# OVERPARTITION PAIRS

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Abstract. An overpartition pair is a combinatorial object associated with the $q$-Gauss identity and the ${}_1\psi_1$ summation. In this paper, we prove identities for certain restricted overpartition pairs using Andrews' theory of $q$-difference equations for well-poised basic hypergeometric series and the theory of Bailey chains.

## 1. Introduction

An *overpartition* of $n$ is a partition of $n$ in which the first occurrence of a number can be overlined. For example, there are 14 overpartitions of 4,

$$(4), (\overline{4}), (3,1), (\overline{3},1), (3,\overline{1}), (\overline{3},\overline{1}), (2,2), (\overline{2},2), (2,1,1),$$
$$(\overline{2},1,1), (2,\overline{1},1), (\overline{2},\overline{1},1), (1,1,1,1), (\overline{1},1,1,1).$$

An *overpartition pair* of $n$ is a pair of overpartitions $(\mu, \lambda)$ where the sum of all the parts is $n$. For example, there are 12 overpartition pairs of 2,

$$((2),\emptyset), ((\overline{2}),\emptyset), ((1,1),\emptyset), ((\overline{1},1),\emptyset), ((1),\overline{1}), ((1),(1)), ((\overline{1}),(\overline{1})), ((\overline{1}),(1)),$$
$$(\emptyset,(\overline{2})), (\emptyset,(2)), (\emptyset,(1,1)), (\emptyset,(\overline{1},1)).$$

Since the overlined parts of an overpartition form a partition into distinct parts and the non-overlined parts of an overpartition form an unrestricted partition, we have the generating functions

$$\sum_{n\geq 0} \overline{p}(n)q^n = \prod_{n\geq 1} \frac{(1+q^n)}{(1-q^n)} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \cdots \tag{1.1}$$

and

$$\sum_{n\geq 0} \overline{pp}(n)q^n = \prod_{n\geq 1} \frac{(1+q^n)^2}{(1-q^n)^2} = 1 + 4q + 12q^2 + 32q^3 + 76q^4 + \cdots, \tag{1.2}$$

where $\overline{p}(n)$ and $\overline{pp}(n)$ denote the number of overpartitions of $n$ and the number of overpartition pairs of $n$, respectively.

Overpartition pairs are naturally associated with $q$-series identities like the $q$-Gauss summation [16, p.236, (II.8)],

$$\sum_{n\geq 0} \frac{(-1/a,-1/b)_n(abcq)^n}{(q,cq)_n} = \frac{(-acq,-bcq)_\infty}{(cq,abcq)_\infty}, \tag{1.3}$$

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and Ramanujan's ${}_1\psi_1$ summation [16, p.239, (II.29)],

$$\sum_{n\in\mathbb{Z}} \frac{(-1/a)_n(azq)^n}{(-bq)_n} = \frac{(-zq,-1/z,abq,q)_\infty}{(azq,b/z,-aq,-bq)_\infty}. \tag{1.4}$$

Here we have employed the standard $q$-series notation,

$$(a_1,\ldots,a_k)_\infty := (a_1,\ldots,a_k;q)_\infty := \prod_{j=0}^{\infty}(1-a_1q^j)\cdots(1-a_kq^j) \tag{1.5}$$

and

$$(a_1,\ldots,a_k)_n := \frac{(a_1,\ldots,a_k)_\infty}{(a_1q^n,\ldots,a_kq^n)_\infty}. \tag{1.6}$$

Specifically, let $p_{\mathcal{O},\mathcal{O}}(n)$, denote the number of *generalized Frobenius partitions*,

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_k \\ b_1 & b_2 & \ldots & b_k \end{pmatrix}, \tag{1.7}$$

having an overpartition in the top row, an overpartition into non-negative parts in the bottom row, and satisfying $\sum a_i + b_i = n$. Then the $q$-Gauss identity and the ${}_1\psi_1$ summation are consequences of the fact that $p_{\mathcal{O},\mathcal{O}}(n)$ is equal to the number overpartition pairs of $n$ [9, 11, 23].

Overpartitions arise in many areas where ordinary partitions play an important role, most notably $q$-series and combinatorics (*e.g.* [5, 10, 12, 19, 20, 21]), but also in mathematical physics (*e.g.* [14, 15]), symmetric functions (*e.g.* [6, 13], and representation theory (*e.g.* [17]). In these subjects overpartitions are variously called standard MacMahon diagrams, joint partitions, superpartitions, jagged partitions, dotted partitions, and probably many other things.

