exists. Let $(\pi', \lambda') = \left(\pi', \begin{pmatrix} c_1 & \cdots & c_{k'-1} & 1 \\ d_1 & \cdots & d_{k'-1} & * \end{pmatrix}\right) \in \mathcal{B}_n$. Then we can find a unique pair $(\pi, \lambda) \in \mathcal{A}_n$ with $\phi(\pi, \lambda) = (\pi', \lambda')$ if $(\pi', \lambda') \in \phi(\mathcal{A}_n)$ as follows:

Case 1: $* = d_{k'}$ is unoverlined.

(1) If $\pi' \neq \emptyset$, $\pi'_1 \neq 2k'$, and $\pi'_1 - \pi'_2 > k'$ (taking $\pi'_2 = 0$ if $\ell(\pi') = 1$), then this is from **Case 1**-(1):

$$(\pi', \lambda') \to \left((\pi'_1 - k', \pi'_2, \ldots, \pi'_{\ell(\pi')}), \begin{pmatrix} c_1 - 1 & \cdots & c_{k'-1} - 1 & 0 \\ d_1 + 2 & \cdots & d_{k'-1} + 2 & d_{k'} + 2 \end{pmatrix}\right).$$

(2) If $\pi' = (k' - 1, 1)$ and $k' > 2$, then this is from **Case 1**-(2):

$$(\pi', \lambda') \to \left(\emptyset, \begin{pmatrix} c_1 - 1 & \cdots & c_{k'-1} - 1 & 0 \\ d_1 + 2 & \cdots & d_{k'-1} + 2 & d_{k'} + 2 \end{pmatrix}\right).$$

(3) If $\pi' = \emptyset$, $k' = 2$, and $c_1 \geq 5$, then this is from **Case 1**-(2):

$$(\pi', \lambda') = \left(\emptyset, \begin{pmatrix} c_1 & 1 \\ d_1 & d_2 \end{pmatrix}\right) \to \left(\emptyset, \begin{pmatrix} c_1 - 3 & 0 \\ d_1 + 2 & d_2 + 2 \end{pmatrix}\right).$$

(4) If $\pi' = (1)$ and $\lambda' = \begin{pmatrix} 3 & 1 \\ d_1 & 0 \end{pmatrix}$, then this is from **Case 1**-(2):

$$(\pi', \lambda') = \left((1), \begin{pmatrix} 3 & 1 \\ d_1 & 0 \end{pmatrix}\right) \to \left(\emptyset, \begin{pmatrix} 0 \\ d_1 + 6 \end{pmatrix}\right).$$

(5) If $\pi'_1 - \pi'_2 = 1$, $\ell(\pi') > 1$, and $\lambda' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then this is from **Case 3**-(4):

$$(\pi', \lambda') \to \left((\pi'_1 + 1, \pi'_2, \ldots, \pi'_{\ell(\pi')}), \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).$$

(6) Otherwise, there is no preimage.

Case 2: $* = \overline{0}$.

(1) If $k' > 1$,

(a) Suppose that $\pi'_1 \neq 2k' - 2$. If $\pi'_1 - \pi'_2 > k' - 2$ for $\ell(\pi') \geq 2$ or $\pi'_1 \geq k' - 2$ for $\ell(\pi') = 1$ or $k' = 2$ for $\pi' = \emptyset$, then this is from **Case 2**-(1)-(a):

$$(\pi', \lambda') \to \left((\pi'_1 - (k' - 2), \pi'_2, \ldots, \pi'_{\ell(\pi')}), \begin{pmatrix} c_1 + 1 & \cdots & c_{k'-1} + 1 & 0 \\ d_1 & \cdots & d_{k'-1} & 0 \end{pmatrix}\right).$$

Note that if $\pi = \{0\}$ as a result, we let $\pi = \emptyset$.

