# PARTIAL INDEFINITE THETA IDENTITIES

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**Abstract**

Using a result of Warnaar, we prove a number of single- and multi-sum identities in the spirit of Ramanujan's partial theta identities, but with partial indefinite binary theta functions in the role of partial theta functions. We also calculate the corresponding residual identities and use a result of Ji and Zhao to recast our identities in terms of indefinite ternary theta functions.

*Keywords and phrases*: partial theta functions, indefinite theta functions, partial indefinite theta functions, Bailey pairs.

## 1. Introduction

Among the most fascinating identities in Ramanujan's lost notebook are those featuring partial theta functions, such as [5, Entry 6.6.1]

$$\sum_{n\geq 0}\frac{(q^{n+1})_n q^n}{(aq,q/a)_n} = (1-a)\sum_{n\geq 0} a^n q^{n^2+n} + \frac{1}{(aq,q/a)_\infty}\sum_{n\geq 0} a^{3n+1}q^{3n^2+2n}(1-aq^{2n+1}) \tag{1.1}$$

or [5, Entry 6.3.11]

$$\sum_{n\geq 0}\frac{(q;q^2)_n q^n}{(aq,q/a)_n} = (1-a)\sum_{n\geq 0} a^n q^{\binom{n+1}{2}} + \frac{(q;q^2)_\infty}{(aq,q/a)_\infty}\sum_{n\geq 0}(-1)^n a^{2n+1}q^{n^2+n}.$$

Here and throughout we use the usual $q$-hypergeometric notation,

$$(a_1,a_2,\ldots,a_k)_n = (a_1,a_2,\ldots,a_k;q)_n := \prod_{j=0}^{n-1}(1-a_1q^j)(1-a_2q^j)\cdots(1-a_kq^j),$$

valid for $n\in\mathbb{N}\cup\{\infty\}$.

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The hidden structure behind Ramanujan's partial theta identities was revealed by Warnaar [26], who showed that if $(\alpha_n, \beta_n)$ is a Bailey pair relative to $q$, then

$$\begin{aligned}\sum_{n\geq 0} \frac{(q)_{2n}q^n}{(a)_{n+1}(q/a)_n}\beta_n &= (1-q)\sum_{n\geq 0}\frac{(-1)^n a^n q^{-\binom{n}{2}}}{1-q^{2n+1}}\alpha_n \\ &\quad + \frac{1}{(q^2,a,q/a)_\infty}\sum_{r\geq 1}(-1)^{r+1}a^r q^{\binom{r}{2}}\sum_{n\geq 0}\frac{q^{(1-r)n}(1-q^{r(2n+1)})}{1-q^{2n+1}}\alpha_n.\end{aligned} \tag{1.2}$$

Recall that $(\alpha_n, \beta_n)$ is said to be a Bailey pair relative to $a$ if

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}}. \tag{1.3}$$

(For a history of Bailey pairs and their classical applications, see [4] or [25].) The right-hand side of (1.2) tends to simplify nicely when $\alpha_n$ contains the special factor $(1-q^{2n+1})/(1-q)$ and an appropriate quadratic power of $q$, which is indeed the case for a large number of known Bailey pairs. This leads to all of Ramanujan's partial theta identities and it also naturally embeds them in infinite families. For example, one has [26, Theorem 1.1]

$$\begin{aligned}\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n}\sum_{n\geq n_{k-1}\geq\cdots\geq n_1\geq 0}\frac{q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}}{(q)_{n-n_{k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}} &= (1-a)\sum_{n\geq 0}a^n q^{kn^2+kn} \\ &\quad + \frac{1}{(q,aq,q/a)_\infty}\sum_{i=1}^{2k}(-1)^{i+1}a^i q^{\binom{i}{2}} j(q^i,q^{2k+1})\sum_{n\geq 0}a^{(2k+1)n}q^{kn((2k+1)n+2i)},\end{aligned} \tag{1.4}$$

which reduces to (1.1) when $k=1$. Here we have employed the Jacobi theta function

$$j(x,q) := (x, q/x, q)_\infty.$$

For much more on this, see Warnaar's paper [26]. For further applications of Warnaar's ideas and applications to conjugate Bailey pairs, see [20]. For extensive background on the partial theta identities in Ramanujan's lost notebook, see [5, Chapter 6]. For applications of Ramanujan's identities to rank differences for unimodal sequences, see [14, 15].

For partial theta functions beyond the world of $q$-hypergeometric identities, we refer to recent papers of Kostov [16]–[18], including his resolution of the Hardy-Petrovitch-Hutchinson problem with Shapiro [19], and to recent work of Sokal [24] and Prellberg [23] on positivity conjectures involving partial theta functions.

In this paper we prove identities in the spirit of Ramanujan and Warnaar, but with indefinite theta functions and their partial analogues occurring in place of some of the theta and/or partial theta functions. We refer to these as *partial indefinite theta identities*. The basic idea is to use Bailey pairs with indefinite quadratic forms in (1.2), though we shall see that the simplification of the final term is considerably more involved than with classical partial theta identities. We begin by presenting three single-sum identities, which we prove in the following section.

Theorem 1.1. *We have*

$$\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n} = (1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}} q^{\frac{3}{2}rs+\frac{1}{2}r+s} + \frac{(q)_\infty}{(aq,q/a)_\infty}\sum_{r\geq 0}(-1)^r a^{2r+1}q^{3r(r+1)/2}, \tag{1.5}$$

$$\sum_{n\geq 0}\frac{(q;q^2)_n(q)_nq^n}{(aq,q/a)_n} = (1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}} q^{rs+\frac{1}{2}r+\frac{1}{2}s} + \frac{(q)_\infty}{(aq,q/a,-q)_\infty}\sum_{r\geq 0}(-1)^r a^{3r+1}q^{3r^2+2r}(1+aq^{2r+1}), \tag{1.6}$$

$$\sum_{n\geq 0}\frac{(q;q^2)_n^2q^{2n}}{(aq^2,q^2/a;q^2)_n} = (1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}} q^{rs+\frac{1}{2}r+\frac{1}{2}s} + \frac{(q;q^2)_\infty^2}{(aq^2,q^2/a;q^2)_\infty}\sum_{r\geq 0}a^{2r+1}q^{2r^2+2r}\frac{1+aq^{2r+1}}{1-aq^{2r+1}}. \tag{1.7}$$

The double sums on the right-hand sides above can also be written in terms of Lambert series, but we keep the two-variable expression in order to emphasize the connection with the indefinite theta series, defined for $ac < b^2$ by

$$f_{a,b,c}(x,y,q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}.$$

Partial and complete versions of these series feature in greater generality in our multisum identities, which we obtain using two different types of Bailey pairs. The first are special Bailey pairs from [21] and the second arise from classical iteration methods. This results in identities like the following, each of which can be reduced to (1.5) when $k = 1$.

Theorem 1.2. *For $k$ a positive integer and $0 \leq \ell < k$ we have*

$$\begin{aligned}
&\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n}\sum_{n\geq n_{2k-1}\geq\cdots\geq n_1\geq 0}\frac{(-1)^{n_k}q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})+\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i}}{(q)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}}\\
&= (1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}}q^{\frac{1}{2}(2k+1)rs+\frac{1}{2}(k-\ell)r+\frac{1}{2}(k+1+\ell)s}\\
&\quad+\frac{1}{(q,aq,q/a)_\infty}\sum_{i=1}^{k}(-1)^i a^i q^{\binom{i+1}{2}}H^1_{k,\ell}(i)\\
&\qquad\times\sum_{r\geq 0}a^{(2k+2)r}q^{(2k^2+3k+1)r^2+(2k+1)ir}(1-a^{2k+2-2i}q^{(2k+1)(2r+1)(k+1-i)}),
\end{aligned}\tag{1.8}$$

*where*

$$H^1_{k,\ell}(i) := f_{1,4k+3,1}(q^{2+k+\ell+i},q^{1+k-\ell+i},q) + q^{2+2k+i}f_{1,4k+3,1}(q^{4+3k+\ell+i},q^{3+3k-\ell+i},q). \tag{1.9}$$

THEOREM 1.3. *For $k$ a positive integer we have*

$$\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n}\sum_{n\geq n_{k-1}\geq\cdots\geq n_1\geq 0}\frac{q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}}{(q)_{n-n_{k-1}}\cdots(q)_{n_2-n_1}}$$
$$= (1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}}q^{\frac{1}{4}(k-1)r^2+\frac{1}{2}(k+2)rs+\frac{1}{4}(k-1)s^2+\frac{1}{2}kr+\frac{1}{2}(k+1)s}$$
$$+\frac{1}{(q,aq,q/a)_\infty}\Bigg(\sum_{i=1}^{2k+2}(-1)^i a^i q^{\binom{i+1}{2}}H_k^1(i)\sum_{r=0}^{\infty}a^{(2k+2)r}q^{(2k^2+3k+1)r^2+(2k+1)ir}$$
$$+\sum_{\substack{1\leq i\leq 2k+2\\ 0\leq n\leq 2k-2\\ 1\leq b\leq 2}}(-1)^{i+n}a^{i+(2k+2)n}q^{Q_1(k,i,n,b)}j(q^{3n+i+b},q^{2k-1})$$
$$\times\sum_{\substack{r\geq 0\\ m\geq 1}}a^{(2k+2)((2k-1)r+m)}q^{R_1(k,i,n,b,r,m)}\Bigg), \tag{1.10}$$

*where*

$$H_k^1(i) := f_{2k-1,2k+5,2k-1}(q^{i+2k+1},q^{i+2k},q) + q^{2+2k+i}f_{2k-1,2k+5,2k-1}(q^{i+4k+3},q^{i+4k+2},q), \tag{1.11}$$
$$Q_1(k,i,n,b) := \binom{i+1}{2}+(1+k)(2i+1)n+2(k+1)^2n^2 - (2k-1)\binom{n+1}{2}-(b+i+3n)(n+1),$$
$$R_1(k,i,n,b,r,m) := 4(k^2-1)(k+1)(2k-1)r^2+2(4k^2-1)(k+1)rm+(2k+1)(k+1)m^2 + 2(k+1)((4k^2-4)n+(2k-2)i-b)r+(2k+1)((2k+1)n+i)m.$$

It is worth taking a moment to compare the partial indefinite theta identities in equations (1.8) and (1.10) above with the partial theta identity in (1.4). Note that the first terms on the right-hand sides of (1.8) and (1.10) are partial indefinite theta functions, while the first term on the right-hand side of (1.4) is a partial theta function. Also note that the second terms on the right-hand sides of (1.8) and (1.10) involve (complete) indefinite theta functions while the the second term on the right-hand side of (1.4) involves ordinary Jacobi theta functions. The final term in (1.10) does not appear in either (1.4) or (1.8). Although it is notationally heavy due to its generality, it is merely a finite sum of products of Jacobi theta functions with partial indefinite binary theta functions.

We prove Theorem 1.2 in Section 3 along with some similar results, and in Section 4 we prove Theorem 1.3 and two related results. See Theorems 3.1–3.3 and Theorems 4.1–4.2. We explicitly state several cases when $k = 2$.

In Section 5 we use a result of Ji and Zhao [13] to recast all of the partial indefinite theta identities from Sections 1–4 in terms of indefinite ternary theta functions. In doing so, we lose the first partial indefinite theta term on the right-hand side but gain considerably in simplicity. See Propositions 5.1–5.10.

We close in Section 6 by computing the *residual identities* associated to each of the partial theta identities in the first part of the paper.

