MORE LACUNARY PARTITION FUNCTIONS

JEREMY LOVEJOY

1. INTRODUCTION

A partition of n is a nonincreasing sequence of natural numbers whose sum is n. The rank of a partition is the largest part minus the number of parts. Inspired by Ramanujan's lost notebook, Andrews, Dyson, and Hickerson were led to study certain partitions weighted according to the rank [5]. They showed that if $R^{\pm}(n)$ is equal to the number of partitions of n into distinct parts with even (odd) rank, then

(i) $R^{+}(n) - R^{-}(n) = k$ has infinitely many solutions for any integer k.
(ii) $R^{+}(n) - R^{-}(n)$ is lacunary. That is, $\#\{k \leq n : R^{+}(k) \neq R^{-}(k)\}/n \to 0$ as $n \to \infty$.

This was rather unlike anything seen before, and there are still only a few examples of partition functions exhibiting such behavior.

Recently we focused on the second property and discussed how to construct infinite families of lacunary q-series which are not theta functions. We interpreted two such families partition theoretically in terms of the Durfee rectangle dissection of a Ferrers diagram [7]. Although this is natural in the context of generalizations of q-series identities arising from the Bailey chain [2], such geometric decompositions require some effort to compute. Therefore, we would like to present another family of lacunary partition functions of a much simpler nature - an arithmetic example in terms of parts in certain congruence classes.

Theorem 1.1. *Let $Q_{m,k}(n)$ denote the number of partitions into distinct non-negative parts which are 0 or m modulo k with the condition that the smallest part λ_1 is a multiple of k and the part $2\lambda_1 + m$ does not occur. Let $Q^{\pm}_{m,k}(n)$ denote the number of such partitions where the number of parts is even (odd). Then*

$$\#\{n \leq N : Q^{+}_{m,k}(n) \neq Q^{-}_{m,k}(n)\} \ll \frac{N}{\sqrt{\log N}}. \tag{1.1}$$

Date: January 21, 2004.

The proof of Theorem 1.1 depends on a lemma that converts a case of the Heine transformation [6, p. 241, (III.1)]

$$\sum_{n=0}^{\infty} \frac{(a,b;q)_n z^n}{(c,q;q)_n} = \frac{(az,b;q)_\infty}{(z,c;q)_\infty} \sum_{n=0}^{\infty} \frac{(c/b,z;q)_n b^n}{(az,q;q)_n} \tag{1.2}$$

into a statement about Bailey pairs. Here we have employed the standard q-series notation

$$(a_1,...,a_j;q)_n := \prod_{k=0}^{n-1}(1-a_1q^k)\cdots(1-a_jq^k). \tag{1.3}$$

We note that statements such as (1.2) are subject to covergence conditions which will not be stated explicitly.

2. Proof of the main theorem

Before proving the key lemma, we recall [3] that to say a pair of sequences (α_n,β_n) forms a Bailey pair with respect to a means (among other things) that

$$\alpha_n = \frac{(1-aq^{2n})}{(1-a)} \sum_{j=0}^{n} \frac{(a;q)_{n+j}(-1)^{n-j}q^{(n-j)(n-j-1)/2}\beta_j}{(q;q)_{n-j}}. \tag{2.1}$$

Lemma 2.2. *If (α_n,β_n) is a Bailey pair with respect to b, then*

$$\sum_{n=0}^{\infty} \frac{(1-b)z^n\alpha_n}{(1-bq^{2n})} = (b,z;q)_\infty \sum_{n,j\geq 0} \frac{b^n z^j q^{2nj}\beta_j}{(z,q;q)_n}. \tag{2.2}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(1-b)z^n\alpha_n}{(1-bq^{2n})} &= \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} \frac{(b;q)_{n+j}(-1)^{n-j}q^{(n-j)(n-j-1)/2}\beta_j}{(q;q)_{n-j}}. \\
&= \sum_{j=0}^{\infty}(-1)^j q^{j(j+1)/2}\beta_j \sum_{n=j}^{\infty} \frac{(-z)^n q^{n(n-1)/2-nj}(b;q)_{n+j}}{(q;q)_{n-j}} \\
&= \sum_{j=0}^{\infty} z^j\beta_j \sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}(b;q)_{n+2j}}{(q;q)_n} \\
&= \sum_{j=0}^{\infty} z^j\beta_j (b;q)_{2j} \sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}(bq^{2j};q)_n}{(q;q)_n} \\
&= \sum_{j=0}^{\infty} z^j\beta_j (z,b;q)_\infty \sum_{n=0}^{\infty} \frac{b^n q^{2jn}}{(z,q;q)_n},
\end{aligned}$$

where in the last step we have employed (1.2) with $z = z/a, c = 0, b = bq^{2j}$, and $a \to \infty$. □

Proof of Theorem 1.1. It is known [7] that (α_n, β_n) form a Bailey pair with respect to q, where

$$\alpha_n = \frac{(1-q^{2n+1})q^{2n^2+n}}{(1-q)} \sum_{j=-n}^{n} (-1)^j q^{-j(3j+1)/2} \quad \text{and} \quad \beta_n = 1. \tag{2.3}$$