(b) If $\pi' = (k' - 2, 2)$, then this is from **Case 2**-(1)-(b):

$$((k' - 2, 2), \lambda') \to \left((2), \begin{pmatrix} c_1 + 1 & \cdots & c_{k'-1} + 1 & 0 \\ d_1 & \cdots & d_{k'-1} & 0 \end{pmatrix}\right).$$

(c) If $\pi' = (k'-1, 2)$ and $k' \geq 4$, then this is from **Case 2**-(1)-(b):

$$((k'-1,2),\lambda') \to \left((2,1), \begin{pmatrix} c_1+1 & \cdots & c_{k'-1}+1 & 0 \\ d_1 & \cdots & d_{k'-1} & 0 \end{pmatrix}\right).$$

(d) If $\pi' = (2,1)$, then this is from **Case 2**-(1)-(b):

$$((2,1),\lambda') \to \left((2), \begin{pmatrix} c_1+1 & \cdots & c_{k'-1}+1 & 0 \\ d_1 & \cdots & d_{k'-1} & 0 \end{pmatrix}\right).$$

(e) If $\pi' = \emptyset$, $k' = 4$, and $c_2 - c_3 \geq 4$, then this is from **Case 2**-(1)-(b):

$$(\emptyset,\lambda') \to \left((2), \begin{pmatrix} c_1-1 & c_2-1 & c_3+1 & 0 \\ d_1 & d_2 & d_3 & 0 \end{pmatrix}\right).$$

(f) If $\pi' = \emptyset$, and $k' = 3$, then this is from **Case 2**-(1)-(b):

$$(\emptyset,\lambda') \to \left((2,1), \begin{pmatrix} c_1-1 & c_2-1 & 0 \\ d_1 & d_2 & 0 \end{pmatrix}\right).$$

(2) If $k' = 1$, i.e. $\lambda' = \begin{pmatrix} 1 \\ \overline{0} \end{pmatrix}$,

(a) If $\pi' \neq \emptyset$ and $\pi'_1 - \pi'_2 > 2$ (letting $\pi'_2 = 0$ if $\ell(\pi') = 1$), then this is from **Case 3**-(1):

$$\left(\pi', \begin{pmatrix} 1 \\ \overline{0} \end{pmatrix}\right) \to \left((\pi'_1+1, \pi'_2, \ldots, \pi'_{\ell(\pi')}), \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).$$

(b) If $\ell(\pi') = 2$ and $\pi'_1 - \pi'_2 = 2$, then this is from **Case 3**-(3):

$$\left(\pi', \begin{pmatrix} 1 \\ \overline{0} \end{pmatrix}\right) \to \left((\pi'_1+1, \pi'_2), \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).$$

(c) If $\ell(\pi') > 2$ and $\pi'_1 - \pi'_2 \leq 2$, then with $\pi' = (\pi'_1, \ldots, \pi'_{\ell(\pi')})$ let g be the largest integer such that $\pi'_{i-1} - \pi'_i = 1$ holds for all $1 \leq i < g$ and $\pi'_{g-1} - \pi'_g = 2$. If no such g exists, then there is no preimage. Otherwise we have $2 \leq g \leq \ell(\pi')$ and this is from **Case 3**-(5):

$$\left(\pi', \begin{pmatrix} 1 \\ \overline{0} \end{pmatrix}\right) \to \left((\pi'_1, \ldots, \pi'_g+1, \ldots, \pi'_{\ell(\pi')}), \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).$$

(3) Otherwise, there is no preimage.

Case 3: $* = \overline{d_{k'}}$ is overlined and $* \neq \overline{0}$.

(1) Suppose that $\pi'_1 \neq 2k'+2$. If $\pi'_1 - \pi'_2 > k'+1$ and $k'+1 \notin \pi'$ for $\ell(\pi') > 1$ or $\pi'_1 \geq k'+1$ for $\ell(\pi') = 1$, then this is from **Case 2**-(2)-(a):

$$(\pi',\lambda') \to \left((\pi'_1-k'-1, \pi'_2, \ldots, \pi'_{\ell(\pi')}), \begin{pmatrix} c_1+1 & \cdots & c_{k'-1}+1 & 2 & 0 \\ d_1 & \cdots & d_{k'-1} & d_{k'} & 0 \end{pmatrix}\right).$$

Note that if $\pi = \{0\}$ as a result, we let $\pi = \emptyset$.