In this paper we shall prove identities for overpartition pairs using Andrews' theory of $q$-difference equations for well-poised basic hypergeometric series [1] and the theory of Bailey pairs [3]. To state the first theorem, we require some definitions. In an overpartition $\lambda$, we say that an overlined part $\overline{k}$ is *accompanied* if there is also at least one occurrence of $k$ non-overlined. In an overpartition pair $(\mu,\lambda)$, we say that a non-overlined part $k$ of $\mu$ is *attached* if $k$ or $\overline{k}$ also appears in the overpartition $\lambda$. Finally, we define the valuation of a natural number $k$ with respect to an overpartition pair $(\mu,\lambda)$ by

$$v_{(\mu,\lambda)}(k) = \begin{cases} 1, & \text{if } k \text{ occurs unattached in } \mu \\ \text{the number of occurrences of } k \text{ and } \overline{k} \text{ in } \lambda, & \text{otherwise} \end{cases} \tag{1.8}$$

For $k \geq 1$ and $1 \leq r \leq 2k$, let us define $a_{2k,r}(n)$ to be the number of overpartition pairs $(\mu,\lambda)$ of $n$ such that

(i) $\mu$ has no overlined parts,

(ii) $v_{(\mu,\lambda)}(1) \leq r-1$,

(iii) all parts of $\lambda$ occur at most $2k-1$ times,

(iv) $v_{(\mu,\lambda)}(j) + v_{(\mu,\lambda)}(j+1)$ is at most $2k-1$, and at most $2k+1$ if $j$ occurs overlined in $\lambda$,

(v) If $v_{(\mu,\lambda)}(j) + v_{(\mu,\lambda)}(j+1)$ attains the maximum allowed above, then

$$jv_{(\mu,\lambda)}(j) + (j+1)v_{(\mu,\lambda)}(j+1) \equiv r-1 \pmod 2.$$

(vi) only numbers congruent to $r-1$ modulo 2 can occur unattached in $\mu$,

(vii) if $\overline{k} \equiv r-1 \pmod 2$ occurs in $\lambda$, then it must be accompanied.

Notice that $a_{2,1}(n)$ is the number of overpartitions of $n$ into even parts, while $a_{2,2}(n)$ is the number of overpartitions into odd parts. Also, if $\mu$ is empty and there are no overlined parts in $\lambda$, then we have a type of partition studied by Bressoud [7, p.64, $B_{2k,r,0}(n)$] in his extension of the Rogers-Ramanujan identities to all moduli. We shall prove the following:

**Theorem 1.1.**

$$\sum_{n\geq 0} a_{2k,2k-1}(n)q^n = \frac{(-q)_\infty(-q^2;q^2)_\infty(q^{2k-1};q^{2k-1})_\infty}{(q)_\infty(q^2;q^2)_\infty(-q^{2k-1};q^{2k-1})_\infty}.$$

*In other words, $a_{2k,2k-1}(n)$ is equal to the number of overpartition pairs $(\mu,\lambda)$ of $n$ where the parts of $\mu$ are even and the parts of $\lambda$ are not divisible by $2k-1$.*

The second main theorem corresponds to the family of $q$-series identities

$$\sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{(-1;q^2)_{n_k} q^{n_k+n_{k-1}^2+\cdots+n_2^2+\chi(k\neq 1)n_1^2+n_1}}{(q^{n_1+1})_{n_k-n_1}(q^2;q^2)_{n_1}} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix} = \frac{(-q)_\infty(-q^2;q^2)_\infty(q^k;q^k)_\infty}{(q)_\infty(q^2;q^2)_\infty(-q^k;q^k)_\infty}. \tag{1.9}$$

Here $\chi(x)$ is the usual characteristic function, equal to 1 if $x$ is true and 0 if $x$ is false. Also, we have employed the $q$-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{(q^{k+1})_{n-k}}{(q)_{n-k}}, & 0\leq k\leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{1.10}$$