## 2. Proof of Theorem 1.1

In this section we prove the three identities in Theorem 1.1 along with some similar identities. Before getting started, we record some relations for indefinite theta series from [10, Section 6.1].

$$f_{a,b,c}(x,y,q) = f_{a,b,c}(y,x,q), \tag{2.1a}$$
$$= -\frac{q^{a+b+c}}{xy} f_{a,b,c}(q^{2a+b}/x, q^{2c+b}/y, q), \tag{2.1b}$$
$$= -y f_{a,b,c}(q^b x, q^c y, q) + j(x, q^a), \tag{2.1c}$$
$$= -x f_{a,b,c}(q^a x, q^b y, q) + j(y, q^c). \tag{2.1d}$$

We also note that for $n \in \mathbb{Z}$ and any $x$,

$$j(q^n, q) = 0, \tag{2.2}$$
$$j(xq^n, q) = (-1)^n q^{-\binom{n}{2}} x^{-n} j(x,q). \tag{2.3}$$

**2.1. Proof of** (1.5) We begin with the Bailey pair [3, Theorem 4 and Lemma 7]

$$\alpha_n = \frac{(1-q^{2n+1})}{1-q} q^{2n^2+n} \sum_{j=-n}^{n} (-1)^j q^{-j(3j+1)/2}, \tag{2.4}$$
$$\beta_n = 1. \tag{2.5}$$

Substituting this into (1.2) we obtain

$$\begin{aligned}\sum_{n\geq 0} \frac{(q)_{2n} q^n}{(aq, q/a)_n} &= (1-a) \sum_{\substack{n\geq 0\\ |j|\leq n}} (-1)^{n+j} a^n q^{3n(n+1)/2 - j(3j+1)/2} \\ &+ \frac{1}{(q, aq, q/a)_\infty} \sum_{r\geq 1} (-1)^r a^r q^{\binom{r+1}{2}} \sum_{\substack{n\geq 0\\ |j|\leq n}} \Big( (-1)^j q^{(1-r)n + 2nr + 2n^2 + n - j(3j+1)/2} \\ &\qquad - (-1)^j q^{(1-r)n - r + 2n^2 + n - j(3j+1)/2} \Big).\end{aligned}$$

In the first sum on $n$ and $j$ we let $n = (u+v)/2$ and $j = (u-v)/2$. This gives the first term on the right-hand side of (1.5). In the second term on the right-hand side we let $n = (u+v)/2$ and $j = (u-v)/2$ in the first summand and $n = (-u-v-2)/2$ and $j = (u-v)/2$ in the second summand. The sum on $r, n$, and $j$ then becomes

$$\sum_{r=1}^{\infty} (-1)^r a^r q^{\binom{r+1}{2}} \Bigg( \sum_{\substack{u,v\geq 0\\ u\equiv v \pmod 2}} - \sum_{\substack{u,v<0\\ u\equiv v \pmod 2}} \Bigg) (-1)^{\frac{u-v}{2}} q^{\frac{1}{8}u^2 + \frac{1}{8}v^2 + \frac{7}{4}uv + \frac{3}{4}u + \frac{5}{4}v + \frac{1}{2}ru + \frac{1}{2}rv}.$$

Letting $(u,v) = (2u, 2v)$ or $(2u+1, 2v+1)$, this can be written

$$\sum_{r=1}^{\infty} (-1)^r a^r q^{\binom{r+1}{2}} \left( f_{1,7,1}(q^{r+2}, q^{r+3}, q) + q^{r+4} f_{1,7,1}(q^{r+6}, q^{r+7}, q) \right).$$

To finish the proof of (1.5), we will show that

$$\begin{aligned} H(r) &:= f_{1,7,1}(q^{r+2}, q^{r+3}, q) + q^{r+4} f_{1,7,1}(q^{r+6}, q^{r+7}, q) \\ &= (-1)^{(r+1)/2} q^{-\binom{(r+3)/2}{2}} (q)_\infty^2 \chi(r \text{ is odd}). \end{aligned} \tag{2.6}$$

We begin with the periodicity

$$H(r) = q^{r+4} H(r+4), \tag{2.7}$$

which follows from

$$\begin{aligned} &H(r) - q^{r+4} H(r+4) \\ &\quad = f_{1,7,1}(q^{2+r}, q^{3+r}, q) - q^{12+2r} f_{1,7,1}(q^{11+r}, q^{10+r}, q) \\ &\quad = -q^{3+r} f_{1,7,1}(q^{9+r}, q^{4+r}, q) - q^{12+2r} f_{1,7,1}(q^{11+r}, q^{10+r}, q) && \text{by (2.1c)} \\ &\quad = q^{12+2r} f_{1,7,1}(q^{10+r}, q^{11+r}, q) - q^{12+2r} f_{1,7,1}(q^{11+r}, q^{10+r}, q) && \text{by (2.1d)} \\ &\quad = 0. \end{aligned}$$

Next we have

$$H(r) = -q^{4-2r} H(4-r), \tag{2.8}$$

which follows from

$$\begin{aligned} H(r) &= f_{1,7,1}(q^{2+r}, q^{3+r}, q) + q^{4+r} f_{1,7,1}(q^{7+r}, q^{6+r}, q) \\ &= -q^{4-2r} f_{1,7,1}(q^{6-r}, q^{7-r}, q) - q^{-r} f_{1,7,1}(q^{2-r}, q^{3-r}, q) && \text{by (2.1b)} \\ &= q^{-r} H(-r) \\ &= -q^{4-2r} H(4-r) && \text{by (2.7).} \end{aligned}$$

Now we calculate $H(i)$ for $0 \leq i \leq 4$. First, by (2.1b) we have

$$f_{1,7,1}(q^3, q^2, q) = -q^4 f_{1,7,1}(q^6, q^7, q),$$

so $H(0) = 0$. Second, $H(2) = 0$ by (2.8). Third, we have

$$\begin{aligned} qH(1) &= q f_{1,7,1}(q^3, q^4, q) + q^6 f_{1,7,1}(q^7, q^8, q) \\ &= q f_{1,7,1}(q^3, q^4, q) - f_{1,7,1}(q^2, q, q) && \text{by (2.1b)} \\ &= -(q)_\infty^2, \end{aligned}$$

from [22, Eqn. (1.13)]. Finally, (2.8) gives $H(3) = q^{-3}(q)_\infty^2$. Induction using (2.7) gives (2.6), completing the proof of (1.5). □

Before continuing, we remark that the first sum on the right-hand side of (1.5) with $a = 1$ is the generating function for 3-core partitions,

$$\frac{(q^3; q^3)_\infty^3}{(q)_\infty}.$$

This follows from

$$\begin{aligned}
\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}} (-1)^r q^{3rs/2+r/2+s} &= \sum_{r,s\geq 0} q^{6rs+s+2r} - \sum_{r,s\geq 0} q^{6rs+5s+4r+3}\\
&= \sum_{n\geq 0}\left(\frac{q^{2n}}{1-q^{6n+1}} - \frac{q^{5n+3}}{1-q^{6n+4}}\right)\\
&= \sum_{n\geq 0}\left(\frac{q^{n}}{1-q^{3n+1}} - \frac{q^{2n+1}}{1-q^{6n+4}} - \frac{q^{5n+3}}{1-q^{6n+4}}\right)\\
&= \sum_{n\geq 0}\left(\frac{q^{n}}{1-q^{3n+1}} - \frac{q^{2n+1}}{1-q^{3n+2}}\right)\\
&= \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty},
\end{aligned}$$

where we use [12, Theorem 1] for the last equality.

**2.2. Proof of** (1.6) For the proof of (1.6), we begin with the Bailey pair from (5.7) of [3],

$$\alpha_n = \frac{(1-q^{2n+1})q^{n(3n+1)/2}}{1-q}\sum_{j=-n}^{n}(-1)^j q^{-j^2}, \tag{2.9}$$

$$\beta_n = \frac{1}{(-q)_n}. \tag{2.10}$$

Plugging this Bailey pair into Warnaar's identity and simplifying the right-hand side as in the previous subsection, we obtain

$$\begin{aligned}
\sum_{n\geq 0}\frac{(q;q^2)_n(q)_n q^n}{(aq,q/a)_n} &= (1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}} (-1)^r a^{\frac{r+s}{2}} q^{rs+\frac{1}{2}r+\frac{1}{2}s}\\
&+ \frac{(q)_\infty}{(aq,q/a,-q)_\infty}\sum_{r\geq 1}(-1)^r a^r q^{\binom{r+1}{2}}\left(f_{1,5,1}(q^{r+2},q^{r+2},q) + q^{r+3}f_{1,5,1}(q^{r+5},q^{r+5},q)\right).
\end{aligned}$$

To finish the proof of (1.6), we will show that

$$\begin{aligned}
H(r) &:= f_{1,5,1}(q^{r+2},q^{r+2},q) + q^{r+3}f_{1,5,1}(q^{r+5},q^{r+5},q)\\
&= \frac{(q;q)_\infty^3}{(q^2;q^2)_\infty}\times\begin{cases} -q^{-k(3k+5)/2-1}, & \text{if } r=3k+1,\\ q^{-k(3k+7)/2-2}, & \text{if } r=3k+2,\\ 0 & \text{if } r\equiv 0 \pmod 3).\end{cases}
\end{aligned} \tag{2.11}$$

We begin with the periodicity

$$H(r) = q^{r+3}H(r+3), \tag{2.12}$$

which follows from

$$\begin{aligned}
&H(r) - q^{r+3}H(r+3) \\
&\quad = f_{1,5,1}(q^{r+2}, q^{r+2}, q) - q^{2r+9} f_{1,5,1}(q^{r+8}, q^{r+8}, q) \\
&\quad = -q^{r+2} f_{1,5,1}(q^{r+7}, q^{r+3}, q) - q^{2r+9} f_{1,5,1}(q^{r+8}, q^{r+8}, q) && \text{by (2.1c)} \\
&\quad = q^{2r+9} f_{1,5,1}(q^{r+8}, q^{r+8}, q) - q^{2r+9} f_{1,5,1}(q^{r+8}, q^{r+8}, q) && \text{by (2.1d)} \\
&\quad = 0.
\end{aligned}$$

Now we calculate $H(i)$ for $0 \leq i \leq 2$. First,

$$\begin{aligned}
H(0) &= f_{1,5,1}(q^2, q^2, q) + q^3 f_{1,5,1}(q^5, q^5, q) \\
&= -q^3 f_{1,5,1}(q^5, q^5, q) + q^3 f_{1,5,1}(q^5, q^5, q) && \text{by (2.1b)} \\
&= 0.
\end{aligned}$$

Second, using the periodicity (2.12) and equation (5.5) of [3], we have

$$H(2) = q^{-2}H(-1) = q^{-2}\left(f_{1,5,1}(q,q,q) + q^2 f_{1,5,1}(q^4, q^4, q)\right) = q^{-2}(q)_\infty^2(q;q^2)_\infty.$$

Finally, we have

$$\begin{aligned}
H(1) &= f_{1,5,1}(q^3, q^3, q) + q^4 f_{1,5,1}(q^6, q^6, q) \\
&= -q f_{1,5,1}(q^4, q^4, q) - q^{-1} f_{1,5,1}(q, q, q) && \text{by (2.1b)} \\
&= -q^{-1}H(-1) \\
&= -q^{-1}(q)_\infty^2(q;q^2)_\infty.
\end{aligned}$$