(2) If $\ell(\pi')>1$, $k'+1\in\pi'$, and $\pi'_1-\pi'_2>k'$, then this is from **Case 2**-(2)-(b):

$$(\pi',\lambda')=\big((\pi'_1,\pi'_2,\dots,k'+1,\dots,\pi'_{\ell(\pi')}),\lambda'\big) \rightarrow \left((\pi'_1-k',\pi'_2,\dots,k',\dots,\pi'_{\ell(\pi')}),\begin{pmatrix}c_1+1 & \cdots & c_{k'-1}+1 & 2 & 0\\ d_1 & \cdots & d_{k'-1} & d_{k'} & 0\end{pmatrix}\right).$$

(3) If $\lambda'=\begin{pmatrix}1\\2\end{pmatrix}$, $\ell(\pi')>2$, and $\pi'_1-\pi'_2=1$, then this is from **Case 3**-(2):

$$(\pi',\lambda')\rightarrow\left((\pi'_1+2,\pi'_2,\dots,\pi'_{\ell(\pi')}+1),\begin{pmatrix}0\\0\end{pmatrix}\right).$$

(4) If $\lambda'=\begin{pmatrix}1\\2\end{pmatrix}$, $\ell(\pi')>1$, $\pi'_1-\pi'_2=2$, and $\pi'_{\ell(\pi')}>2$, then this is from **Case 3**-(2):

$$(\pi',\lambda')\rightarrow\left((\pi'_1+1,\pi'_2,\dots,\pi'_{\ell(\pi')},2),\begin{pmatrix}0\\0\end{pmatrix}\right)$$

(5) If $\lambda'=\begin{pmatrix}1\\4\end{pmatrix}$, $\ell(\pi')=2$, and $\pi'_1-\pi'_2=2$, then this is from **Case 3**-(6):

$$(\pi',\lambda')\rightarrow\left((\pi'_1+2,\pi'_2+3),\begin{pmatrix}0\\0\end{pmatrix}\right).$$

(6) If $\lambda'=\begin{pmatrix}1\\4\end{pmatrix}$, $\ell(\pi')\geq 2$, $\pi'_{\ell(\pi')}=4$, and $\pi'_i-\pi'_{i+1}=1$ for all i, i.e $\pi'=(\dots,5,4)$, then this is from **Case 3**-(7):

$$(\pi',\lambda')=\left((\dots,5,4),\begin{pmatrix}1\\4\end{pmatrix}\right)\rightarrow\left((\dots,5,4,3,2),\begin{pmatrix}0\\0\end{pmatrix}\right).$$

(7) Otherwise, there is no preimage.

Here are some examples from the case $n=14$.

$$\left(\emptyset,\begin{pmatrix}2&0\\6&4\end{pmatrix}\right)\xrightarrow[\mathbf{1}\text{-(2)}]{\phi}\left(\emptyset,\begin{pmatrix}5&1\\4&2\end{pmatrix}\right)\xrightarrow[\mathbf{1}\text{-(3)}]{\phi^{-1}}\left(\emptyset,\begin{pmatrix}2&0\\6&4\end{pmatrix}\right)$$
$$\left(\emptyset,\begin{pmatrix}8&0\\4&0\end{pmatrix}\right)\xrightarrow[\mathbf{2}\text{-(1)-(a)}]{\phi}\left(\emptyset,\begin{pmatrix}7&1\\4&\overline{0}\end{pmatrix}\right)\xrightarrow[\mathbf{2}\text{-(1)-(a)}]{\phi^{-1}}\left(\emptyset,\begin{pmatrix}8&0\\4&0\end{pmatrix}\right)$$
$$\left((3,1),\begin{pmatrix}6&0\\2&0\end{pmatrix}\right)\xrightarrow[\mathbf{2}\text{-(1)-(a)}]{\phi}\left((3,1),\begin{pmatrix}5&1\\2&\overline{0}\end{pmatrix}\right)\xrightarrow[\mathbf{2}\text{-(1)-(a)}]{\phi^{-1}}\left((3,1),\begin{pmatrix}6&0\\2&0\end{pmatrix}\right)$$
$$\left((8),\begin{pmatrix}2&0\\2&0\end{pmatrix}\right)\xrightarrow[\mathbf{2}\text{-(2)-(a)}]{\phi}\left((10),\begin{pmatrix}1\\\overline{2}\end{pmatrix}\right)\xrightarrow[\mathbf{3}\text{-(1)}]{\phi^{-1}}\left((8),\begin{pmatrix}2&0\\2&0\end{pmatrix}\right)$$
$$\left((7,4,2),\begin{pmatrix}0\\0\end{pmatrix}\right)\xrightarrow[\mathbf{3}\text{-(2)}]{\phi}\left((6,4),\begin{pmatrix}1\\\overline{2}\end{pmatrix}\right)\xrightarrow[\mathbf{3}\text{-(4)}]{\phi^{-1}}\left((7,4,2),\begin{pmatrix}0\\0\end{pmatrix}\right).$$

