LACUNARY PARTITION FUNCTIONS

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Abstract. We combine the theory of Bailey chains and the theory of binary quadratic forms to show that there are large classes of q-series which are not theta series but whose coefficients are almost all 0. We interpret some examples in terms of simple partition functions.

1. introduction

A partition of n into distinct parts is a decreasing sequence of natural numbers whose sum is n. The rank of such a partition is the largest part minus the number of parts. Let $E(n)$ (resp. $O(n)$) denote the number of partitions of n into an even (resp. odd) number of distinct parts. It is a celebrated fact in the theory of partitions that

$$E(n) - O(n) = \begin{cases} 1, & n = k(6k+1) \\ -1, & n = (2k+1)(3k+2) \\ 0, & \text{otherwise} \end{cases}$$

This can be proven combinatorially [10] and is equivalent to Euler's pentagonal number theorem,

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}. \tag{1.1}$$

In particular, we have $E(n) - O(n) = 0$ for almost all natural numbers. There are, in fact, numerous examples like (1.1) of simple partition functions given by theta functions or false theta functions (see [3], for instance) which are trivially almost always 0. A function on $\mathbb{N}$ which is almost always 0 is called *lacunary*, and the modern study of lacunary q-series has its origins in the work of Ramanujan on eta functions.

If one asks for lacunary q-series and partition functions which are not simply theta or false theta series, then one can identify several examples by using the theory of complex multiplication [14]. For instance, Serre [15] and Gordon and Robins [12] have continued the work of Ramanujan and identified approximately 60 pairs (r, s) for which

$$\prod_{n=1}^{\infty}(1-q^n)^r(1-q^{2n})^s$$

is lacunary. Unfortunately, the associated partition functions can be rather unnatural. (See also [16, 17].)

It turns out that lacunary partition functions can also arise from q-series with "real multiplication." In a more combinatorially relevant example, Andrews, Dyson, and Hickerson [6]

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used the arithmetic of $\mathbb{Q}(\sqrt{6})$ to show that the number of partitions into distinct parts with even rank minus the number of partitions into distinct parts with odd rank is almost always 0 (and, in fact, is equal to any given integer infinitely often). (See also [9].)

Here we shall observe how a theorem on quadratic forms can be combined with the theory of Bailey chains [4] to give a general technique for producing families of lacunary q-series which correspond to natural and reasonably simple partition functions. Theorems 1 and 2 below are just two such examples. We employ the standard q-series notation,

$$(a_1, ..., a_j; q)_n = \prod_{k=0}^{n-1} (1 - a_1 q^k)...(1 - a_j q^k).$$

Theorem 1. *For $k \geq 1$ we have*

$$\sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^{n+j} q^{kn^2+(k-1)n+j^2} (1 - q^{2n+1})$$
$$= \quad 2 \sum_{n_k \geq ... \geq n_1 \geq 0} \frac{(q;q)_{n_k} (-1)^{n_k} q^{n_k(n_k+1)/2+n_{k-1}(n_{k-1}+1)+...+n_1(n_1+1)}}{(q;q)_{n_k-n_{k-1}} ... (q;q)_{n_2-n_1} (q;q)_{n_1} (1 + q^{n_1})}.$$

Theorem 2. *For $k \geq 1$ we have*

$$\sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^{n+j} q^{((2k+3)n^2+(2k+1)n)/2-j(3j+1)/2} (1 - q^{2n+1})$$
$$= \sum_{n_k \geq ... \geq n_1 \geq 0} \frac{(q;q)_{n_k} (-1)^{n_k} q^{n_k(n_k+1)/2+n_{k-1}(n_{k-1}+1)+...+n_1(n_1+1)}}{(q;q)_{n_k-n_{k-1}} ... (q;q)_{n_2-n_1}}.$$

The lacunarity of each series in Theorem 1 and almost every series in Theorem 2 will follow immediately from

Theorem 3. *Take a set $S \subset \mathbb{Z} \times \mathbb{Z}$ and a function $\omega : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. If*

$$\sum a(n) q^n = \sum_{(x,y) \in S} \omega(x,y) q^{ax^2+bx+cy^2+dy} \tag{1.2}$$

converges in some disk around $q = 0$ and $-ac$ is not a square, then

$$|\{n \leq N : a(n) \neq 0\}| \ll \frac{N}{\sqrt{\log N}}.$$

The partition theoretic interpretation of the generating functions in Theorems 1 and 2 for arbitrary k is detailed in §3 and relies on the Durfee dissection of a partition [2]. The case $k = 1$ of Theorem 1 is a classical result which relates the number of solutions to $x^2 + y^2 = n$ to the number of ways to write the natural number n as the sum of consecutive positive integers with even largest summand minus the number of ways to write to write n as the sum of consecutive positive integers with odd largest summand.