To state the second theorem, we need some more definitions. For $k=1$, let $f_k(n)$ denote the number of generalized Frobenius partitions (1.7) counted by $p_{\mathcal{O},\mathcal{O}}(n)$ where if we add 2 to each part on the bottom row then we obtain the top row. For $k\geq 2$, we will appeal to the "Durfee dissection" introduced by Andrews [2] and commonly associated with identities like (1.9). Recall that the Ferrers diagram of a partition $\pi$ contains a largest upper-left justified square called the Durfee square. Similarly, a partition has a largest upper-left justified $k\times k+1$ rectangle, called the Durfee rectangle. To the right of the Durfee square is another partition, which itself has a Durfee square, called the second Durfee square of $\pi$. Continuing in this way, we obtain a sequence of Durfee squares, and this sequence can be of any length we choose if we allow squares of size 0.

Now let $f_k(n)$ denote the number of generalized Frobenius partitions (1.7) counted by $p_{\mathcal{O},\mathcal{O}}(n)$ such that when we decompose the top (bottom) row into a partition into distinct parts $\nu_1$ ($\nu_2$) and an ordinary partition $\pi_1$ ($\pi_2$) into $n_k$ parts (parts less than or equal to $n_k$), then

(i) $\nu_1 = \nu_2$,

(ii) if we subtract one from each part of $\pi_1$ then, in the resulting partition, to the right of the $k-2$nd Durfee square is either nothing or a partition with Durfee rectangle of size $n_1$,

(iii) exactly $n_1$ parts to the right of the $k-2$nd Durfee square have size at least $n_1+1$,

(iv) the columns to the right of this Durfee rectangle of size $n_1$ are identical to the parts at most $n_1$ in $\pi_2$.

Then we have

**Theorem 1.2.** *For all $k \geq 1$,*

$$\sum_{n \geq 0} f_k(n) q^n = \frac{(-q)_\infty (-q^2;q^2)_\infty (q^k;q^k)_\infty}{(q)_\infty (q^2;q^2)_\infty (-q^k;q^k)_\infty}. \tag{1.11}$$

We should remark that the decomposition referred to in the definition of $f_k(n)$, that of an overpartition with $n$ parts into a partition with $n$ parts and a partition into distinct non-negative parts less than $n$, is the standard Joichi-Stanton algorithm [12, Proposition 2.1].

Andrews [3, Chapter 9] has promoted in a precise way the philosophy that whenever a $q$-series can be written as an infinite product, then related $q$-series should also be "interesting." In the case of overpartitions, it seems that there are frequently connections to the arithmetic of real quadratic fields [8, 19], in the spirit of [4]. In the present case, we'll discover a connection between $a_{2,2}(n)$, whose generating function is the infinite product $(-q;q^2)_\infty/(q;q^2)_\infty$, and the arithmetic of $\mathbb{Q}(\sqrt{2})$. This connection manifests itself in the following theorem:

**Theorem 1.3.** *Let $a^{\pm}_{2,2}(n)$ denote the number of overpartitions of $n$ into odd parts where the largest part is congruent to $1/3 \pmod 4$. If $n$ has the prime factorization $n = 2^a p_1^{e_1} \cdots p_j^{e_j} q_1^{f_1} \cdots q_k^{f_k}$, where the $p_i$ are congruent to $\pm 1$ modulo $8$ and the $q_i$ are congruent to $\pm 3$ modulo $8$, then $a^{+}_{2,2}(n) - a^{-}_{2,2}(n)$ is equal to $0$, if some $f_i$ is odd, and $-2i^{n^2+n}(e_1+1)\cdots(e_j+1)$ otherwise.*

The organization of this paper is straightforward. We prove Theorem 1.1 in Section 2, Theorem 1.2 in Section 3, and Theorem 1.3 in Section 4. Some concluding remarks are offered in Section 5.