We can see that $\phi^{-1}(\phi(\pi,\lambda)) = (\pi,\lambda)$ for all $(\pi,\lambda) \in \mathcal{A}_n$. We also note that there are elements in $\mathcal{B}_n \backslash \phi(\mathcal{A}_n)$. For example,

$$\left((1), \begin{pmatrix} a_1(>3) & 1 \\ b_1 & b_2 \end{pmatrix}\right), \left(\emptyset, \begin{pmatrix} 3 & 1 \\ b_1 & b_2 \end{pmatrix}\right) \in \mathcal{B}_n \backslash \phi(\mathcal{A}_n),$$

which implies that the coefficients are positive for $n \geq 14$. By checking the first few coefficients up to q^{13}, we can conclude the desired inequality. □

Finally, we prove Theorem 3. To apply the Tauberian theorem, we first need to show that both $p_o(n)$ and $p_e(n)$ are weakly increasing. Note that a function $a : \mathbb{N} \to \mathbb{Z}$ is weakly increasing if and only if $(1-q)\sum_{n\geq 1} a(n)q^n$ has non-negative coefficients.

Lemma 5. *We have that $p_o(n)$ is weakly increasing for all positive integers n and $p_e(n)$ is weakly increasing for all integers $n > 5$.*

Proof. Using (3.1) and (3.2) we have

$$(1-q)P_o(q) = \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2-n}}{(q^2;q^2)_n (q^2;q^2)_{n-1}}, \tag{3.6}$$

$$(1-q)P_e(q) = \frac{1}{(q^3;q^2)_\infty} \sum_{n\geq 1} \frac{q^{2n^2}(1-q^n)}{(q^2;q^2)_n^2}. \tag{3.7}$$

The q-series on the right-hand side of (3.6) clearly has non-negative coefficients, and so $p_o(n)$ is weakly increasing. On the right hand side of (3.7), the nth summand has non-negative coefficients for $n \geq 2$. The first summand is

$$\frac{q^2(1-q)^2}{(1-q^2)^2(q;q^2)_\infty},$$

which is q times (3.4). Therefore $p_e(n)$ is weakly increasing for $n > 5$. □

Now we are ready to prove the asymptotic result.

Proof of Theorem 3. We use the constant term method on

$$G_b(q) := \frac{1}{(q;q^2)_\infty} \sum_{n\geq 0} \frac{q^{2n^2+bn}}{(q^2;q^2)_n^2} = \operatorname{coeff}[x^0]\left(\frac{(-xq;q^2)_\infty(-q^{1+b}/x;q^2)_\infty}{(q;q^2)_\infty}\right)$$

for $b = -1, 0,$ and 1, where the identity comes from

$$(-yq;q^2)_\infty = \sum_{n\geq 0} \frac{y^n q^{n^2}}{(q^2;q^2)_n}.$$

Let $q = e^{-z}$ and $x = e^{2\pi i u}$, where $z, u \in \mathbb{C}$ and $\mathrm{Re}(z) > 0$. For the case $b = 1$,

$$G_1(q) = \operatorname{coeff}[x^0]\left(\frac{(-xq;q^2)_\infty(-q^2/x;q^2)_\infty(q^2;q^2)_\infty(-x;q^2)_\infty^2}{(q;q^2)_\infty(q^2;q^2)_\infty(-x;q^2)_\infty^2}\right)$$

$$= \operatorname{coeff}[x^0]\left(\frac{(-x;q)_\infty}{(q)_\infty(-x;q^2)_\infty^2}\theta(q;x/q)\right)$$

where $\theta(q;x)$ is Jacobi theta function defined by $\theta(q;x) := \sum_{n\in\mathbb{Z}} q^{n^2}x^n$. Now we need the modular inversion formula for Jacobi theta function (see e.g. [8, eq. (2.5)])

$$\theta(e^{-z};e^{2\pi iu}) = \sqrt{\frac{\pi}{z}}\sum_{n\in\mathbb{Z}} e^{-\frac{\pi^2(n+u)^2}{z}} \tag{3.8}$$

and the quantum dilogarithm [17, p.28] defined for $|x|<1$ by

$$\mathrm{Li}_2(x;q) := -\log(x)_\infty = \sum_{m\geq 1}\frac{x^m}{m(1-q^m)}. \tag{3.9}$$