Induction using (2.12) gives (2.11), completing the proof of (1.6). □

**2.3. Proof of** (1.7) After replacing $q$ by $q^2$ in the definition of a Bailey pair, from the case $(a,b,c) = (q^2, -1, q)$ of [6, Theorem 2.2] we have a Bailey pair relative to $q^2$,

$$\alpha_n = \frac{q^{2n^2}(1-q^{4n+2})}{1-q^2} \sum_{j=-n}^{n} (-1)^j q^{-j^2}, \tag{2.13}$$

$$\beta_n = \frac{(q;q^2)_n}{(q^4;q^4)_n(-q;q^2)_n}. \tag{2.14}$$

Inserting this into Warnaar's identity (remembering to replace $q$ by $q^2$) and using the usual substitutions, we obtain

$$\begin{aligned}
\sum_{n\geq 0} \frac{(q;q^2)_n^2 q^{2n}}{(aq^2, q^2/a; q^2)_n} &= (1-a) \sum_{\substack{r,s\geq 0 \\ r\equiv s \pmod 2}} (-1)^r a^{\frac{r+s}{2}} q^{rs+\frac{1}{2}r+\frac{1}{2}s} \qquad (2.15) \\
&+ \frac{1}{(q^2, aq^2, q^2/a; q^2)_\infty} \sum_{r=1}^{\infty} (-1)^r a^r q^{2\binom{r+1}{2}} \left(f_{2,6,2}(q^{3+2r}, q^{3+2r}, q) + q^{4+2r} f_{2,6,2}(q^{7+2r}, q^{7+2r}, q)\right).
\end{aligned}$$

As usual, define

$$H(r) := f_{2,6,2}(q^{3+2r}, q^{3+2r}, q) + q^{4+2r} f_{2,6,2}(q^{7+2r}, q^{7+2r}, q).$$

To simplify the sum over $r$, we first note the periodicity

$$H(r) = q^{4+2r}H(r+2) + 2(-1)^{r+1}q^{-(r+1)^2}j(q,q^2), \tag{2.16}$$

which follows routinely from the relations (2.1a)–(2.3) as follows,

$$\begin{aligned}
H(r) - q^{4+2r}H(r+2) &= f_{2,6,2}(q^{3+2r},q^{3+2r},q) - q^{12+4r}f_{2,6,2}(q^{11+2r},q^{11+2r},q)\\
&= -q^{3+2r}f_{2,6,2}(q^{9+2r},q^{5+2r},q) + j(q^{3+2r},q^2)\\
&\quad - q^{12+4r}f_{2,6,2}(q^{11+2r},q^{11+2r},q)\\
&= q^{12+4r}f_{2,6,2}(q^{11+2r},q^{11+2r},q) - q^{3+2r}j(q^{5+2r},q^2)\\
&\quad + j(q^{3+2r},q^2) - q^{12+4r}f_{2,6,2}(q^{11+2r},q^{11+2r},q)\\
&= -q^{3+2r}j(q^{5+2r},q^2) + j(q^{3+2r},q^2)\\
&= 2(-1)^{r+1}q^{-(r+1)^2}j(q,q^2).
\end{aligned}$$

Next we calculate $H(1)$ and $H(2)$. We have

$$\begin{aligned}
H(2) &= q^{-4}H(0) + 2q^{-5}j(q,q^2) && \text{by (2.16)}\\
&= q^{-4}f_{2,6,2}(q^3,q^3,q) + f_{2,6,2}(q^7,q^7,q) + 2q^{-5}j(q,q^2)\\
&= q^{-4}f_{2,6,2}(q^3,q^3,q) - q^{-4}f_{2,6,2}(q^3,q^3,q) + 2q^{-5}j(q,q^2) && \text{by (2.1b)}\\
&= 2q^{-5}j(q,q^2),
\end{aligned}$$

and

$$\begin{aligned}
H(1) &= q^{-2}H(-1) - 2q^{-2}j(q,q^2) && \text{by (2.16)}\\
&= q^{-2}f_{2,6,2}(q,q,q) + f_{2,6,2}(q^5,q^5,q) - 2q^{-2}j(q,q^2)\\
&= -q^6f_{2,6,2}(q^9,q^9,q) - f_{2,6,2}(q^5,q^5,q) - 2q^{-2}j(q,q^2) && \text{by (2.1b)}\\
&= -H(1) - 2q^{-2}j(q,q^2),
\end{aligned}$$

so that $H(1) = -q^{-2}j(q,q^2)$.

We now apply the periodicity (2.16) to deduce that for $r \geq 1$,

$$H(2r) = 2q^{-2r^2-2r-1}j(q,q^2)\sum_{j=0}^{r-1}q^{-2j^2-2j}, \tag{2.17}$$

$$H(2r-1) = -q^{-2r^2}j(q,q^2)\sum_{j=-(r-1)}^{r-1}q^{-2j^2}. \tag{2.18}$$

Thus the sum on $r$ in (2.15) is

$$
\begin{aligned}
&\sum_{r\geq 1}(-1)^r a^r q^{r(r+1)}H(r)\\
&= j(q,q^2)\sum_{r\geq 1}\left(a^{2r-1}q^{2r^2-2r}\sum_{j=-(r-1)}^{r-1}q^{-2j^2}+2a^{2r}q^{2r^2-1}\sum_{j=0}^{r-1}q^{-2j^2-2j}\right)\\
&= -j(q,q^2)\sum_{r\geq 0}a^{2r+1}q^{2r^2+2r}\\
&\quad + 2j(q,q^2)\sum_{r\geq 0}\left(a^{2r+1}q^{2r^2+2r}\sum_{j=0}^{r}q^{-2j^2}+a^{2r+2}q^{2r^2+4r+1}\sum_{j=0}^{r}q^{-2j^2-2j}\right).
\end{aligned}
$$

Letting $r = r + j$ in the final two sums and then summing over $j$ gives

$$
\begin{aligned}
\sum_{r\geq 1}(-1)^r a^r q^{r(r+1)}H(r) &= j(q,q^2)\sum_{r\geq 0}a^{2r+1}q^{2r^2+2r}\left(-1+\frac{2}{1-a^2q^{4r+2}}+\frac{2aq^{2r+1}}{1-a^2q^{4r+2}}\right)\\
&= j(q,q^2)\sum_{r\geq 0}a^{2r+1}q^{2r^2+2r}\frac{1+aq^{2r+1}}{1-aq^{2r+1}},
\end{aligned}
$$

and this finishes the proof of (1.7). □

**2.4. Further identities** There are other simple Bailey pairs in the literature like the ones considered so far, but it is not necessarily the case that these lead to identities as elegant as those in Theorem 1.1. We close this section by giving one example which illustrates some of the complications which may arise, and leave further investigations to the interested reader.

Consider the Bailey pair relative to $q$,

$$\alpha_n = \frac{q^{n^2}(1-q^{2n+1})}{1-q}\sum_{j=-n}^{n}(-1)^j q^{-j^2}, \tag{2.19}$$

$$\beta_n = \frac{(-1)^n}{(q^2;q^2)_n}, \tag{2.20}$$

which is the case $(a,b,c)\to(q,-1,0)$ of [6, Theorem 2.2]. If we insert this directly into (1.2), then the first term on the right-hand side diverges. To remedy this, we need to move along the *Bailey chain*. Recall (see [6], for example) that if $(\alpha_n,\beta_n)$ is a Bailey pair relative to $a$, then

$$\alpha_n' = a^n q^{n^2}\alpha_n, \tag{2.21}$$

$$\beta_n' = \sum_{j=0}^{n}\frac{a^j q^{j^2}}{(q)_{n-j}}\beta_j \tag{2.22}$$

is also a Bailey pair relative to $a$. Thus we obtain a new Bailey pair relative to $q$,

$$\alpha_n = \frac{q^{2n^2+n}(1-q^{2n+1})}{1-q}\sum_{j=-n}^{n}(-1)^j q^{-j^2}, \tag{2.23}$$

$$\beta_n = \sum_{j=0}^{n}\frac{(-1)^j q^{j^2+j}}{(q^2;q^2)_j(q)_{n-j}}. \tag{2.24}$$

Now we may use Warnaar's identity. Simplifying in the usual way we obtain

$$\sum_{n\geq j\geq 0}\frac{(-1)^j q^{j^2+j+n}(q)_{2n}}{(q^2;q^2)_j(q)_{n-j}(aq,q/a)_n} = (1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}} a^{\frac{r+s}{2}}(-1)^r q^{\frac{1}{8}r^2+\frac{5}{4}rs+\frac{1}{8}s^2+\frac{3}{4}r+\frac{3}{4}s}$$
$$+\frac{1}{(q,aq,q/a)_\infty}\sum_{r=1}^{\infty}(-1)^r a^r q^{\binom{r+1}{2}}\left(f_{2,6,2}(q^{r+3},q^{r+3},q)+q^{r+4}f_{2,6,2}(q^{r+7},q^{r+7},q)\right). \tag{2.25}$$

Define

$$G(r) := f_{2,6,2}(q^{r+3},q^{r+3},q)+q^{r+4}f_{2,6,2}(q^{r+7},q^{r+7},q).$$

Note that $G(2r)=H(r)$, with $H(r)$ defined in the previous subsection.

For the odd case, we will show that

$$\begin{aligned} G(2r-1) &= f_{2,6,2}(q^{2r+2},q^{2r+2},q)+q^{2r+3}f_{2,6,2}(q^{2r+6},q^{2r+6},q)\\ &= (-1)^r q^{-r(r+1)/2}(q)_\infty(q^2;q^2)_\infty. \end{aligned} \tag{2.26}$$

We begin by noting that by (2.1c) and (2.1d) we have

$$G(2r-1)-q^{2r+3}G(2r+3) = f_{2,6,2}(q^{2r+2},q^{2r+2},q)-q^{4r+10}f_{2,6,2}(q^{2r+10},q^{2r+10},q)=0.$$

We also note that

$$f_{2,6,2}(q^4,q^4,q) = -q^2 f_{2,6,2}(q^6,q^6,q) \qquad \text{by (2.1b)}$$

and

$$\begin{aligned} f_{2,6,2}(q^8,q^8,q) &= -q^{-6}f_{2,6,2}(q^2,q^2,q) && \text{by (2.1b)}\\ &= q^{-4}f_{2,6,2}(q^8,q^4,q) && \text{by (2.1d)}\\ &= -q^4 f_{2,6,2}(q^{10},q^{10},q). && \text{by (2.1c)} \end{aligned}$$

To show (2.26) then, it is enough to show that

$$G(1) = f_{2,6,2}(q^4,q^4,q)+q^5 f_{2,6,2}(q^8,q^8,q) = -q^{-1}(q)_\infty(q^2;q^2)_\infty.$$

This is equivalent to showing that

$$\sum_{\substack{n\geq 0\\ |j|\leq n}}(-1)^j q^{2n^2+n-j^2}(1-q^{2n+1}) = (q)_\infty(q^2;q^2)_\infty,$$

which is equation (3.16) in [2].

Returning to the even case, we use (2.17) and (2.18) to evaluate

$$\begin{aligned} \sum_{r\geq 1}a^{2r}q^{2r^2+r}G(2r) &= \sum_{r\geq 1}a^{2r}q^{2r^2+r}H(r)\\ &= j(q,q^2)\sum_{r\geq 1}a^{4r}q^{6r^2-1}\sum_{j=-r}^{r-1}q^{-2j^2-2j} - j(q,q^2)\sum_{r\geq 1}a^{4r-2}q^{6r^2-6r+1}\sum_{j=-(r-1)}^{r-1}q^{-2j^2}\\ &= j(q,q^2)\sum_{r\geq 0}\sum_{j=-r+1}^{r}q^{4r+4}q^{6r^2+12r+5-2j^2-2j} - j(q,q^2)\sum_{r\geq 0}\sum_{j=-r}^{r}a^{4r+2}q^{6r^2+6r+1-2j^2}. \end{aligned}$$

Now in the first sum we replace $(r,j)$ by $(\frac{u+v-1}{2},\frac{u-v-1}{2})$ and in the second sum we replace $(r,j)$ by $(\frac{u+v}{2},\frac{u-v}{2})$. We obtain

$$\begin{aligned}
j(q,q^2) & \sum_{\substack{u,v\ge 0\\ u\not\equiv v \pmod 2}} a^{2u+2v+2}q^{u^2+4uv+v^2+3u+3v+1} \\
- j(q,q^2) & \sum_{\substack{u,v\ge 0\\ u\equiv v \pmod 2}} a^{2u+2v+2}q^{u^2+4uv+v^2+3u+3v+1} \\
& = -j(q,q^2)\sum_{u,v\ge 0}(-1)^{u+v}a^{2u+2v+2}q^{u^2+4uv+v^2+3u+3v+1}.
\end{aligned}$$

Combining this with (2.26), we obtain the following double-sum partial indefinite theta identity.