## 2. Proof of Theorem 1.1

Following Andrews [1], we define for any real numbers $k$ and $i$,

$$J_{k,i}(a;x;q) = \frac{(xq/a)_\infty}{(xq)_\infty} \sum_{n \geq 0} \frac{x^{kn} q^{kn^2+kn-in+n} (xq,a)_n}{a^n (q, xq/a)_n} \left(1 + \frac{x^i q^{(2n+1)i-n}(1-aq^n)}{a(1-xq^{n+1}/a)}\right). \tag{2.1}$$

We shall be concerned with

$$L_{k,r}(x) = \frac{(-xq)_\infty}{(xq)_\infty} J_{\frac{k-1}{2},\frac{r}{2}}(-1;x^2;q^2). \tag{2.2}$$

From [1, (2.1)-(2.4)], it may be deduced that

$$L_{k,1}(x) = L_{k,k}(xq), \tag{2.3}$$

$$L_{k,2}(x) = \frac{(1+xq)}{(1-xq)} L_{k,k-1}(xq), \tag{2.4}$$

and, for $k \geq 3$,

$$L_{k,r}(x) - L_{k,r-2}(x) = \frac{(1+xq)}{(1-xq)}(xq)^{r-2}L_{k,k-r+1}(xq) + \frac{(1+xq)}{(1-xq)}(xq)^{r-2}L_{k,k-r+3}(xq). \tag{2.5}$$

**Lemma 2.1.** *For $k \geq 2$ we have*

$$L_{k,k-1}(1) = \frac{(-q)_\infty(-q^2;q^2)_\infty(q^{k-1};q^{k-1})_\infty}{(q)_\infty(q^2;q^2)_\infty(-q^{k-1};q^{k-1})_\infty}. \tag{2.6}$$

*Proof.*

$$\begin{aligned}
L_{k,k-1}(1) &= \frac{(-q)_\infty}{(q)_\infty}\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}\frac{2q^{(k-1)n^2+2n}(-1)^n}{(1+q^{2n})}\left(1-\frac{q^{(2n+1)(k-1)-2n}(1+q^{2n})}{(1+q^{2n+2})}\right) \\
&= \frac{(-q)_\infty(-q^2;q^2)_\infty}{(q)_\infty(q^2;q^2)_\infty}\left(\sum_{n\geq 0}\frac{2q^{(k-1)n^2+2n}(-1)^n}{(1+q^{2n})} - \sum_{n\geq 0}\frac{2q^{(k-1)n^2+(2n+1)(k-1)}(-1)^n}{(1+q^{2n+2})}\right) \\
&= \frac{(-q)_\infty(-q^2;q^2)_\infty}{(q)_\infty(q^2;q^2)_\infty}\left(1+\sum_{n\geq 1}\frac{2q^{(k-1)n^2+2n}(-1)^n}{(1+q^{2n})} + \sum_{n\geq 1}\frac{2q^{(k-1)n^2}(-1)^n}{(1+q^{2n})}\right) \\
&= \frac{(-q)_\infty(-q^2;q^2)_\infty}{(q)_\infty(q^2;q^2)_\infty}\left(1+\sum_{n\geq 1}\frac{2q^{(k-1)n^2}(-1)^n(1+q^{2n})}{(1+q^{2n})}\right) \\
&= \frac{(-q)_\infty(-q^2;q^2)_\infty}{(q)_\infty(q^2;q^2)_\infty}\sum_{n\in\mathbb{Z}}(-1)^n q^{(k-1)n^2} \\
&= \frac{(-q)_\infty(-q^2;q^2)_\infty(q^{k-1};q^{k-1})_\infty}{(q)_\infty(q^2;q^2)_\infty(-q^{k-1};q^{k-1})_\infty},
\end{aligned}$$

by the triple product identity [16, p. 239, Eq. II.28],

$$\sum_{n\in\mathbb{Z}} z^n q^{n(n+1)/2} = (-zq, -1/z, q)_\infty. \tag{2.7}$$

□

We remark that the $L_{k,r}(1)$ seem to be expressible as infinite products only in the case $r = k-1$ (or $r = k = 2$). Such a one-parameter family of products is what we have come to expect when dealing with overpartitions [18, 20].