To interpret the generating function for the $k = 1$ case of Theorem 2, let $A(n)$ denote the number of partitions of n into distinct parts which differ by at most 2 and whose smallest part is at most 2. The difference condition is the opposite of that occurring in the famous

Rogers-Ramanujan identities. These partitions are enumerated by one of Ramanujan's fifth order mock theta functions [5],

$$\psi_1(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}(-q;q)_n,$$

and the right hand side of Theorem 2 for $k = 1$ is then the generating function for the number of partitions counted by $A(n)$ with largest part even minus the number of partitions counted by $A(n)$ with largest part odd.

If one investigates the values of the relevant partition functions, it quickly becomes apparent that there is little hope of finding a combinatorial explanation for their lacunarity. This property arises instead from estimates involving norm functions in quadratic fields which imply Theorem 3. These estimates and Bailey chains are considered in the following section before the main theorems are established in §3.

2. Quadratic forms and Bailey chains

The following theorem seems to have first appeared in print in [13] and is credited there to one P. Bernays [8].

Theorem 4. *Let $Q(x, y) = Ax^2 + Bxy + Cy^2$ be a quadratic form which is not negative-definite and whose discriminant is not a square. If $P(N)$ denotes the number of positive integers less than N which are integrally representable by Q, then*

$$P(N) \ll \frac{N}{\sqrt{\log N}}.$$

Proof of Theorem 3. Multiplying by appropriate powers of q and making substitutions of the form $q \to q^\ell$ in (1.2) allows one to complete the square and apply the above theorem. □

In order to use Theorem 3 to establish the lacunarity of q-series and partition functions, we need to produce representations for q-series which resemble counting functions for binary quadratic forms. Below we give details about one general method by which this can be accomplished.

Two sequences (α_n, β_n) form a Bailey pair with respect to a if the follwing two equivalent conditions hold:

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}, \tag{2.1}$$

$$\alpha_n = \frac{(1-aq^{2n})(a;q)_n(-1)^n q^{n(n-1)/2}}{(1-a)(q;q)_n} \sum_{j=0}^{n} (q^{-n};q)_j(aq^n;q)_j q^j \beta_j. \tag{2.2}$$

The condition (2.1) is the original definition of a Bailey pair, while the equivalent form (2.2) is demonstrated in [1].

Proposition 5. *If (α_n, β_n) form a Bailey pair with respect to a, then so do*

$$\alpha_n' = \frac{(b_1, c_1;q)_n (aq/b_1c_1)^n \alpha_n}{(aq/b_1, aq/c_1;q)_n} \tag{2.3}$$

and

$$\beta_n' = \frac{1}{(aq/b_1, aq/c_1; q)_n} \sum_{j=0}^{n} \frac{(b_1, c_1; q)_j (aq/b_1c_1; q)_{n-j} (aq/b_1c_1)^j \beta_j}{(q;q)_{n-j}}. \tag{2.4}$$

Since each Bailey pair leads to a new Bailey pair, we can indefinitely iterate to obtain the so-called Bailey chain.

Theorem 6 (Andrews, [3]). *If (α_n, β_n) form a Bailey pair with respect to a, then*

$$\frac{(\frac{aq}{b_k}, \frac{aq}{c_k}; q)_n}{(aq, \frac{aq}{b_k c_k}; q)_n} \sum_{r \geq 0} \frac{(b_1, c_1, ..., b_k, c_k, q^{-n}; q)_r}{(\frac{aq}{b_1}, \frac{aq}{c_1}, ... \frac{aq}{b_k}, \frac{aq}{c_k}, aq^{n+1}; q)_r} \left(\frac{-a^k q^{k+n}}{b_1 c_1 ... b_k c_k} \right)^r q^{r(r-1)/2} \alpha_r$$