Inserting this pair into (2.2), we obtain the transformation

$$(z,q;q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(z,q;q)_n(1-zq^{2n})} = \sum_{\substack{n\geq 0\\ |j|\leq n}} z^n(-1)^j q^{2n^2+n-j(3j-1)/2}, \tag{2.4}$$

which with $q = q^k$ and $z = zq^m$ becomes

$$\sum_{n=0}^{\infty} \frac{q^{kn}(zq^{kn+m}, q^{kn+k}; q^k)_\infty}{(1-zq^{2kn+m})} = \sum_{\substack{n\geq 0\\ |j|\leq n}} z^n(-1)^j q^{2kn^2+(k+m)n-kj(3j-1)/2}. \tag{2.5}$$

When $z = 1$ the elementary theory of partitions [1] tells us that the left hand side generates the partitions described in the theorem, while Theorem 3 from [7] provides the asymptotic estimate. □

3. Further results

A number of comments are in order regarding the equations in the previous section. First, we may set $z = -1$ in (2.5) to obtain another family like the one in Theorem 1.1.

Theorem 3.3. *Let $T^{\pm}_{m,k}(n)$ equal the number of partitions counted by $Q_{m,k}(n)$ where the number of parts divisible by k is even (odd). Then*

$$\#\{n \leq N : T^{+}_{m,k}(n) \neq T^{-}_{m,k}(n)\} \ll \frac{N}{\sqrt{\log N}}. \tag{3.1}$$

Moreover, we could consider the coefficient of z^ℓ, which is a polynomial on the right side and again a simple generating function on the left. Specifically, we have

Theorem 3.4. *Let $Q^{\pm}_{m,k,\ell}(n)$ denote the number of partitions counted by $Q^{\pm}_{m,k}(n)$ in which the number of parts not divisible by k is equal to ℓ. Then*

$$Q^{+}_{m,k,\ell}(n) - Q^{-}_{m,k,\ell}(n) = \begin{cases} (-1)^j, & \text{if } n = 2k\ell^2 + (k+m)\ell - kj(3j-1)/2,\ |j| \leq \ell \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Next it is worth noting that the condition that $2\lambda_1 + m$ does not occur in the partitions counted by $Q_{m,k}(n)$ is not superfluous. For if we set $q = q^k, z = zq^m/a, b = q^k$, and let $a \to \infty$ in (1.2), we have simply a theta-type series:

$$\sum_{n=0}^{\infty} (zq^{kn+m}, q^{kn+k}; q^k)_\infty q^{kn} = \sum_{n=0}^{\infty} (-z)^n q^{kn(n-1)/2+mn}. \tag{3.3}$$

This may also be regarded as arising from inserting into (2.2) the unit Bailey pair with respect to q (which follows from (2.1)),

$$\alpha_n = \frac{(1-q^{2n+1})(-1)^n q^{n(n-1)/2}}{(1-q)} \quad \text{and} \quad \beta_n = \begin{cases} 1, & n=0 \\ 0, & n>0 \end{cases}. \tag{3.4}$$

In fact, there are a number of different pairs that can be inserted into (2.2) which allow the sum over j on the right to be simplified by an application the $b=c$ case of (1.2),

$$\sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{3.5}$$

For example, (α_n, β_n) form a Bailey pair with respect to q [8] if

$$\alpha_n = \frac{(1-q^{2n+1})(-1)^n q^{n^2}}{(1-q)} \quad \text{and} \quad \beta_n = \frac{1}{(q^2;q^2)_n}. \tag{3.6}$$

This results in the transformation

$$\sum_{n=0}^{\infty} (-z)^n q^{n^2} = (q;q)_\infty (zq;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^n (z;q^2)_n}{(q,z;q)_n}, \tag{3.7}$$

which is (5.3) in [9].

Finally we emphasize that although it was sufficient for our purposes, Lemma 2.2 does not represent the full extent of the interaction between Heine's transformation and Bailey pairs. One may similarly prove a companion to (2.2) in which the roles of α_n and β_n are reversed, and these transformations can be extended to include the well-poised Bailey pairs [4].

References

[1] G.E. Andrews, The Theory of Partitions, Cambrige Univ. Press, Cambridge, 1998.
[2] G.E. Andrews, Partitions and Durfee dissection, *Amer. J. Math.* **101** (1979), no.3, 735-742.
[3] G.E. Andrews, q-series: Their Development and Application in Analysis, Number Theory, Combinatorics, and Computer Algebra, CBMS **66**, American Mathematical Society, Providence, 1986.
[4] G.E. Andrews and A. Berkovich, WP Bailey pairs, preprint.
[5] G.E. Andrews, F.J. Dyson, and D. Hickerson, Partitions and indefinite quadratic forms, *Invent. Math.* **91** (1988), 391-407.
[6] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge Univ. Press, Cambridge, 1990.
[7] J. Lovejoy, Lacunary partitions functions, *Math. Res. Lett.* **9** (2002), 191-198.
[8] L.J. Slater, A new proof of Rogers's transformations of infinite series, *Proc. London Math. Soc.* (2) **53** (1951), 460-475.
[9] S.O. Warnaar, Partial theta functions. I. Beyond the lost notebook, preprint.

CNRS, LaBRI, 351 cours de la Libération, 33405 Talence Cedex, FRANCE
E-mail address: lovejoymath.wisc.edu