Now write

$$L_{2k,r}(x) = \sum_{m,n\geq 0} b_{2k,r}(m,n)x^m q^n.$$

Then, the three equations (2.3) - (2.5) above imply that

$$b_{2k,1}(m,n) = b_{2k,2k}(m,n-m), \tag{2.8}$$

$$b_{2k,2}(m,n) = b_{2k,2k-1}(m,n-m) + 2\sum_{t\geq 1} b_{2k,2k-1}(m-t,n-m), \tag{2.9}$$

and

$$\begin{aligned} b_{2k,r}(m,n) - b_{2k,r-2}(m,n) &= \sum_{t\geq 0} b_{2k,2k-r+1}(m-r+2-t,n-m) \\ &+ \sum_{t\geq 0} b_{2k,2k-r+1}(m-r+1-t,n-m) \\ &+ \sum_{t\geq 0} b_{2k,2k-r+3}(m-r+2-t,n-m) \\ &+ \sum_{t\geq 0} b_{2k,2k-r+3}(m-r+1-t,n-m). \end{aligned} \tag{2.10}$$

These three facts, together with

$$b_{2k,r}(m,n) = \begin{cases} 1, & (m,n) = (0,0), \\ 0, & m \leq 0 \text{ or } n \leq 0, \text{ but } (m,n) \neq (0,0), \end{cases} \tag{2.11}$$

uniquely define the $b_{2k,r}(m,n)$.

Now let $a_{2k,r}(m,n)$ denote the number of pairs of overpartitions counted by $a_{2k,r}(n)$ having exactly $m$ parts. We shall demonstrate that the $a_{2k,r}(m,n)$ satisfy the same defining equations as the $b_{2k,r}(m,n)$. We first treat condition (2.8).

If $(\mu,\lambda)$ is an overpartition pair counted by $a_{2k,1}(m,n)$, then there are no ones whatsoever, so we may subtract 1 from each part. According to the definition of $a_{2k,1}(m,n)$, the valuation $v_{(\mu,\lambda)}(2)$ could have been as much as $2k-1$, so the result of subtracting one from each part is an overpartition pair counted by $a_{2k,2k}(m,n-m)$. This operation is reversible and therefore establishes a one-to-one correspondence between overpartition pairs counted by $a_{2k,1}(m,n)$ and overpartition pairs counted by $a_{2k,2k}(m,n-m)$. Notice here that the the second subscript has changed in parity (from 1 to $2k$). Since we have subtracted 1 from each part, this change in parity is compatible with the conditions $(v)-(vii)$ in the definition of the $a_{2k,r}(n)$. This will happen throughout the proof, although we shall not emphasize it again.

We now treat condition (2.9). If $(\mu,\lambda)$ is an overpartition pair counted by $a_{2k,2}(m,n)$, then $v_{(\mu,\alpha)}(1)$ is 0 or 1. We consider the two cases separately:

-If $v_{(\mu,\alpha)}(1) = 0$, then there are again no ones whatsoever, so we may subtract 1 from each part. Here, since $r = 2$, the valuation $v_{(\mu,\alpha)}(2)$ could not have been $2k-1$, but it could have been as much as $2k-2$. Hence, we have an overpartition pair counted by $a_{2k,2k-1}(m,n-m)$.

-If $v_{(\mu,\alpha)}(1) = 1$, then this may be for two reasons. First, it may be that 1 occurs unattached in $\mu$, and if so, it may occur any number of times $t \geq 1$. In this case, we remove all of the ones and then subtract one from all of the remaining parts. The result is an overpartition pair counted by $a_{2k,2k-1}(m-t,n-m)$. Second, it could be that the valuation is 1 because 1 occurs in $\lambda$ (note that $\overline{1}$ cannot occur because it would have to be accompanied!). In this case, we remove this one as well as any of the $t \geq 0$ ones that may occur in $\mu$, and then subtract one from each of the remaining parts. The result is an overpartition pair counted by $a_{2k,2k-1}(m-t-1,n-m)$. These operations are again reversible, establishing condition (2.9).

Finally, we tackle condition (2.10). We observe that $a_{2k,r}(m,n) - a_{2k,r-2}(m,n)$ counts those overpartition pairs $(\mu,\lambda)$ that are counted by $a_{2k,r}(m,n)$ and have either $r-1$ ones or $r-2$ ones. The possible overlining of one of these ones leads to four cases:

-If there are $r-1$ ones and there is not an overlined one, then the valuation of 2 could be as much as $2k-r$. Removing the $r-1$ ones and the $t \geq 0$ ones occurring in $\mu$ and then subtracting one from each part leaves an overpartition pair counted by $a_{2k,2k-r+1}(m-t-r+1, n-m)$.

-If there are $r-1$ ones, one of which is overlined, then the valuation of 2 could be as much as $2k-r+2$. Removing the $r-1$ ones and the $t \geq 0$ ones occurring in $\mu$ and then subtracting one from each part leaves an overpartition pair counted by $a_{2k,2k-r+3}(m-t-r+1, n-m)$.