$$= \sum_{n_k \geq n_{k-1} \geq ... \geq n_1 \geq 0} \frac{(q^{-n}; q)_{n_k} (b_k, c_k; q)_{n_k} ... (b_1, c_1; q)_{n_1}}{(\frac{b_k c_k q^{-n}}{a})_{n_k} (\frac{aq}{b_{k-1}}, \frac{aq}{c_{k-1}}; q)_{n_k} ... (\frac{aq}{b_1}, \frac{aq}{c_1}; q)_{n_2}}$$

$$\times \quad \frac{(\frac{aq}{b_{k-1}c_{k-1}}; q)_{n_k - n_{k-1}} ... (\frac{aq}{b_1 c_1}; q)_{n_2 - n_1}}{(q;q)_{n_k - n_{k-1}} ... (q;q)_{n_2 - n_1}} \left(\frac{aq}{b_{k-1} c_{k-1}} \right)^{n_{k-1}} ... \left(\frac{aq}{b_1 c_1} \right)^{n_1} q^{n_k} \beta_{n_1}.$$

All general statements about q-series such as Theorem 6 are subject to convergence conditions, though we shall not state them explicitly.

Theorem 7. *If*

$$\beta_n = \frac{(d, a/bc; q)_n}{(q, dq, a/b, a/c; q)_n} \tag{2.5}$$

and

$$\alpha_n = \frac{(a/d;q)_n (1 - aq^{2n})(-d)^n q^{n(n-1)/2}}{(1-a)(dq;q)_n} \sum_{j=0}^{n} \frac{(a;q)_{j-1} (b,c,d;q)_j (1 - aq^{2j-1}) a^j}{(q, a/b, a/c, a/d; q)_j (bcd)^j}, \tag{2.6}$$

then (α_n, β_n) form a Bailey pair with respect to a.

Proof. In Watson's q-analogue of Whipple's transformation [11, p.242, Eq. (III.18)], let $a = a/q$ and then let $e = aq^n$. The theorem then follows from the second definition of a Bailey pair (2.2). □

Theorem 8. *If*

$$\beta_n = \frac{(adq/bc; q)_n}{(bq, cq, dq; q)_n} \tag{2.7}$$

and

$$\alpha_n = \frac{(a/b, a/c, a/d; q)_n (1 - aq^{2n})(-bcdq)^n q^{n(n-1)/2}}{(1-a)(bq, cq, dq; q)_n a^n} \sum_{j=0}^{n} \frac{(a;q)_{j-1} (b,c,d;q)_j (1 - aq^{2j-1}) a^j}{(q, a/b, a/c, a/d; q)_j (bcd)^j}, \tag{2.8}$$

then (α_n, β_n) form a Bailey pair with respect to a.

Proof. Apply Proposition 5 to the Bailey pair from the previous theorem with $b_1 = a/b$ and $c_1 = a/c$. Then (2.8) is straightforward. To get (2.7), notice that if $[z^n] \sum a(j) z^j := a(n)$, then we can write (2.4) as

$$\begin{aligned}\beta_n' &= \frac{1}{(bq,cq;q)_n}[z^n]\sum_j \frac{(d,a/bc;q)(bcqz/a)^j}{(q,dq;q)_j}\sum_j \frac{(bcq/a;q)_j z^j}{(q;q)_j} \\ &= [z^n]\frac{(bcqz/a;q)_\infty}{(bq,cq;q)_n(z;q)_\infty}\sum_j \frac{(d,a/bc;q)(bcqz/a)^j}{(q,dq;q)_j} \qquad \text{(by [11, p. 236 Eq. (II.3)])} \\ &= \frac{1}{(bq,cq;q)_n}[z^n]\sum_j \frac{(dqa/bc;q)_j z^j}{(dq;q)_j} \qquad \text{(by [11, p. 241 Eq. (III.3)])}\end{aligned}$$

□

Theorems 7 and 8 can now be employed to give numerous Bailey pairs which are related to quadratic forms. The case $d = 0$ has been useful in establishing relationships between Ramanujan's fifth, sixth, and seventh order mock theta functions and indefinite quadratic forms, and several examples of Bailey pairs corresponding to indefinite forms are given in [5, 7]. By letting some of the variables tend to ∞ we instead find Bailey pairs with positive definite quadratic forms. The following lemmas give two examples, the positive definite form used to prove Theorem 1 and the indefinite form which leads to Theorem 2.