Cauchy's integral formula together with (3.8) and (3.9) gives us

$$\begin{aligned}
G_1(q) &= \frac{1}{(q)_\infty}\sqrt{\frac{\pi}{z}}\int_{[0,1]+iC}\sum_{n\in\mathbb{Z}}\exp\left(-\frac{\pi^2}{z}\left(n+u-\frac{iz}{2\pi}\right)^2 - \mathrm{Li}_2(-x;q) + 2\mathrm{Li}_2(-x;q^2)\right)du \\
&= \frac{1}{(q)_\infty}\sqrt{\frac{\pi}{z}}\int_{\mathbb{R}+iC}\exp\left(-\frac{\pi^2}{z}\left(u-\frac{iz}{2\pi}\right)^2 - \mathrm{Li}_2(-x;q) + 2\mathrm{Li}_2(-x;q^2)\right)du,
\end{aligned}$$

where $C = \frac{\mathrm{Re}(z)}{2\pi} > 0$. The asymptotic expansion of the quantum dilogarithm for $|x|<1$ and $a\geq 0$ is

$$\begin{aligned}
\mathrm{Li}_2(xe^{-az};e^{-z}) &= \sum_{n\geq 1}\frac{x^n e^{-anz}}{n(1-e^{-nz})} = \frac{1}{z}\sum_{n\geq 1}\frac{x^n}{n^2}\left(1 - nz(a-\frac{1}{2}) + \mathcal{O}(z^2)\right) \\
&= \frac{1}{z}\mathrm{Li}_2(x) + (a-\frac{1}{2})\log(1-x) + \mathcal{O}(z),
\end{aligned}$$

where the series converges uniformly in x as $z\to 0$ with $|z|\leq 1$ [17, p.28]. Using the change of variable $v = u - \frac{iz}{2\pi}$ with the asymptotic expansion of the quantum dilogarithm, we obtain

$$\begin{aligned}
G_1(q) &= \frac{1}{(q)_\infty}\sqrt{\frac{\pi}{z}}\int_{\mathbb{R}}\exp\left(-\frac{\pi^2}{z}v^2 - \mathrm{Li}_2(-e^{2\pi iv}q;q) + 2\mathrm{Li}_2(-e^{2\pi iv}q;q^2)\right)dv \\
&= \frac{1}{(q)_\infty}\sqrt{\frac{\pi}{z}}\int_{\mathbb{R}}\exp\left(-\frac{\pi^2}{z}v^2 - \frac{1}{2}\log(1+e^{2\pi iv}) + \mathcal{O}(z)\right)dv.
\end{aligned}$$

Therefore, as $z\to 0$ with $\mathrm{Re}(z)>0$ in the restricted region $\mathrm{Im}(z)\leq \Delta\mathrm{Re}(z)$ for $\Delta>0$,

$$\begin{aligned}
G_1(q) &= \frac{1}{(q)_\infty}\sqrt{\frac{\pi}{z}}(1+\mathcal{O}(z))\int_{\mathbb{R}}\frac{e^{-\frac{\pi^2}{z}v^2}}{\sqrt{1+e^{2\pi iv}}}dv \\
&= \frac{1}{(q)_\infty}\sqrt{\frac{\pi}{z}}(1+\mathcal{O}(z))\int_{\mathbb{R}}e^{-\frac{\pi^2}{z}v^2}\left(\frac{1}{\sqrt{2}} - \frac{i\pi v}{2\sqrt{2}} + \frac{\pi^2v^2}{8\sqrt{2}} - \frac{5i\pi^3v^3}{48\sqrt{2}} + \frac{17\pi^4v^4}{384\sqrt{2}} + \cdots\right)dv \\
&= \frac{1}{(q)_\infty}\sqrt{\frac{\pi}{z}}(1+\mathcal{O}(z))\sqrt{\frac{z}{2\pi}}\left(1 + \frac{z}{16} + \frac{51z^2}{768\sqrt{2}} + \cdots\right)
\end{aligned}$$