THEOREM 2.1. *We have*

$$\begin{aligned}
\sum_{n\ge j\ge 0}\frac{(-1)^j q^{j^2+j+n}(q)_{2n}}{(q^2;q^2)_j(q)_{n-j}(aq,q/a)_n} &= (1-a)\sum_{\substack{r,s\ge 0\\ r\equiv s \pmod 2}} a^{\frac{r+s}{2}}(-1)^r q^{\frac18 r^2+\frac54 rs+\frac18 s^2+\frac34 r+\frac34 s} \\
&\quad + \frac{(q^2;q^2)_\infty}{(aq,q/a)_\infty}\sum_{r\ge 0}(-1)^r a^{2r+1}q^{3\binom{r+1}{2}} \qquad (2.27)\\
&\quad - \frac{a^2q(q;q^2)_\infty}{(aq,q/a)_\infty}\sum_{u,v\ge 0}(-a^2)^{u+v}q^{u^2+4uv+v^2+3u+3v}.
\end{aligned}$$

## 3. Proof of Theorem 1.2 and related results

In this section we prove Theorem 1.2 and three related results, which we state below. The proofs use four families of Bailey pairs with indefinite quadratic forms established by the second author [21].

THEOREM 3.1. *For $k$ a positive integer and $0\le \ell < k$, we have*

$$\begin{aligned}
&\sum_{n\ge 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n}\sum_{n\ge n_{2k-1}\ge\cdots\ge n_1\ge 0}\frac{(-1)^{n_k}q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i+\binom{n_1+1}{2}}(-q)_{n_k}}{(q)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q^2;q^2)_{n_1}(-q)_{n_{k+1}}} \\
&= (1-a)\sum_{\substack{r,s\ge 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}}q^{krs+\frac12(k-\ell)r+\frac12(k+\ell)r} \\
&\qquad + \frac{1}{(q,aq,q/a)_\infty}\sum_{i=1}^{k}(-1)^i a^i q^{\binom{i+1}{2}}H_{k,\ell}^2(i) \\
&\qquad\quad \times\sum_{r\ge 0}(-1)^r a^{(2k+1)r}q^{(2k^2+k)r^2+2kir}(1+a^{2k+1-2i}q^{(2k^2+k-2ki)(2r+1)}),
\end{aligned}$$

*where*

$$H_{k,\ell}^2(i) := f_{1,4k+1,1}(q^{k+1+\ell+i},q^{k+1-\ell+i},q)+q^{1+2k+i}f_{1,4k+1,1}(q^{3k+2+\ell+i},q^{3k+2-\ell+i},q). \quad (3.1)$$

Theorem 3.2. *For $k$ a positive integer and $0 \leq \ell < k$, we have*

$$
\begin{aligned}
&\sum_{n\geq 0} \frac{(q)_{2n} q^n}{(aq,q/a)_n} \sum_{n\geq n_{2k-1}\geq\cdots\geq n_1\geq 0} \frac{(-1)^{n_k} q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})+\binom{n_k+1}{2}-\sum_{i=1}^{k-1} n_i n_{i+1}-\sum_{i=1}^{\ell} n_i+\binom{n_1+1}{2}}}{(q)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q^2;q^2)_{n_1}} \\
&= (1-a) \sum_{\substack{r,s\geq 0 \\ r\equiv s \pmod 2}} (-1)^r a^{\frac{r+s}{2}} q^{\frac{1}{8}r^2+(k+\frac{1}{4})rs+\frac{1}{8}s^2+(\frac{k}{2}-\frac{1}{2}\ell+\frac{1}{4})r+(k+\frac{1}{2}\ell+\frac{1}{4})s} \\
&\quad + \frac{1}{(q,aq,q/a)_\infty} \sum_{i=1}^{2k+2} (-1)^i a^i q^{\binom{i+1}{2}} H_{k,\ell}^3(i) \sum_{r=0}^{\infty} a^{(2k+2)r} q^{(2k+1)(k+1)r^2+(2k+1)ir} \\
&\quad + \frac{1}{(-q,aq,q/a)_\infty} \sum_{\substack{i=1 \\ i\not\equiv k+\ell \pmod 2}}^{2k+2} (-1)^{\frac{k+3i+\ell-1}{2}} a^i q^{\binom{i+1}{2}-\left(\frac{k+i+\ell+1}{2}\right)^2} \\
&\quad \times \sum_{r\geq 0, m\geq 1} (-1)^{r(k+1)} a^{(2k+2)(r+m)} q^{(k+1)^2r^2+2(k+1)(2k+1)rm+(k+1)(2k+1)m^2} \\
&\quad \times q^{i(2k+1)m+(k+1)(i-k-\ell)r} \left(1+(-1)^\ell q^{\ell((2r+1)(k+1)+i)}\right),
\end{aligned}
$$

*where*

$$
H_{k,\ell}^3(i) := f_{2,4k+2,2}(q^{2+k+\ell+r}, q^{2+k-\ell+r}, q) + q^{2+2k+r} f_{2,4k+2,2}(q^{4+3k+\ell+r}, q^{4+3k-\ell+r}, q). \tag{3.2}
$$

Theorem 3.3. *For $k$ a positive integer and $0 \leq \ell < k$, we have*

$$
\begin{aligned}
&\sum_{n\geq 0} \frac{(q)_{2n} q^n}{(aq,q/a)_n} \sum_{n\geq n_{2k}\geq n_{2k-1}\geq\cdots\geq n_1\geq 0} \frac{(-1)^{n_k} q^{\sum_{i=1}^{k}(n_{k+i}^2+n_{k+i})-\sum_{i=1}^{k-1} n_i n_{i+1}-\sum_{i=1}^{\ell} n_i}(-q)_{n_k}}{(q)_{n-n_{2k}}(q)_{n_{2k}-n_{2k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}(-q)_{n_{k+1}}} \\
&= (1-a) \sum_{\substack{r,s\geq 0 \\ r\equiv s \pmod 2}} (-1)^r a^{\frac{r+s}{2}} q^{\frac{1}{8}r^2+(k+\frac{3}{4})rs+\frac{1}{8}s^2+(k-\frac{1}{2}\ell-\frac{1}{4})r+(k+\frac{1}{2}\ell+\frac{1}{4})s} \\
&\quad + \frac{1}{(q,aq,q/a)_\infty} \sum_{i=1}^{2k+3} (-1)^i a^i q^{\binom{i+1}{2}} H_{k,\ell}^4(i) \sum_{r=0}^{\infty} (-1)^r a^{(2k+3)r} q^{(2k^2+5k+3)r^2+2(k+1)ir} \\
&\quad + \frac{1}{(-q,aq,q/a)_\infty} \sum_{i=1}^{2k+3} (-1)^{\frac{k+3i-\ell-1}{2}} a^i q^{\binom{i+1}{2}-\left(\frac{k+i-\ell+1}{2}\right)^2} \\
&\quad \times \sum_{\substack{r\geq 0, m\geq 1 \\ r\not\equiv k+i-\ell \pmod 2}} (-1)^{(2k+1)r/2+m} a^{(2k+3)(r+m)} q^{(k^2+3k+9/4)r^2+(2k^2+5k+3)m^2} \\
&\quad \times q^{(4k^2+10k+6)rm+(2k+3)(\ell+i-k)r/2+2(k+1)im} \\
&\quad + \frac{1}{(-q,aq,q/a)_\infty} \sum_{i=1}^{2k+3} (-1)^{\frac{k+3i+\ell}{2}} a^i q^{\binom{i+1}{2}-\left(\frac{k+i+\ell+2}{2}\right)^2} \\
&\quad \times \sum_{\substack{r\geq 0, m\geq 1 \\ r\equiv k+i+\ell \pmod 2}} (-1)^{(2k+1)r/2+m} a^{(2k+3)(r+m)} q^{(k^2+3k+9/4)r^2+(2k^2+5k+3)m^2} \\
&\quad \times q^{(4k^2+10k+6)rm-(2k+3)(\ell-i+k+1)r/2+2(k+1)im},
\end{aligned}
$$

*where*

$$H^4_{k,\ell}(i) := f_{2,4k+4,2}(q^{k-\ell+r+2}, q^{k+\ell+r+3}, q) + q^{2k+3+r} f_{2,4k+4,2}(q^{3k-\ell+r+5}, q^{3k+\ell+r+6}, q). \quad (3.3)$$

**3.1. Proof of Theorem 1.2** Let $0 \le \ell < k$. From [21, Theorem 1.1, $K = k$] we have that $\alpha_n^{(k,\ell)}, \beta_n^{(k,\ell)}$ is Bailey pair relative to $q$, where

$$\alpha_n^{(k,\ell)} := \frac{q^{(k+1)n^2+kn}(1-q^{2n+1})}{1-q} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$

and

$$\beta_n^{(k,\ell)} := \sum_{n \ge n_{2k-1} \ge \cdots \ge n_1 \ge 0} \frac{q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})+\binom{n_k+1}{2}-\sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i}(-1)^{n_k}}{(q)_{n-n_{2k-1}} \cdots (q)_{n_2-n_1}(q)_{n_1}}.$$

Note that when $k = 1$ and $\ell = 0$, the sum on $n_1$ in $\beta_n^{(1,0)}$ is identically 1 by the $q$-binomial theorem,

$$\sum_{k=0}^{n} \frac{z^k q^{\binom{k+1}{2}}}{(q)_k (q)_{n-k}} = \frac{(-zq)_n}{(q)_n},$$

and we have the Bailey pair used in Section 2.1.

We use the Bailey pairs $\alpha_n^{(k,\ell)}$ and $\beta_n^{(k,\ell)}$ in Warnaar's identity (1.2). The left-hand side and the first sum on the right-hand side of (1.8) are easily obtained. For the second sum on the right-hand side, we have

$$\frac{1}{(q, aq, q/a)_\infty} \sum_{r \ge 1} (-1)^{r+1} a^r q^{\binom{r}{2}} \sum_{\substack{n \ge 0 \\ |j| \le n}} (-1)^j q^{(1-r)n}(1-q^{r(2n+1)}) q^{(k+1)n^2+kn-((2k+1)j^2+(2\ell+1)j)/2}.$$

Carrying out the usual substitutions for $n$ and $j$ in terms of $u$ and $v$ (i.e. $n = (u+v)/2$, $j = (u-v)/2$, and so on), the sum becomes

$$\sum_{r \ge 1} (-1)^r a^r q^{\binom{r+1}{2}} \times \left( \sum_{\substack{u,v \ge 0 \\ u \equiv v \pmod 2}} - \sum_{\substack{u,v < 0 \\ u \equiv v \pmod 2}} \right) (-1)^{\frac{u-v}{2}} q^{\frac{1}{8}u^2+\frac{1}{8}v^2+\frac{4k+3}{4}uv+u(\frac{1}{2}r+\frac{3}{4}+\frac{1}{2}k+\frac{1}{2}\ell)+v(\frac{1}{2}r+\frac{1}{4}+\frac{1}{2}k-\frac{1}{2}\ell)},$$

which can be written in terms of the indefinite theta functions $f_{a,b,c}(x, y, q)$ (by replacing $(u, v)$ by $(2u, 2v)$ and then $(2u+1, 2v+1)$) as

$$\sum_{r \ge 1} (-1)^r a^r q^{\binom{r+1}{2}} H^1_{k,\ell}(r), \quad (3.4)$$

where $H^1_{k,\ell}(r)$ is defined by (1.9).