Lemma 9. *If*

$$\beta_n = \frac{2}{(q;q)_n(1+q^n)}$$

and

$$\alpha_n = \frac{(1-q^{2n+1})q^{n(n-1)/2}}{(1-q)}\sum_{j=-n}^{n}(-1)^j q^{j^2},$$

then (α_n, β_n) *is a Bailey pair with respect to* q.

Proof. Let $d = -1, a = q$, and $b, c \to \infty$ in Theorem 7. □

Lemma 10. *If*

$$\beta_n = 1$$

and

$$\alpha_n = \frac{q^{2n^2+n}(1-q^{2n+1})}{(1-q)}\sum_{j=-n}^{n}(-1)^j q^{-j(3j+1)/2},$$

then (α_n, β_n) *is a Bailey pair with respect to* q.

Proof. Let $a = q$ and $d = 0$ in Theorem 8, and then let $b, c \to 0$. (See also [5, p.121, (5.11)].) □

3. Durfee dissection and proofs of the main theorems

Proofs of Theorems 1 and 2. Simply insert the Bailey pairs from Lemmas 9 and 10 into the Bailey chain (Theorem 6), let $a, b_k = q$, and let $c_k, b_{k-1}, c_{k-1}, ..., b_1, c_1, n \to \infty$. □

To establish the partition-theoretic interpretation of Theorems 1 and 2, we require the Durfee rectangle dissection of a partition [2]. In the Ferrers diagram of a partition λ, we call the largest $n \times (n+1)$ rectangle (upper left justified) the Durfee rectangle. To obtain

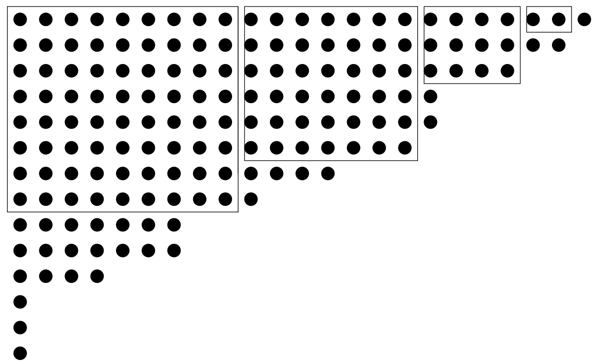

Figure 1. A Durfee rectangle dissection

the Durfee rectangle dissection of λ, notice that to the right of the Durfee rectangle will be another partition, and we call its Durfee rectangle the second Durfee rectangle of λ. Clearly we can continue until we run out of rectangles.

Definition 11. *A partition λ is called k-admissible provided*

(i) *If there are less than k Durfee rectangles, then there is nothing to the right of the final rectangle.*
(ii) *For $1 \leq j \leq k$, there is nothing directly below the lower right dot of the jth Durfee rectangle.*

The Durfee rectangle dissection of a partition which is $1-, 2-$, and $3-$admissible, but not $k-$admissible for $k \geq 4$, is shown in Figure 1. We should point out that to facilitate our discussion the definition of admissibility differs slightly from that in [2].

For any partition λ into j distinct parts, let the associated partition be that which is obtained by subtracting i from the ith part for $1 \leq i \leq j$. Let the k-largest part of λ denote the number of parts plus the number of columns to the right of the kth Durfee rectangle in the associated partition. The 0-largest part is just the largest part.

We begin with the partitions enumerated in Theorem 2. Let $A_k(n)$ denote the number of partitions λ of n into distinct parts whose associated partition is k-admissible and such that the columns to the right of the kth Durfee rectangle of the associated partition form a partition into distinct parts. Let $A_k^+(n)$ (resp. $A_k^-(n)$) be the number of partitions counted by $A_k(n)$ with even (resp. odd) k-largest part.