$$\sim \frac{1}{2}\sqrt{\frac{z}{\pi}}e^{\frac{\pi^2}{6z}}$$

using the asymptotic formula [7, eq. (3.2)]

$$\frac{1}{(q)_\infty} \sim \sqrt{\frac{z}{2\pi}}e^{\frac{\pi^2}{6z}}$$

as $z \to 0$ in this restricted region . Similarly, for $b = -1$ we find

$$G_{-1}(e^{-z}) \sim \sqrt{\frac{z}{\pi}}e^{\frac{\pi^2}{6z}}$$

as $z \to 0$ in the restricted region $\mathrm{Im}(z) \leq \Delta\mathrm{Re}(z)$ for $\Delta > 0$. The $b = 0$ case is simpler since we have

$$G_0(q) = \frac{1}{(q)_\infty}.$$

Hence, we have the asymptotics of $P_o(q)$ and $P_e(q)$: as $z \to 0$ with $\mathrm{Im}(z) \leq \Delta\mathrm{Re}(z)$ for $\Delta > 0$,

$$P_o(e^{-z}) = G_{-1}(e^{-z}) - G_1(e^{-z}) \sim \frac{1}{2}\sqrt{\frac{z}{\pi}}\exp\left(\frac{\pi^2}{6z}\right),$$

$$P_e(e^{-z}) = G_0(e^{-z}) - G_1(e^{-z}) \sim \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)\sqrt{\frac{z}{\pi}}\exp\left(\frac{\pi^2}{6z}\right).$$

Since $p_o(n)$ and $p_e(n)$ are weakly increasing for $n > 5$, by Theorem 4,

$$p_o(n) \sim \frac{1}{4\sqrt{6}n}\exp\left(\pi\sqrt{\frac{2n}{3}}\right) \sim \frac{1}{\sqrt{2}}p(n),$$

$$p_e(n) \sim \frac{\sqrt{2}-1}{4\sqrt{6}n}\exp\left(\pi\sqrt{\frac{2n}{3}}\right) \sim \frac{\sqrt{2}-1}{\sqrt{2}}p(n),$$

as $n \to \infty$.

□

4. Partitions without repeated odd parts

Let $pod_o(n)$ (resp. $pod_e(n)$) denote the number of partitions of n without repeated odd parts having more odd (resp. even) parts than even (resp. odd) parts. Using elementary combinatorial arguments we have

$$POD_o(q) := \sum_{n\geq 0} pod_o(n)q^n = \sum_{n\geq 0}\frac{q^{n^2}}{(q^2;q^2)_n^2} - \sum_{n\geq 0}\frac{q^{n^2+2n}}{(q^2;q^2)_n^2} \tag{4.1}$$
$$= q + q^3 + q^4 + q^5 + 2q^6 + q^7 + 4q^8 + 2q^9 + \cdots,$$

$$POD_e(q) := \sum_{n\geq 0} pod_e(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} - \sum_{n\geq 0}\frac{q^{n^2}}{(q^2;q^2)_n^2} \tag{4.2}$$

$$= q^2 + 2q^4 + q^5 + 3q^6 + 3q^7 + 5q^8 + 7q^9 + 8q^{10} + \cdots .$$

Given the fact that odd parts may not repeat, we naturally expect $pod_e(n)$ to be larger than $pod_o(n)$. Our goal in this section is to prove a combinatorial lower bound for $pod_e(n) - pod_o(n)$.

To describe the lower bound, recall that $\mathcal{D}_e$ is the set of partitions into distinct non-negative even parts and $\mathcal{POD}$ is the set of partitions into non-negative parts where odd parts are distinct. Define $F_2(n)$ to be the number of Frobenius symbols counted by $F_{\mathcal{D}_e,\mathcal{POD}}(n)$ having at least two odd parts in the bottom row. Note that the generating function for $F_2(n)$ is

$$\sum_{n\geq 1} F_2(n)q^n = \sum_{n\geq 1} \frac{q^{n^2}\{(-q;q^2)_n - 1\}}{(q^2;q^2)_n^2} - \sum_{n\geq 1} \frac{q^{n^2+1}(1-q^{2n})}{(1-q^2)(q^2;q^2)_n^2} \tag{4.3}$$

$$= q^8 + 2q^{10} + 5q^{12} + q^{13} + 8q^{14} + 3q^{15} + 14q^{16} + \cdots ,$$

the first (resp. second) series on the right hand side of (4.3) corresponding to Frobenius symbols with at least (resp. exactly) one odd part in the bottom row. Our lower bound is the following.