While this is a nice, clean expression, the infinite sum of indefinite theta functions is rather undesirable. To eliminate this, we take advantage of the periodicity in these functions using (2.1a)–(2.1d). We first note that

$$
\begin{aligned}
&H^1_{k,\ell}(r) - q^{2+2k+r} H^1_{k,\ell}(2k+2+r) \\
&= f_{1,4k+3,1}(q^{1+k-\ell+r}, q^{2+k+\ell+r}, q) - q^{6k+6+2r} f_{1,4k+3,1}(q^{5k+6+\ell+r}, q^{5k+5-\ell+r}, q) \\
&= -q^{2+k+\ell+r} f_{1,4k+3,1}(q^{4+5k-\ell+r}, q^{3+k+\ell+r}, q) - q^{6k+6+2r} f_{1,4k+3,1}(q^{5k+6+\ell+r}, q^{5k+5-\ell+r}, q) \\
&= q^{5k+6+2r} f_{1,4k+3,1}(q^{5+5k-\ell+r}, q^{6+5k+\ell+r}, q) - q^{6k+6+2r} f_{1,4k+3,1}(q^{5k+6+\ell+r}, q^{5k+5-\ell+r}, q) \\
&= 0,
\end{aligned}
$$

which implies that $H^1_{k,\ell}((2k+2)r+i) = q^{-(2k+2)\binom{r+1}{2}-ri} H^1_{k,\ell}(i)$ for $1 \le i \le 2k+2$. Thus, upon replacing $r$ by $(2k+2)r+i$ in (3.4), the sum becomes

$$
\sum_{i=1}^{2k+2} (-1)^i a^i q^{\binom{i+1}{2}} H^1_{k,\ell}(i) \sum_{r=0}^{\infty} a^{(2k+2)r} q^{(2k+1)r((k+1)r+i)}. \tag{3.5}
$$

Moreover, since

$$
\begin{aligned}
f_{1,4k+3,1}(q^{3k+3-\ell}, q^{3k+4+\ell}, q) &= -q^{-2k-2} f_{1,4k+3,1}(q^{k+2+\ell}, q^{k+1-\ell}, q) \\
&= q^{-k-1-\ell} f_{1,4k+3,1}(q^{5k+5+\ell}, q^{k+2-\ell}, q) \\
&= -q^{4k+4} f_{1,4k+3,1}(q^{5k+6+\ell}, q^{5k+5-\ell}, q),
\end{aligned}
$$

we find that $H^1_{k,\ell}(2k+2) = 0$. We also have $H^1_{k,\ell}(i) + q^{2k+2-2i} H^1_{k,\ell}(2k+2-i) = 0$ for $1 \le i \le k+1$ from the fact that

$$
f_{1,4k+3,1}(q^{2+k+\ell+i}, q^{1+k-\ell+i}, q) = -q^{2k+2-2i} f_{1,4k+3,1}(q^{3k+3-\ell-i}, q^{3k+4+\ell-i}, q)
$$

and

$$
\begin{aligned}
f_{1,4k+3,1}(q^{4+3k+\ell+i}, q^{3+3k-\ell+i}, q) &= -q^{-2k-2-2i} f_{1,4k+3,1}(q^{k+1-\ell-i}, q^{k+2+\ell-i}, q) \\
&= q^{-k+\ell-3i} f_{1,4k+3,1}(q^{5k+4-\ell-i}, q^{k+3+\ell-i}, q) \\
&= -q^{4k+4-4i} f_{1,4k+3,1}(q^{5k+5-\ell-i}, q^{5k+6+\ell-i}, q).
\end{aligned}
$$

Note also that this implies that $H^1_{k,\ell}(k+1) = 0$. Thus, we can further simplify (3.5) to

$$
\begin{aligned}
&\sum_{i=1}^{k} (-1)^i a^i q^{\binom{i+1}{2}} H^1_{k,\ell}(i) \sum_{r=0}^{\infty} a^{(2k+2)r} q^{(2k+1)r((k+1)r+i)} \\
&+ \sum_{i=1}^{k} (-1)^{i+1} a^{2k+2-i} q^{\binom{i+1}{2}} H^1_{k,\ell}(i) \sum_{r=0}^{\infty} a^{(2k+2)r} q^{(2k+1)(r+1)((k+1)(r+1)-i)} \\
&= \sum_{i=1}^{k} (-1)^i a^i q^{\binom{i+1}{2}} H^1_{k,\ell}(i) \sum_{r=0}^{\infty} a^{(2k+2)r} q^{(2k^2+3k+1)r^2+(2k+1)ir} (1 - a^{2k+2-2i} q^{(2k+1)(2r+1)(k+1-i)}).
\end{aligned}
$$

This completes the proof of Theorem 1.2. □

**3.2. Some special cases of Theorem 1.2** It turns out that $H^1_{k,\ell}(i)$ is always modular. This follows from a result of Hickerson and Mortenson [10, Theorem 1.3]. Using their formulas together with classical methods for proving modular form identities, we find the following simple infinite products when $k=2$:

$$\begin{aligned}
H^1_{2,0}(1) &= -q^{-1}\frac{(q)_\infty j(q^3,q^{10})j(q^4,q^{20})}{(q^{20};q^{20})_\infty},\\
H^1_{2,0}(2) &= -q^{-2}\frac{(q)_\infty (q^2;q^2)_\infty (q^{20};q^{20})_\infty}{j(q^4,q^{20})},\\
H^1_{2,1}(1) &= -q^{-1}\frac{(q)_\infty j(q,q^{10})j(q^8,q^{20})}{(q^{20};q^{20})_\infty},\\
H^1_{2,1}(2) &= -q^{-1}\frac{(q)_\infty j(q^4,q^{10})j(q^2,q^{20})}{(q^{20};q^{20})_\infty}.
\end{aligned}$$

This gives the following two identities.

Corollary 3.4. *We have*

$$\begin{aligned}
&\sum_{n\geq n_3\geq n_2\geq n_1\geq 0}\frac{(-1)^{n_2}(q)_{2n}q^{n+n_3^2+n_3+\binom{n_2+1}{2}-n_1n_2}}{(aq,q/a)_n(q)_{n-n_3}(q)_{n_3-n_2}(q)_{n_2-n_1}(q)_{n_1}}\\
&\quad=(1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}}q^{\frac{5}{2}rs+r+\frac{3}{2}s}\\
&\qquad+\frac{j(q^3,q^{10})j(q^4,q^{20})}{(aq,q/a)_\infty(q^{20};q^{20})_\infty}\sum_{r\geq 0}a^{4r+1}q^{15r^2+5r}(1-a^4q^{10(2r+1)})\\
&\qquad-\frac{(q^2;q^2)_\infty(q^{20};q^{20})_\infty}{(aq,q/a)_\infty j(q^4,q^{20})}\sum_{r\geq 0}a^{4r+2}q^{15r^2+10r+1}(1-a^2q^{5(2r+1)})
\end{aligned}$$

*and*

$$\begin{aligned}
&\sum_{n\geq n_3\geq n_2\geq n_1\geq 0}\frac{(-1)^{n_2}(q)_{2n}q^{n+n_3^2+n_3+\binom{n_2+1}{2}-n_1n_2-n_1}}{(aq,q/a)_n(q)_{n-n_3}(q)_{n_3-n_2}(q)_{n_2-n_1}(q)_{n_1}}\\
&\quad=(1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}}q^{\frac{5}{2}rs+r+\frac{3}{2}s}\\
&\qquad+\frac{j(q,q^{10})j(q^8,q^{20})}{(aq,q/a)_\infty(q^{20};q^{20})_\infty}\sum_{r\geq 0}a^{4r+1}q^{15r^2+5r}(1-a^4q^{10(2r+1)})\\
&\qquad-\frac{j(q^4,q^{10})j(q^2,q^{20})}{(aq,q/a)_\infty(q^{20};q^{20})_\infty}\sum_{r\geq 0}a^{4r+2}q^{15r^2+10r+2}(1-a^2q^{5(2r+1)}).
\end{aligned}$$

**3.3. Sketch of proof of Theorem 3.1** From [21, Theorem 1.4, $K=k$], we have that $\alpha_n^{(k,\ell)},\beta_n^{(k,\ell)}$ is a Bailey pair relative to $q$, where

$$\alpha_n^{(k,\ell)} := \frac{q^{((2k+1)n^2+(2k-1)n)/2}(1-q^{2n+1})}{1-q}\sum_{j=-n}^{n}(-1)^j q^{-kj^2-\ell j}$$

and

$$\beta_n^{(k,\ell)} := \sum_{n \geq n_{2k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i}) - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i + \binom{n_1+1}{2}} (-1)^{n_k} (-q)_{n_k}}{(q)_{n-n_{2k-1}} \cdots (q)_{n_2-n_1} (q^2;q^2)_{n_1} (-q)_{n_{k+1}}}.$$

Again using the $q$-binomial theorem, we have that $\beta_n^{(1,0)} = 1/(-q)_n$, so that the case $k=1$ and $\ell = 0$ corresponds to the Bailey pair in Section 2.2.

Using these Bailey pairs in (1.2), the first two terms give the first two terms in Theorem 3.1 while the final term is equal to

$$\frac{1}{(q,aq,q/a)_\infty} \sum_{r\geq 1} (-1)^{r+1} a^r q^{\binom{r}{2}} \sum_{\substack{n\geq 0 \\ |j|\leq n}} q^{(1-r)n}(1-q^{r(2n+1)}) q^{((2k+1)n^2+(2k-1)n)/2} (-1)^j q^{-kj^2-\ell j}.$$

This can be written in terms of the $f_{a,b,c}(x,y,q)$'s as

$$\frac{1}{(q,aq,q/a)_\infty} \sum_{r\geq 1} (-1)^r a^r q^{\binom{r+1}{2}} H_{k,\ell}^2(r), \tag{3.6}$$

where $H_{k,\ell}^2(r)$ is defined by (3.1). Then, as before, we easily see that

$$H_{k,\ell}^2(r) - q^{2k+1+r} H_{k,\ell}^2(2k+1+r) = 0.$$

Moreover, we find that $H_{k,\ell}^2(2k+1) = 0$. Thus, we can rewrite the sum in (3.6) as

$$\sum_{i=1}^{2k} (-1)^i a^i q^{\binom{i+1}{2}} H_{k,\ell}^2(i) \sum_{r=0}^{\infty} (-1)^r a^{(2k+1)r} q^{(2k^2+k)r^2+2kir}. \tag{3.7}$$

To simplify further, we now note that $H_{k,\ell}^2(i) + q^{2k+1-2i} H_{k,\ell}^2(2k+1-i) = 0$ for $1 \leq i \leq k$. This is because

$$f_{1,4k+1,1}(q^{1+k+\ell+i}, q^{1+k-\ell+i}, q) = -q^{2k+1-2i} f_{1,4k+1,1}(q^{3k+2-\ell-i}, q^{3k+2+\ell-i}, q)$$

and

$$\begin{aligned} f_{1,4k+1,1}(q^{2+3k+\ell+i}, q^{2+3k-\ell+i}, q) &= -q^{-2k-1-2i} f_{1,4k+1,1}(q^{k+1-\ell-i}, q^{k+1+\ell-i}, q) \\ &= q^{-k+\ell-3i} f_{1,4k+1,1}(q^{5k+2-\ell-i}, q^{k+2+\ell-i}, q) \\ &= -q^{4k+2-4i} f_{1,4k+1,1}(q^{5k+3-\ell-i}, q^{5k+3+\ell-i}, q). \end{aligned}$$

Thus, in summary, we find that (3.7) is equal to

$$\sum_{i=1}^{k} (-1)^i a^i q^{\binom{i+1}{2}} H_{k,\ell}^2(i) \sum_{r=0}^{\infty} (-1)^r a^{(2k+1)r} q^{(2k^2+k)r^2+2kir} (1 + a^{2k+1-2i} q^{(4k^2-4ki+2k)r+2k^2+k-2ki}).$$

This completes the proof of Theorem 3.1. $\square$

**3.4. Some special cases of Theorem 3.1** Once again the $H^2_{k,\ell}(i)$ are always modular and once again we have simple infinite products when $k=2$:

$$
\begin{aligned}
H^2_{2,0}(1) &= -q^{-1}\frac{(q)^2_\infty (q^{10};q^{10})^2_\infty (q^{20};q^{20})_\infty}{j(q,q^{10})j(q^8,q^{20})j(q^5,q^{20})},\\
H^2_{2,0}(2) &= -q^{-2}\frac{(q)^2_\infty (q^{10};q^{10})^2_\infty (q^{20};q^{20})_\infty}{j(q^3,q^{10})j(q^4,q^{20})j(q^5,q^{20})},\\
H^2_{2,1}(1) &= -q^{-1}\frac{(q)_\infty j(q^8,q^{20})}{(-q)_\infty},\\
H^2_{2,1}(2) &= -q^{-1}\frac{(q)_\infty j(q^4,q^{20})}{(-q)_\infty}.
\end{aligned}
$$

This gives the following two identities.