-If there are $r-2$ ones, all non-overlined, then the valuation of 2 could be $2k-r$. Removing the $r-2$ ones and the $t \geq 0$ ones occurring in $\mu$ and then subtracting one from each part leaves an overpartition pair counted by $a_{2k,2k-r+1}(m-t-r+2, n-m)$.

- If there are $r-2$ ones, one of which is overlined, then the valuation of 2 could be $2k-r+2$. Removing the $r-2$ ones and the $t \geq 0$ ones occurring in $\mu$ and then subtracting one from each part leaves an overpartition pair counted by $a_{2k,2k-r+3}(m-t-r+2, n-m)$.

Since all of these operations are reversible, we have established (2.10). For overpartition pairs the condition (2.11) is immediate, and so we may now deduce the equality of $b_{2k,r}(m,n)$ and $a_{2k,r}(m,n)$ for all $m, n \geq 0$. To finish the proof, we have

$$\begin{aligned}
\sum_{n\geq 0} a_{2k,2k-1}(n)q^n &= \sum_{n\geq 0} b_{2k,2k-1}(n)q^n \\
&= L_{2k,2k-1}(1) \\
&= \frac{(-q)_\infty(-q^2;q^2)_\infty(q^{2k-1};q^{2k-1})_\infty}{(q)_\infty(q^2;q^2)_\infty(-q^{2k-1};q^{2k-1})_\infty},
\end{aligned}$$

and the proof of Theorem 1.1 is complete. □

## 3. Proof of Theorem 1.2

We now turn to the proof of Theorem 1.2, beginning with the establishment of (1.9). We say that two sequences $(\alpha_n, \beta_n)$ form a Bailey pair with respect to $a$ if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}, \tag{3.12}$$

The following is known as Bailey's lemma, which shows how each Bailey pair generates new Bailey pairs.

**Lemma 3.2.** *If $(\alpha_n, \beta_n)$ form a Bailey pair with respect to $a$, then so do*

$$\alpha_n' = \frac{(b,c;q)_n(aq/bc)^n\alpha_n}{(aq/b,aq/c;q)_n} \tag{3.13}$$

*and*

$$\beta_n' = \frac{1}{(aq/b,aq/c;q)_n}\sum_{j=0}^{n}\frac{(b,c;q)_j(aq/bc;q)_{n-j}(aq/bc)^j\beta_j}{(q;q)_{n-j}}. \tag{3.14}$$

One may then indefinitely iterate Bailey's lemma to obtain a chain of Bailey pairs, as specified below.

**Theorem 3.3** (Andrews, [3]). *If $(\alpha_n, \beta_n)$ form a Bailey pair with respect to $a$, then*

$$\frac{(\frac{aq}{b_k}, \frac{aq}{c_k}; q)_m}{(aq, \frac{aq}{b_kc_k}; q)_m} \sum_{r\geq 0} \frac{(b_1, c_1, ..., b_k, c_k, q^{-m}; q)_r}{(\frac{aq}{b_1}, \frac{aq}{c_1}, ... \frac{aq}{b_k}, \frac{aq}{c_k}, aq^{m+1}; q)_r} \left(\frac{-a^k q^{k+m}}{b_1c_1...b_kc_k}\right)^r q^{r(r-1)/2}\alpha_r$$

$$= \sum_{n_k\geq n_{k-1}\geq ...\geq n_1\geq 0} \frac{(q^{-m}; q)_{n_k}(b_k, c_k; q)_{n_k}...(b_1, c_1; q)_{n_1}}{(\frac{b_kc_kq^{-m}}{a})_{n_k}(\frac{aq}{b_{k-1}}, \frac{aq}{c_{k-1}}; q)_{n_k}...(\frac{aq}{b_1}, \frac{aq}{c_1}; q)_{n_2}}$$

$$\times \frac{(\frac{aq}{b_{k-1}c_{k-1}}; q)_{n_k-n_{k-1}}...(\frac{aq}{b_1c_1}; q)_{n_2-n_1}}{(q; q)_{n_k-n_{k-1}}...(q; q)_{n_2-n_1}} \left(\frac{aq}{b_{k-1}c_{k-1}}\right)^{n_{k-1}} ... \left(\frac{aq}{b_1c_1}\right)^{n_1} q^{n_k}\beta_{n_1}.$$