Theorem 12. *When $3(2k+5)$ is not a square, we have*

$$|\{n \leq N : A_k^+(n) \neq A_k^-(n)\}| \ll \frac{N}{\sqrt{\log N}}.$$

Proof. From Andrews [2], we know that the generating function for k-admissible partitions with at most N parts and nothing to the right of the final rectangle is

$$\sum_{N \geq n_k \geq n_{k-1} \geq \ldots n_1 \geq 0} \frac{(q;q)_N q^{n_k(n_k+1)+\ldots+n_1(n_1+1)}}{(q;q)_{N-n_k}(q;q)_{n_k-n_{k-1}} \ldots (q;q)_{n_2-n_1}(q;q)_{n_1}} \tag{3.1}$$

To make a partition into N distinct parts with a k-admissible associated partition and track the parity of the number of parts, we multiply by $(-1)^N q^{N(N+1)/2}$. Then we multiply by $(q;q)_{n_1}$ to ensure that the columns to the right of the kth Durfee rectangle form a partition into distinct parts and to track the parity of the number of such columns. If we sum over all N, then we find that the k-fold summation on the right side of Theorem 2 is the generating function for $A^+_{k-1}(n) - A^-_{k-1}(n)$. To complete the proof, invoke Theorem 3. □

A similar argument reveals the nature of the partitions generated by the q-series in Theorem 1. Let $R_k(n)$ denote the number of partitions λ of n into distinct parts whose associated partition is k-admissible with less than k Durfee rectangles plus twice the number of partitions λ of n into distinct parts whose associated partition is k-admissible with exactly k Durfee rectangles such that the first n_1 parts of λ are n_1 consecutive integers, where n_1 is the size of the kth Durfee rectangle in the associated partition. Let $R^+_k(n)$ (resp. $R^-_k(n)$) be the number of partitions counted by $R(n)$ with even (resp. odd) k-largest part.

Theorem 13. *We have*

$$|\{n \leq N : R^+_k(n) \neq R^-_k(n)\}| \ll \frac{N}{\sqrt{\log N}}.$$

Proof. The argument is nearly the same as in Theorem 12, a notable difference being that the positive-definite forms require no restrictions on values of k to guarantee lacunarity. After multiplying (3.1) by $(-1)^N q^{N(N+1)/2}$, we multiply by by $2/(1+q^{n_1})$, which is 1 if there aren't k Durfee rectangles. If there are k Durfee rectangles this ensures that the first n_1 parts are consecutive integers, tracks the parity of the number of columns to the right of the final rectangle in the associated partition, and counts the partition twice. □

References

[1] G.E. Andrews, Connection coefficient problems and partitions, Proc. Symp. Pure Math. **34** (1979), 1-24.
[2] G.E. Andrews, Partitions and Durfee dissection, Amer. J. Math. **101** (1979), 735-742.
[3] G.E. Andrews, Partitions: yesterday and today, New Zealand Math. Soc., Wellington, 1979.
[4] G.E. Andrews, q-series: Their development and application in analysis, number theory, combinatorics, physics, and computer algebra, CBMS 66, 1984.
[5] G.E. Andrews, The fifth and seventh order mock theta functions, Trans. Amer. Math. Soc. **293** (1986), 113-134.
[6] G.E. Andrews, F.J. Dyson, and D. Hickerson, Partitions and indefinite quadratic forms, Invent. Math. **91** (1988), 391-407.
[7] G.E. Andrews and D. Hickerson, Ramanujan's lost notebook VII: the sixth order mock theta functions, Adv. Math. **89** (1991), 60-105.
[8] P. Bernays, Über die Darstellung ... durch primitiven binären quadratischen Formen, Dissertation, Göttingen, 1912.
[9] H. Cohen, q-identities for Maass waveforms, Invent. Math. **91** (1988), 409-422.
[10] F. Franklin, Sur le développement du produit infini $(1-x)(1-x^2)(1-x^3)$..., Comptes Rendus **82** (1881), 448-450.
[11] G. Gasper and M. Rahman, Basic hypergeometric series, Cambridge Univ. Press, Cambridge, 1990.
[12] B. Gordon and S. Robins, Lacunarity of Dedekind η-products, Glasgow Math. J. **37** (1995), 1-14.
[13] R.W.K. Odoni, On norms of integers in a full module of an algebraic number field and the distribution of values of binary integral quadratic forms, Mathematika **22** (1975), 108-111.
[14] K. Ribet, Galois representations attached to eigenformas with Nebentypus, Lect. Notes Math. **601** (1977), 17-52.
[15] J.-P. Serre, Sur la lacunarité des puissances de η, Glasgow Math. J. **27** (1985), 203-221.

[16] M. Tamba, On a partition function which assumes all integral values, J. Number Th. **41** (1992), 77-86.
[17] M. Tamba, Note on a partition function, J. Number Th. **41** (1992), 280-282.

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