Corollary 3.5. *We have*

$$
\begin{aligned}
&\sum_{n\geq n_3\geq n_2\geq n_1\geq 0}\frac{(-1)^{n_2}(q)_{2n}q^{n+n_3^2+n_3-n_1n_2+\binom{n_1+1}{2}}(-q)_{n_2}}{(aq,q/a)_n(q)_{n-n_3}(q)_{n_3-n_2}(q)_{n_2-n_1}(q^2;q^2)_{n_1}(-q)_{n_3}}\\
&\quad= (1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}}q^{2rs+r+s}\\
&\qquad+\frac{(q)_\infty(q^{10};q^{10})^2_\infty(q^{20};q^{20})_\infty}{(aq,q/a)_\infty j(q,q^{10})j(q^8,q^{20})j(q^5,q^{20})}\sum_{r\geq 0}(-1)^r a^{5r+1}q^{10r^2+4r}(1+a^3q^{12r+6})\\
&\qquad-\frac{(q)_\infty(q^{10};q^{10})^2_\infty(q^{20};q^{20})_\infty}{(aq,q/a)_\infty j(q^3,q^{10})j(q^4,q^{20})j(q^5,q^{20})}\sum_{r\geq 0}(-1)^r a^{5r+2}q^{10r^2+8r+1}(1+aq^{4r+2})
\end{aligned}
$$

*and*

$$
\begin{aligned}
&\sum_{n\geq n_3\geq n_2\geq n_1\geq 0}\frac{(-1)^{n_2}(q)_{2n}q^{n+n_3^2+n_3-n_1n_2+\binom{n_1}{2}}(-q)_{n_2}}{(aq,q/a)_n(q)_{n-n_3}(q)_{n_3-n_2}(q)_{n_2-n_1}(q^2;q^2)_{n_1}(-q)_{n_3}}\\
&\quad= (1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}}q^{2rs+\frac{1}{2}r+\frac{3}{2}s}\\
&\qquad+\frac{j(q^8,q^{20})}{(-q,aq,q/a)_\infty}\sum_{r\geq 0}(-1)^r a^{5r+1}q^{10r^2+4r}(1+a^3q^{12r+6})\\
&\qquad-\frac{j(q^4,q^{20})}{(-q,aq,q/a)_\infty}\sum_{r\geq 0}(-1)^r a^{5r+2}q^{10r^2+8r+2}(1+aq^{4r+2}).
\end{aligned}
$$

**3.5. Sketch of proof of Theorem 3.2** From [21, Theorem 1.2, $K=k$], for $0\leq \ell<k$, $\alpha^{(k,\ell)}_n, \beta^{(k,\ell)}_n$ is Bailey pair relative to $q$, where

$$
\alpha^{(k,\ell)}_n := \frac{q^{(k+1)n^2+kn}(1-q^{2n+1})}{1-q}\sum_{j=-n}^{n}(-1)^j q^{-kj^2-\ell j}
$$

and

$$\beta_n^{(k,\ell)} := \sum_{n\geq n_{2k-1}\geq\cdots\geq n_1\geq 0} \frac{(-1)^{n_k} q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})+\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i+\binom{n_1+1}{2}}}{(q)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q^2;q^2)_{n_1}}.$$

Note that the case $k=1$ and $\ell=0$ corresponds to the Bailey pair in Section 2.4.

In Warnaar's identity the first two terms are the first two terms in Theorem 3.2, as usual. The final term is

$$\frac{1}{(q,aq,q/a)_\infty}\sum_{r\geq 1}(-1)^{r+1}a^rq^{\binom{r}{2}}\sum_{n\geq 0}q^{(1-r)n}(1-q^{r(2n+1)})q^{(k+1)n^2+kn}\sum_{j=-n}^{n}(-1)^jq^{-kj^2-\ell j}.$$

This can be written in terms of the $f_{a,b,c}(x,y,q)$ as

$$\frac{1}{(q,aq,q/a)_\infty}\sum_{r\geq 1}(-1)^ra^rq^{\binom{r+1}{2}}H_{k,\ell}^3(r), \tag{3.8}$$

where $H_{k,\ell}^3(r)$ is defined by (3.2). Then, using relations (2.1a)–(2.1d), we find that

$$H_{k,\ell}^3(r)-q^{2+2k+r}H_{k,\ell}^3(2k+2+r)=j(q^{2+k+r+\ell},q^2)+j(q^{2+k+r-\ell},q^2),$$

which implies that

$$\begin{aligned}H_{k,\ell}^3((2k+2)r+i) &= q^{-r(r+1)(k+1)-ri}H_{k,\ell}^3(i)\\ &\quad -\sum_{m=1}^{r}q^{-mr(2k+2)-mi+m(m-1)(k+1)}\\ &\qquad\times\left(j(q^{k+2+(r-m)(2k+2)+i+\ell},q^2)+j(q^{k+2+(r-m)(2k+2)+i-\ell},q^2)\right).\end{aligned}$$

Using this in (3.8), the sum becomes

$$\begin{aligned}&\sum_{i=1}^{2k+2}(-1)^ia^iq^{\binom{i+1}{2}}H_{k,\ell}^3(i)\sum_{r=0}^{\infty}a^{(2k+2)r}q^{(2k^2+3k+1)r^2+(2k+1)ir}\\ &-\sum_{i=1}^{2k+2}(-1)^ia^iq^{\binom{i+1}{2}}\sum_{r=1}^{\infty}a^{(2k+2)r}q^{2(k+1)^2r^2+(k+1)(2i+1)r}\sum_{m=1}^{r}q^{(k+1)m^2-im-(k+1)(2r+1)m}\\ &\qquad\times\left(j(q^{k+2+(r-m)(2k+2)+i+\ell},q^2)+j(q^{k+2+(r-m)(2k+2)+i-\ell},q^2)\right).\end{aligned} \tag{3.9}$$

Note that $j(q^{k+2+(r-m)(2k+2)+i\pm\ell},q^2)=0$ unless $i+\ell\not\equiv k \pmod 2$. In the second sum in (3.9), we make the change of variables $r=r+m$ and then apply (2.3) with $n=1+r(k+1)+\frac{k+i\pm\ell-1}{2}$ and $x=q$. This gives $j(q,q^2)$ times a finite sum of partial theta series,

$$\begin{aligned}&\sum_{\substack{i=1\\ i\not\equiv k+\ell \pmod 2}}^{2k+2}(-1)^{\frac{k+3i-\ell-1}{2}}a^iq^{\binom{i+1}{2}-\left(\frac{k+i-\ell+1}{2}\right)^2}\sum_{r\geq 0,m\geq 1}(-1)^{r(k+1)}a^{(2k+2)(r+m)}\\ &\qquad\times q^{(k+1)^2r^2+2(k+1)(2k+1)rm+(k+1)(2k+1)m^2+(k+1)(i-k)r+(2k+1)im+\ell(1+k)r}\\ &+\sum_{\substack{i=1\\ i\not\equiv k+\ell \pmod 2}}^{2k+2}(-1)^{\frac{k+3i+\ell-1}{2}}a^iq^{\binom{i+1}{2}-\left(\frac{k+i+\ell+1}{2}\right)^2}\sum_{r\geq 0,m\geq 1}(-1)^{r(k+1)}a^{(2k+2)(r+m)}\\ &\qquad\times q^{(k+1)^2r^2+(k+1)(2k+1)m^2+2(k+1)(2k+1)rm+(k+1)(i-k)r+(2k+1)im-\ell(1+k)r}.\end{aligned}$$

Combining this with the first part of (3.9) gives the desired result. $\square$

**3.6. Sketch of proof of Theorem 3.3** We now consider the Bailey pair relative to $q$ [21, Theorem 1.3, $K=k+1$]:

$$\alpha_n^{(k,\ell)} := \frac{q^{((2k+3)n^2+(2k+1)n)/2}(1-q^{2n+1})}{1-q}\sum_{j=-n}^{n}(-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$

and

$$\beta_n^{(k,\ell)} := \sum_{n\geq n_{2k}\geq n_{2k-1}\geq\cdots\geq n_1\geq 0}\frac{q^{\sum_{i=1}^{k}(n_{k+i}^2+n_{k+i})-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i}(-1)^{n_k}(-q)_{n_k}}{(q)_{n-n_{2k}}(q)_{n_{2k}-n_{2k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}(-q)_{n_{k+1}}}.$$

We follow the usual line of reasoning. Using these Bailey pairs in (1.2), we need to examine the final term, which is equal to

$$\frac{1}{(q,aq,q/a)_\infty}\sum_{r\geq 1}(-1)^r a^r q^{\binom{r+1}{2}}H_{k,\ell}^4(r), \tag{3.10}$$

where $H_{k,\ell}^4(r)$ is defined by (3.3). From the relations (2.1a)–(2.1d), we calculate that

$$H_{k,\ell}^4(r) - q^{2k+3+r}H_{k,\ell}^4(2k+3+r) = j(q^{k+r+2-\ell},q^2) + j(q^{k+r+3+\ell},q^2),$$

which implies that

$$\begin{aligned}H_{k,\ell}^4((2k+3)r+i) &= q^{-(2k+3)r(r+1)/2-ir}H_{k,\ell}^4(i)\\ &\quad -\sum_{m=1}^{r}q^{-mr(2k+3)-mi+m(m-1)(2k+3)/2}\\ &\qquad\times\left(j(q^{k+2+i-\ell+(r-m)(2k+3)},q^2)+j(q^{k+3+i+\ell+(r-m)(2k+3)},q^2)\right).\end{aligned}$$

Using this in (3.10), the sum is equal to

$$\begin{aligned}&\sum_{i=1}^{2k+3}(-1)^i a^i q^{\binom{i+1}{2}}H_{k,\ell}^4(i)\sum_{r=0}^{\infty}(-1)^r a^{(2k+3)r}q^{(2k^2+5k+3)r^2+2(k+1)ir}\\ &-\sum_{i=1}^{2k+3}(-1)^i a^i q^{\binom{i+1}{2}}\sum_{r=1}^{\infty}(-1)^r a^{(2k+3)r}q^{(2k^2+6k+3)r^2+(2ki+3i+k)r+3r(r+1)/2}\\ &\qquad\times\sum_{m=1}^{r}q^{km^2+3m(m-1)/2-(k+i+3r+2kr)m}\\ &\qquad\qquad\times\left(j(q^{k+2+i-\ell+(r-m)(2k+3)},q^2)+j(q^{k+3+i+\ell+(r-m)(2k+3)},q^2)\right).\end{aligned}$$

By shifting the summation ($r$ by $r+m$) and using the periodicity of $j$ function, we obtain the desired identity. $\square$

## 4. Proof of Theorem 1.3 and related results

In this section we prove Theorem 1.3 and two related results which we state below. The proofs use Bailey pairs obtained by iterating the Bailey pairs in Sections 2.1 - 2.3 along the Bailey chain. The two results below generalize identities (1.6) and (1.7).