Consider the Bailey pair [22, E(4)]

$$\alpha_n = \begin{cases} 0, & \text{if } n = 0 \\ (-1)^n q^{n^2-n}(1+q^{2n}), & \text{if } n \geq 1 \end{cases} \quad \text{and} \quad \beta_n = \frac{q^n}{(q^2; q^2)_n}. \tag{3.15}$$

Inserting this into the Bailey chain with $b_k = \sqrt{-1}$, $c_k = -\sqrt{-1}$, $m \to \infty$, and $b_j, c_j \to \infty$ for $j < k$, we have

$$\sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{(-1; q^2)_{n_k} q^{n_k+n_{k-1}^2+n_{k-2}^2+\cdots+n_2^2+\chi(k\neq 1)n_1^2+n_1}}{(q)_{n_k-n_{k-1}}\cdots(q)_{n_2-n_1}(q^2; q^2)_{n_1}} = \frac{(-q^2; q^2)_\infty(q^k; q^k)_\infty}{(q)_\infty^2(-q^k; q^k)_\infty}. \tag{3.16}$$

Multiplying the top and the bottom of the sum side by

$$(q^{n_1+1})_{n_2-n_1}(q^{n_2+1})_{n_3-n_2}\cdots(q^{n_{k-1}+1})_{n_k-n_{k-1}}$$

and simplifying using (1.10) gives (1.9).

Having proven this family of identities, we are left with the task of interpreting the sum side as the generating function for $f_k(n)$ detailed in the introduction. For $k = 1$, we have the identity

$$\sum_{n_1\geq 0} \frac{(-1; q^2)_{n_1} q^{2n_1}}{(q^2; q^2)_{n_1}} = \frac{(-q^2; q^2)_\infty}{(q^2; q^2)_\infty}. \tag{3.17}$$

The term $(-1; q^2)_{n_1}$ generates two copies of a partition $\nu$ into distinct non-negative parts less than $n_1$, while the term $1/(q^2; q^2)_{n_1}$ generates two copies of an ordinary partition $\pi$ into $n_1$ non-negative parts. Using the Joichi-Stanton algorithm [12, Proposition 2.1], $\nu$ and $\pi$ may be assembled into an overpartition into exactly $n_1$ non-negative parts. We have then two copies of this overpartition, one for each row of (1.7). The term $q^{2n_1}$ adds two to each part in the top row and the result is a generalized Frobenius partition counted by the function $f_1(m)$, where $m$ is $2n_1$ plus twice the number of parts in $\nu$ plus twice the number of parts in $\pi$.

Now, on the sum side of (1.9) for $k \geq 2$, the term $(-1; q^2)_{n_k}$ generates two copies of a partition $\nu$ into distinct non-negative parts less than $n_k$. From [2] we know that

$$\sum_{n_k\geq\cdots\geq n_1} q^{n_{k-1}^2+\cdots+n_2^2+n_1^2+n_1} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}$$

is the generating function for partitions $\pi_1$ into at most $n_k$ parts with $k-2$ successive Durfee squares of sizes $n_{k-1}, n_{k-2}, \ldots, n_2$, and an $n_1 \times n_1 + 1$ Durfee rectangle to the right of the $k-2$nd square (if $n_1 = 0$ the rectangle is empty), where there are exactly $n_1$ parts to the right of the $k-2$nd square that are at least $n_1 + 1$. The term $q^{n_k}$ adds 1 to each part of $\pi_1$.

Next, the term

$$\frac{1}{(q^{n_1+1})_{n_k-n_1}}$$

is the generating function for partitions $\pi_2$ with parts greater than $n_1$ and at most $n_k$. Then, the term $1/(q^2;q^2)_{n_1}$ contributes two copies of a partition $\pi_3$ into parts at most $n_1$. One of these copies we place in columns to the right of the $n_1 \times n_1 + 1$ Durfee rectangle in $\pi_1$ and the other we put with $\pi_2$ to make $\pi_2$ a partition into parts at most $n_k$. Finally, we use the Joichi-Stanton algorithm [12, Proposition 2.1] to combine $\pi_1$ and one copy of $\nu$ into the top row and $\pi_2$ and the other copy of $\nu$ into the bottom row of a generalized Frobenius partition (1.7) counted by $f_k(m)$, where $m$ is the sum of all the parts in $\pi_1, \pi_2, \nu$, and $\nu$. $\square$