Theorem 6. *For all positive integers $n > 12$,*

$$pod_e(n) - pod_o(n) > F_2(n).$$

Before proving Theorem 6, we establish an inequality for certain Frobenius symbols. Recall that $\mathcal{P}_e'$ denotes the set of partitions into non-negative even parts with at least one 0 and $\mathcal{POD}_1$ denotes the set of partitions into non-negative parts having exactly one odd part.

Lemma 7. *For all $n \neq 1, 3$ we have $F_{\mathcal{D}_e,\mathcal{POD}_1}(n) \geq F_{\mathcal{D}_e,\mathcal{P}_e'}(n)$, the inequality being strict for all odd $n > 5$ and for all even $n > 12$.*

Proof. To prove the result, we define a weight-preserving injection from $\mathcal{F}_{\mathcal{D}_e,\mathcal{P}_e'}$ to $\mathcal{F}_{\mathcal{D}_e,\mathcal{POD}_1}$. Let $\begin{pmatrix} a_1 & a_2 & \cdots & a_{k-1} & a_k \\ b_1 & b_2 & \cdots & b_{k-1} & 0 \end{pmatrix} \in \mathcal{F}_{\mathcal{D}_e,\mathcal{P}_e'}$.

Case 1: $a_k = 0$ and $k > 1$

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{k-1} & a_k \\ b_1 & b_2 & \cdots & b_{k-1} & 0 \end{pmatrix} \to \begin{pmatrix} a_1 & a_2 & \cdots & a_{k-1} \\ b_1+1 & b_2 & \cdots & b_{k-1} \end{pmatrix}.$$

Note that $a_{k-1} > 0$.

Case 2: $a_k = 2$ and $k > 2$

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{k-1} & a_k \\ b_1 & b_2 & \cdots & b_{k-1} & 0 \end{pmatrix} \to \begin{pmatrix} a_1-2 & a_2-2 & \cdots & a_{k-1}-2 \\ b_1+4 & b_2+3 & \cdots & b_{k-1}+2 \end{pmatrix}.$$

Note that $a_{k-1} - 2 > 0$.

Case 3: $a_k = 2$ and $k = 2$

$$\begin{pmatrix} a_1 & 2 \\ b_1 & 0 \end{pmatrix} \to \begin{pmatrix} a_1-2 & 2 & 0 \\ b_1+1 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ b_1+7 \end{pmatrix},$$

where the latter case is only for $a_1 = 4$.

Case 4: $a_k > 2$

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{k-1} & a_k \\ b_1 & b_2 & \cdots & b_{k-1} & 0 \end{pmatrix} \to \begin{pmatrix} a_1-2 & a_2-2 & \cdots & a_{k-1}-2 & a_k-2 & 0 \\ b_1+2 & b_2+2 & \cdots & b_{k-1}+2 & 1 & 0 \end{pmatrix}.$$

Note that the map is not defined on the Frobenius symbols $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, which explains the exceptional cases $n = 1$ and $n = 3$. It is straightforward to check that the map is indeed a weight-preserving injection from $\mathcal{F}_{\mathcal{D}_e,\mathcal{P}'_e}$ to $\mathcal{F}_{\mathcal{D}_e,\mathcal{POD}_1}$. Moreover, for $m \geq 1$ the Frobenius symbols

$$\begin{pmatrix} 2m & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2m+2 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix} \in \mathcal{F}_{\mathcal{D}_e,\mathcal{POD}_1}$$

are not in the image of the map, which gives the strict inequality. □

We are now ready to prove Theorem 6.