Theorem 4.1. *For $k$ a positive integer we have*

$$
\begin{aligned}
&\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n}\sum_{n\geq n_{k-1}\geq\cdots\geq n_1\geq 0}\frac{q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}}{(q)_{n-n_{k-1}}\cdots(q)_{n_2-n_1}(-q)_{n_1}}\\
&= (1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}}q^{\frac14(k-1)r^2+\frac12(k+1)rs+\frac14(k-1)s^2+\frac12 kr+\frac12 ks}\\
&+\frac{1}{(q,aq,q/a)_\infty}\left(\sum_{i=1}^{2k+1}(-1)^i a^i q^{\binom{i+1}{2}}H_k^2(i)\sum_{r=0}^{\infty}(-1)^r a^{(2k+1)r}q^{(2k^2+k)r^2+2kir}\right.\\
&\qquad +2\sum_{\substack{1\leq i\leq 2k+1\\ 0\leq n\leq 2k-2}}(-1)^i a^{i+(2k+1)n}q^{Q_2(k,i,n)}j(q^{2n+i+1},q^{2k-1})\\
&\qquad\qquad\left.\times\sum_{\substack{r\geq 0\\ m\geq 1}}(-1)^m a^{(2k+1)((2k-1)r+m)}q^{R_2(k,i,n,r,m)}\right),
\end{aligned}
\tag{4.1}
$$

*where*

$$
\begin{aligned}
H_k^2(i) &:= f_{2k-1,2k+3,2k-1}(q^{i+2k},q^{i+2k},q)\\
&\qquad + q^{1+2k+i}f_{2k-1,2k+3,2k-1}(q^{i+4k+1},q^{i+4k+1},q),
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
Q_2(k,i,n) &:= \binom{i+1}{2}+(2k^2+k-1)n^2+(2ik-2)n-i-1,\\
R_2(k,i,n,r,m) &:= (4k^2-1)(2k+1)(k-1)r^2+(8k^3-2k)mr+(2k^2+k)m^2\\
&\qquad +((4k^2-2k-2)n+(2k-2)i-1)r+2k(2kn+n+i)m.
\end{aligned}
$$

Theorem 4.2. *For $k$ a positive integer we have*

$$
\begin{aligned}
&\sum_{n\geq 0}\frac{(q^2;q^2)_{2n}q^{2n}}{(aq^2,q^2/a;q^2)_n}\sum_{n\geq n_{k-1}\geq\cdots\geq n_1\geq 0}\frac{q^{2n_{k-1}^2+2n_{k-1}+\cdots+2n_1^2+2n_1}(q;q^2)_{n_1}}{(q^2;q^2)_{n-n_{k-1}}\cdots(q^2;q^2)_{n_2-n_1}(q^4;q^4)_{n_1}(-q;q^2)_{n_1}}\\
&= (1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}}q^{\frac12(k-1)r^2+krs+\frac12(k-1)s^2+\frac12(2k-1)r+\frac12(2k-1)s}\\
&+\frac{1}{(q^2,aq^2,q^2/a;q^2)_\infty}\left(\sum_{i=1}^{2k}(-1)^i a^i q^{i^2+i}H_k^3(i)\sum_{r\geq 0}a^{2kr}q^{(4k^2-2k)r^2+(4k-2)ir}\right.\\
&\left.+2\sum_{\substack{1\leq i\leq 2k\\ 0\leq n\leq 2k-2}}(-1)^{i+n}a^{2kn+i}q^{Q_3(k,i,n)}j(q^{2n+2i+1},q^{4k-2})\sum_{\substack{r\geq 0\\ m\geq 1}}a^{2k((2k-1)r+m)}q^{R_3(k,i,n,r,m)}\right),
\end{aligned}
$$

*where*

$$\begin{aligned}
H_k^3(i) &:= f_{4k-2,4k+2,4k-2}(q^{2i+4k-1}, q^{2i+4k-1}, q) \\
&\qquad + q^{4k+2i} f_{4k-2,4k+2,4k-2}(q^{2i+8k-1}, q^{2i+8k-1}, q), \qquad (4.3)\\
Q_3(k,i,n) &:= i^2 + i + (4k-2)in + (4k^2-2k-1)n^2 - 2n - 2i - 1, \\
R_3(k,i,n,r,m) &:= (16k^4 - 24k^3 + 8k^2)r^2 + (16k^3 - 16k^2 + 4k)rm + (4k^2-2k)m^2 \\
&\qquad + (16k^3 n - 16k^2 n + 8k^2 i - 8ki - 2k)r + (8k^2 n - 4kn + 4ki - 2i)m.
\end{aligned}$$

**4.1. Proof of Theorem 1.3** Iterating (2.21) and (2.22) beginning with (2.4) and (2.5), we obtain the Bailey pairs

$$\alpha_n^{(k)} = \frac{(1-q^{2n+1})q^{(k+1)n^2+kn}}{1-q} \sum_{|j|\le n} (-1)^j q^{-j(3j+1)/2}$$

and

$$\beta_n^{(k)} = \sum_{n\ge n_{k-1}\ge \cdots \ge n_1 \ge 0} \frac{q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}}{(q)_{n-n_{k-1}}\cdots(q)_{n_2-n_1}}.$$

Using Warnaar's identity, we have

$$\begin{aligned}
\sum_{n\ge 0} \frac{(q)_{2n} q^n}{(aq, q/a)_n} \beta_n^{(k)} &= (1-a) \sum_{\substack{n\ge 0\\ |j|\le n}} (-1)^{n+j} a^n q^{(2k+1)\binom{n+1}{2} - j(3j+1)/2} \\
&+ \frac{1}{(q,aq,q/a)_\infty} \sum_{r=1}^{\infty} (-1)^{r+1} a^r q^{\binom{r}{2}} \sum_{n\ge 0} q^{(1-r)n}(1-q^{r(2n+1)}) q^{(k+1)n^2+kn} \sum_{|j|\le n} (-1)^j q^{-j(3j+1)/2}.
\end{aligned}$$

Using the usual substitutions, the first sum on the right becomes the first sum on the right of (1.10) and the three-fold sum on the right can be written as

$$\sum_{r\ge 1} (-1)^r a^r q^{\binom{r+1}{2}} H_k^1(r), \qquad (4.4)$$

where $H_k^1(r)$ is defined by (1.11). Arguing as usual using the relations (2.1a)–(2.1d), we find that

$$H_k^1(r) - q^{2+2k+r} H_k^1(2k+2+r) = j(q^{r+2k}, q^{2k-1}) + j(q^{r+2k+1}, q^{2k-1}),$$

which implies that

$$\begin{aligned}
H_k^1((2k+2)r+i) &= q^{-(k+1)r(r+1)-ir} H_k^1(i) \\
&\quad - \sum_{m=1}^{r} q^{-(2k+2)mr+(k+1)m(m-1)-mi} \\
&\qquad \times \left( j(q^{(2k+2)(r-m)+2k+i}, q^{2k-1}) + j(q^{(2k+2)(r-m)+2k+i+1}, q^{2k-1}) \right).
\end{aligned}$$

Using this in (4.4), we find that (after shifting $r$ by $r+m$ on the right-hand side)

$$\sum_{r\geq 1}(-1)^r a^r q^{\binom{r+1}{2}} H_k^1(r) = \sum_{i=1}^{2k+2}(-1)^i a^i q^{\binom{i+1}{2}} H_k^1(i) \sum_{r=0}^{\infty} a^{(2k+2)r} q^{(2k^2+3k+1)r^2+(2k+1)ir}$$
$$+\sum_{i=1}^{2k+2}(-1)^{i+1} a^i q^{\binom{i+1}{2}}$$
$$\times \sum_{r\geq 0, m\geq 1} a^{(2k+2)(r+m)} q^{2(k+1)^2 r^2+(2+6k+4k^2)rm+(1+3k+2k^2)m^2+(2k+1)im+(1+k)(2i+1)r}$$
$$\times \left(j(q^{(2k+2)r+2k+i}, q^{2k-1}) + j(q^{(2k+2)r+2k+i+1}, q^{2k-1})\right).$$

To use the periodicity of $j$ function, we set $r=(2k-1)r+n$. Then the second summand on the right-hand side above equals

$$\sum_{\substack{1\leq i\leq 2k+2\\ 0\leq n\leq 2k-2}} (-1)^{i+1} a^{i+(2k+2)n} q^{\binom{i+1}{2}+(1+k)(2i+1)n+2(k+1)^2n^2}$$
$$\times \sum_{\substack{r\geq 0\\ m\geq 1}} a^{(2k+2)((2k-1)r+m)} q^{2(k+1)^2(2k-1)^2r^2+4(k+1)^2(2k-1)rn+(2+6k+4k^2)((2k-1)r+n)m}$$
$$\times q^{(1+3k+2k^2)m^2+(2k+1)im+(1+k)(2i+1)(2k-1)r}$$
$$\times \left(j(q^{(2k+2)((2k-1)r+n)+2k+i}, q^{2k-1}) + j(q^{(2k+2)((2k-1)r+n)+2k+i+1}, q^{2k-1})\right).$$

Note that

$$j(q^{(2k+2)((2k-1)r+n)+2k+i+b-1}, q^{2k-1})$$
$$= (-1)^{n+1} q^{-(2k-1)\binom{(2k+2)r+n+1}{2}} q^{-(b+i+3n)((2k+2)r+n+1)} j(q^{i+3n+b}, q^{2k-1}),$$

where $b=1$ or $2$. Using this, after some simplification, we obtain the desired result. □

**4.2. A special case of Theorem 1.3** When $k=2$ the $H_k^1(i)$ are modular and once again we have simple infinite products:

$$H_2^1(1) = -q^{-1}\frac{(q)_\infty (q^4;q^4)_\infty^2}{(q^2,q^2)_\infty} = (q)_\infty \sum_{n\geq 0} q^{n(n+1)},$$
$$H_2^1(2) = -q^{-2}(q)_\infty \sum_{n\geq 0} q^{n^2} = -\frac{1}{2}q^{-2}\left(\frac{(q^2;q^2)_\infty^5}{(q)_\infty (q^4;q^4)_\infty^2} + (q)_\infty\right).$$

Using

$$H_2^1(i) + q^{6-2i} H_2^1(6-i) = -q^{-i} j(q^{4-i}, q^3) + q^{5-2i} j(q^{8-i}, q^3),$$

we obtain the following identity after some simplification.