## 4. Proof of Theorem 1.3

As remarked in the introduction, $a_{2,2}(n)$ is just the number of overpartitions of $n$ into odd parts. Hence we have the generating function

$$\sum_{n\geq 0} a_{2,2}(n)q^n = \sum_{n\geq 0} \frac{(-1;q^2)_n q^n}{(q^2;q^2)_n} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}. \tag{4.1}$$

Using the elementary theory of overpartitions [12], we find that

$$\sum_{n\geq 1}(a^+_{2,2}(n) - a^-_{2,2}(n))q^n = 2\sum_{n\geq 1} \frac{(q^2;q^2)_{n-1}q^n}{(-q^2;q^2)_n}. \tag{4.2}$$

Using the theory of Bailey pairs, we shall prove an identity for the final sum above. In [19], it was shown that if we change $q$ to $q^2$ in all of the Bailey statements, then if $\alpha_0 = \beta_0 = 0$ and if, for $n \geq 1$,

$$\alpha_n = \frac{(-1)^n q^{n^2-n}(1-q^{4n+2})}{(1-q^2)} \sum_{r=1}^{n} \sum_{j=-r+1}^{r} (-1)^{r+j} q^{r^2-j^2} \tag{4.3}$$

and

$$\beta_n = \frac{-1}{(-q)_{2n}(1-q^{2n})}, \tag{4.4}$$

then $(\alpha_n, \beta_n)$ is a Bailey pair with respect to $q^2$. Substituting this pair into Theorem 3.3 (remembering to change $q$ to $q^2$) with $k = 1, b_1 = q^2, c_1 = -q$, and $m \to \infty$, we have

$$\begin{aligned} 2\sum_{n\geq 1} \frac{(q^2;q^2)_{n-1}(-q)^n}{(-q^2;q^2)_n} &= -2\sum_{n\geq 1}\sum_{r=1}^{n}\sum_{j=-r+1}^{r} (-1)^{r+j} q^{n^2+r^2-j^2}(1-q^{2n+1}) \\ &= 2\sum_{n=1}^{\infty}\sum_{j=-n+1}^{n} (-1)^{n+j+1} q^{2n^2-j^2}. \end{aligned}$$

Now our work is considerably simplified by the fact that this last series is precisely the series in [19, Eq. (2.11)]. Replacing $q$ by $-q$, Theorem 1.1 of [19] implies our Theorem 1.3. $\square$

## 5. Discussion

Before concluding, we wish to make several comments. First, it may have been observed that Theorem 1.2 is valid for all natural numbers $k$ while in Theorem 1.1 we required $k$ even. Indeed, the evenness was essential given the conditions $(v) - (vii)$ in the definition of the $a_{2k,r}(n)$ and the combinatorial mappings employed in the proof of Theorem 1.1. What is true is that one can appropriately define a function $a_{k,r}(n)$ for $k$ odd and develop an argument similar to the one in Section 2 to obtain a companion to Theorem 1.2 for odd $k$. However, it turns out that the resulting theorem follows rather easily from Theorem 1.1 of [18]. It seems that only when $(k-1)/2$ is odd do we obtain something substantially new.

Second, in Section 4 we focused on a $q$-series related to $a_{2,2}(n)$. It turns out that series related to $a_{2,1}(n)$, *i.e.*, series related to

$$\sum_{n\geq 0} \frac{(-1;q^2)_n q^{2n}}{(q^2;q^2)_n} = \sum_{n\geq 0} a_{2,1}(n)q^n, \tag{5.1}$$

also have some interesting number theoretic connections, primarily to divisor functions. Such series were extensively studied in [12] (with $q^2 = q$), so we have not discussed them here.

Finally, we emphasize that Andrews' work on $q$-difference equations for well-poised basic hypergeometric series is still a gold mine of combinatorial information, as is his work on Bailey chains. In [18] and [20], we discovered overpartitions occurring naturally in these settings, and now we have found overpartition pairs. A systematic study would surely uncover much more about these objects, and probably about ordinary partitions as well.

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