Proof of Theorem 6. Using (4.1), (4.2), and (2.1) we have

$$\begin{aligned} POD_e(q) - POD_o(q) &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} - 2\sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n^2} + \sum_{n\geq 0} \frac{q^{n^2+2n}}{(q^2;q^2)_n^2} \\ &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} - \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n^2} - \sum_{n\geq 0} \frac{q^{n^2}(1-q^{2n})}{(q^2;q^2)_n^2} \\ &= \sum_{n\geq 0} \frac{q^{n^2}\{(-q;q^2)_n - 1\}}{(q^2;q^2)_n^2} - \sum_{n\geq 0} \frac{q^{(n+1)^2}}{(q^2;q^2)_n (q^2;q^2)_{n+1}} \\ &= \sum_{\lambda \in \mathcal{F}_{\mathcal{D}_e,\mathcal{POD}_{\geq 1}}} q^{|\lambda|} - \sum_{\lambda \in \mathcal{F}_{\mathcal{D}_e,\mathcal{P}'_e}} q^{|\lambda|}, \end{aligned}$$

recalling that $\mathcal{POD}_{\geq 1}$ is the set of partitions into non-negative integers without repeated odd parts having at least one odd part. Applying Lemma 7 gives the result. □

5. Overpartitions

Define $\overline{p}_u(n)$ (resp. $\overline{p}_o(n)$) to be the number of overpartitions of n with more unoverlined (resp. overlined) parts than overlined (resp. unoverlined) parts. Here we prove the following result.

Theorem 8. *The difference $\overline{p}_u(n) - \overline{p}_o(n)$ is equal to the number of overpartitions of n where the number of unoverlined parts is at least two more than the number of overlined parts.*

For example, there are 8 overpartitions of 3,

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1.$$

Here $\overline{p}_u(3) = 4$, $\overline{p}_o(3) = 2$, and thus $\overline{p}_u(3) - \overline{p}_o(3) = 2$, which corresponds to the partitions $2+1$ and $1+1+1$.

For $x = u$ or o, let $\overline{P}_x(q)$ be the generating function for $\overline{p}_x(n)$. We have

$$\overline{P}_o(q) := \sum_{n\geq 0} \overline{p}_o(n)q^n = \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n^2} - \sum_{n\geq 0} \frac{q^{n(n+1)/2+n}}{(q)_n^2},$$

$$\overline{P}_u(q) := \sum_{n\geq 0} \overline{p}_u(n)q^n = \frac{(-q)_\infty}{(q)_\infty} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n^2}.$$

Proof of Theorem 8. Note that

$$\begin{aligned}
\overline{P}_u(q) - \overline{P}_o(q) &= \frac{(-q)_\infty}{(q)_\infty} - 2\sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n^2} + \sum_{n\geq 0} \frac{q^{n(n+1)/2+n}}{(q)_n^2} \\
&= \frac{(-q)_\infty}{(q)_\infty} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n^2} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}(1-q^n)}{(q)_n^2} \\
&= \frac{(-q)_\infty}{(q)_\infty} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n^2} - \sum_{n\geq 0} \frac{q^{n(n+1)/2+n+1}}{(q)_n(q)_{n+1}} \\
&= \frac{(-q)_\infty}{(q)_\infty} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n^2} \left(1 + \frac{q^{n+1}}{1-q^{n+1}}\right) \\
&= \frac{(-q)_\infty}{(q)_\infty} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n(q)_{n+1}}.
\end{aligned}$$

The above sum is the generating function for overpartitions where the number of unoverlined parts at most one more than the number of overlined parts. This gives the result. □

6. Concluding Remarks

We close with some open problems. First, it is natural to wonder about parity bias in other types of partitions. For example, if $d_o(n)$ (resp. $d_e(n)$) denotes the number of partitions into distinct parts having more odd (resp. even) parts than even (resp. odd) parts, then numerics say that

$$d_o(n) > d_e(n)$$

for $n > 19$. (This has been checked up to $n = 2000$.) We leave this as a conjecture.

Second, some interesting behavior arises if we attach weights to some of the generating functions considered in this paper. For example, let $a(n)$ denote the number of partitions λ

counted by $p_o(n)$, each weighted by $(-1)^w$, where

$$w = \frac{1}{2}(\text{the largest even part of } \lambda + \text{the largest odd part of } \lambda - 1).$$

Then we have

$$\sum_{n\geq 1} a(n)q^n = \sum_{n\geq 1} \frac{q^n(1+q^{2n})}{(-q^2;q^2)_n^2},$$

and it appears that $a(n) = 0$ if and only if $n \in \{3, 5, 9, 17, 20, 23, 24, 26, 28, 51, 125, 233\}$. (This has been checkd up to $n = 2000$.) We leave this as a conjecture, as well.

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