Corollary 4.3. *We have*

$$\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n}\sum_{n_1=0}^{n}\frac{q^{n_1^2+n_1}}{(q)_{n-n_1}} = (1-a)\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}(-1)^r a^{\frac{r+s}{2}}q^{\frac{1}{4}r^2+2rs+\frac{1}{4}s^2+r+\frac{3}{2}s}$$
$$+\frac{(q^4;q^4)_\infty}{(aq,q/a)_\infty(q^2;q^4)_\infty}\sum_{r\geq 0}a^{6r+1}q^{15r^2+5r}(1-a^4q^{20r+10})$$
$$-\frac{(q^2;q^2)_\infty^5}{2(q,q,aq,q/a)_\infty(q^4;q^4)_\infty^2}\sum_{r\geq 0}a^{6r+2}q^{15r^2+10r+1}(1-a^2q^{10r+5})$$
$$-\frac{1}{2(aq,q/a)_\infty}\sum_{r\geq 0}a^{6r+2}q^{15r^2+10r+1}\Big(1-2aq^{5r+2}+a^2q^{10r+5}$$
$$+2a^3q^{15r+8}-2a^4q^{20r+12}-2a^4a^{20r+13}\Big)$$
$$-\frac{1}{(aq,q/a)_\infty}\sum_{r\geq 0,m\geq 1}a^{6(r+m)+1}q^{12r^2+30mr+15m^2+5m+2r-1}$$
$$\times\Big(1+aq^{5m+2r-1}+a^2q^{10m+6r}+a^2q^{10m+8r+2}+a^3q^{15m+12r+4}$$
$$-a^4q^{20m+14r+4}+a^5q^{25m+18r+7}+a^5q^{25m+20r+10}\Big).$$

**4.3. Sketch of proof of Theorem 4.1** By considering the Bailey pair

$$\alpha_n^{(k)} = \frac{(1-q^{2n+1})q^{\binom{n}{2}+k(n^2+n)}}{1-q}\sum_{|j|\leq n}(-1)^j q^{-j^2}$$

and

$$\beta_n^{(k)} = \sum_{n\geq n_{k-1}\geq\cdots\geq n_1\geq 0}\frac{q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}}{(q)_{n-n_{k-1}}\cdots(q)_{n_2-n_1}(-q)_{n_1}},$$

we find

$$\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n}\beta_n^{(k)} = (1-a)\sum_{\substack{n\geq 0\\ |j|\leq n}}(-1)^{n+j}a^n q^{k(n^2+n)-j^2}$$
$$+\frac{1}{(q,aq,q/a)_\infty}\sum_{r=1}^{\infty}(-1)^{r+1}a^r q^{\binom{r}{2}}\sum_{n\geq 0}q^{(1-r)n}(1-q^{r(2n+1)})q^{\binom{n}{2}+k(n^2+n)}\sum_{|j|\leq n}(-1)^j q^{-j^2}.$$

Using the usual substitutions, the first sum on the right becomes the first sum on the right of (4.1) and the three-fold sum on the right can be written as

$$\sum_{r\geq 1}(-1)^r a^r q^{\binom{r+1}{2}}H_k^2(r), \tag{4.5}$$

where $H_k^2(r)$ is defined in (4.2).

Then, we calculate that

$$H_k^2(r)-q^{1+2k+r}H_k^2(2k+1+r)=2j(q^{r+2k},q^{2k-1}),$$

and hence we find that

$$\begin{aligned}H_k^2((2k+1)r+i)&=q^{-(2k+1)r(r+1)/2-ir}H_k^2(i)\\&\quad-2\sum_{m=1}^{r}q^{-(2k+1)mr+(2k+1)m(m-1)/2-mi}j(q^{(2k+1)(r-m)+2k+i},q^{2k-1}).\end{aligned}$$

Using this in (4.5), we have that

$$\begin{aligned}\sum_{r\ge1}(-1)^ra^rq^{\binom{r+1}{2}}H_k^2(r)&=\sum_{i=1}^{2k+1}(-1)^ia^iq^{\binom{i+1}{2}}H_k^2(i)\sum_{r=0}^{\infty}(-1)^ra^{(2k+1)r}q^{(2k^2+k)r^2+2ikr}\\&\quad-2\sum_{i=1}^{2k+1}(-1)^ia^iq^{\binom{i+1}{2}}\sum_{r=0}^{\infty}(-1)^ra^{(2k+1)r}q^{(2k^2+2k)r^2+r(r+1)/2}\\&\qquad\times\sum_{m=1}^{r}q^{m(m-1)/2+km^2-(2k+1)(m-i)r-im+k(r-m)}\\&\qquad\quad\times j(q^{(2k+1)(r-m)+2k+i},q^{2k-1}).\end{aligned}$$

By shifting the variable $r$ by $r+m$, the second summand on the right-hand side equals

$$\begin{aligned}&-2\sum_{i=1}^{2k+1}(-1)^ia^iq^{\binom{i+1}{2}}\sum_{r\ge0,m\ge1}(-1)^{r+m}a^{(2k+1)(r+m)}\\&\qquad\times q^{r(r+1)/2+(2k^2+2k)r^2+(4k^2+2k)mr+(2k^2+k)m^2+(2i+1)kr+2ikm+ir}j(q^{(2k+1)r+2k+i},q^{2k-1}).\end{aligned}$$

To use the periodicity of $j$ function, we shift $r$ by $(2k-1)r+n$. Then, from (2.3), we observe that

$$\begin{aligned}j(q^{(2k+1)((2k-1)r+n)+2k+i},q^{2k-1})&=j(q^{(2k-1)((2k+1)r+n+1)+2n+i+1},q^{2k-1})\\&=(-1)^{n+r+1}q^{-(2k-1)\binom{(2k+1)r+n+1}{2}}\\&\qquad\times q^{-(2n+i+1)((2k+1)r+n+1)}j(q^{2n+i+1},q^{2k-1}),\end{aligned}$$

which implies the desired identity after some simplification.

**4.4. Sketch of proof of Theorem 4.2** By iterating Bailey pair (2.13) and (2.14), we obtain a Bailey pair relative to $q^2$,

$$\alpha_n^{(k)}=\frac{q^{2kn^2+2(k-1)n}(1-q^{4n+2})}{1-q^2}\sum_{j=-n}^{n}(-1)^jq^{-j^2},\tag{4.6}$$

$$\beta_n^{(k)}=\sum_{n\ge n_{k-1}\ge\cdots\ge n_1\ge0}\frac{q^{2n_{k-1}^2+2n_{k-1}+\cdots+2n_1^2+2n_1}(q;q^2)_{n_1}}{(q^2;q^2)_{n-n_{k-1}}\cdots(q^2;q^2)_{n_2-n_1}(q^4;q^4)_{n_1}(-q;q^2)_{n_1}}.\tag{4.7}$$

Plugging these to Warnaar's identity and proceeding as before, the final term becomes

$$\frac{1}{(q^2, aq^2, q^2/a)_\infty} \sum_{r\geq 1} (-1)^r a^r q^{r(r+1)} H_k^3(r), \tag{4.8}$$

where $H_k^3$ is defined in (4.3). Then, by employing relations for $f_{a,b,c}$ function, we find that

$$H_k^3(r) - q^{4k+2r} H_k^3(2k+r) = 2j(q^{4k-1+2r}, q^{4k-2}),$$

which implies that

$$H_k^3(2kr+i) = q^{-2kr(r+1)-2ir} H_k^3(i) - 2\sum_{m=1}^{r} q^{-4kmr+2km(m-1)-2mi} j(q^{4k(r-m)+4k-1+2i}, q^{4k-2}).$$

Therefore, after omitting the product, (4.8) is equal to

$$\begin{aligned}
&\sum_{i=1}^{2k} (-1)^i a^i q^{i^2+i} H_k^3(i) \sum_{r\geq 0} a^{2kr} q^{(4k^2-2k)r^2+(4k-2)ir} \\
&\quad - 2\sum_{i=1}^{2k} (-1)^i a^i q^{i^2+i} \sum_{r\geq 0, m\geq 1} a^{2k(r+m)} q^{4k^2r^2+(8k^2-4k)mr+(4k^2-2k)m^2+(4ki+2k)r+(4k-2)im} \\
&\qquad\qquad \times j(q^{(4k-2)(r+1)+2r+2i+1}, q^{4k-2}) \\
&= \sum_{i=1}^{2k} (-1)^i a^i q^{i^2+i} H(i) \sum_{r\geq 0} a^{2kr} q^{(4k^2-2k)r^2+(4k-2)ir} \\
&\quad + 2 \sum_{\substack{1\leq i\leq 2k \\ 0\leq n\leq 2k-2}} (-1)^{i+n} a^{2kn+i} q^{Q_3(k,i,n)} j(q^{2n+2i+1}, q^{4k-2}) \sum_{\substack{r\geq 0 \\ m\geq 1}} a^{2k((2k-1)r+m)} q^{R_3(k,i,n,r,m)},
\end{aligned}$$

where we set $r = (2k-1)r + n$ for the last equality and $Q_3(k,i,n)$ and $R_3(k,i,n,r,m)$ are defined in the statement of the theorem.

## 5. The Ji-Zhao identity and three-variable indefinite theta functions

In this section we recast the partial indefinite theta identities from Sections 1–4 of the paper in terms of indefinite ternary theta series. We define

$$g_{a,b,c,d,e,f}(x,y,z,q) := \left( \sum_{r,s,t\geq 0} + \sum_{r,s,t<0} \right) (-1)^{r+s+t} x^r y^s z^t q^{a\binom{r}{2}+brs+c\binom{s}{2}+drt+est+f\binom{t}{2}}.$$

Special cases of this function have recently occurred in the study of torus knots [11] and the Gromov-Witten theory of elliptic orbifolds [7], and as we shall see shortly, a number of $q$-hypergeometric series can be expressed in this way. For a general theory of multivariable indefinite theta functions, see [27].

We make use of an identity of Ji and Zhao [13]. Arguing as in Warnaar's proof of (1.2) they proved that if $(\alpha_n, \beta_n)$ is a Bailey pair relative to $q$, then

$$\sum_{n\geq 0} \frac{(q)_{2n}q^n}{(aq,q/a)_n}\beta_n = \frac{1}{(q^2,aq,q/a)_\infty}\sum_{n\geq 0}\frac{q^n\alpha_n}{(1-q^{2n+1})}\left(1+\sum_{r=1}^{\infty}(-1)^r q^{\binom{r}{2}}((aq^{n+1})^r+(q^{n+1}/a)^r)\right). \tag{5.1}$$

Note that the left-hand side is the same as in Warnaar's result but the right-hand side is different.

We begin with identity (1.5). If we use the Bailey pair in (2.4) and (2.5) in (5.1), we obtain

$$\begin{aligned}\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n} &= \frac{1}{(q,aq,q/a)_\infty}\sum_{\substack{n\geq 0\\ |j|\leq n\\ r\geq 0}}(-1)^{r+j}a^r q^{2n^2+2n-j(3j+1)/2+\binom{r}{2}+(n+1)r}\\ &\quad+\frac{1}{(q,aq,q/a)_\infty}\sum_{\substack{n\geq 0\\ |j|\leq n\\ r\geq 1}}(-1)^{r+j}a^{-r}q^{2n^2+2n-j(3j+1)/2+\binom{r}{2}+(n+1)r}.\end{aligned}$$

Letting $n=(u+v)/2$ and $j=(u-v)/2$ in the first sum and letting $n=(-u-v-2)/2$, $j=(u-v)/2$, and $r=-r$ in the second sum, we have

$$\begin{aligned}\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n} &= \frac{1}{(q,aq,q/a)_\infty}\\ &\quad\times\left(\sum_{\substack{r,u,v\geq 0\\ u\equiv v \pmod 2}}+\sum_{\substack{r,u,v<0\\ u\equiv v \pmod 2}}\right)a^r(-1)^{r+\frac{u-v}{2}}q^{\binom{r+1}{2}+\frac{1}{8}u^2+\frac{7}{4}uv+\frac{1}{8}v^2+\frac{3}{4}u+\frac{5}{4}v+\frac{1}{2}ru+\frac{1}{2}rv}.\end{aligned}$$

Replacing $(u,v)$ by $(2u,2v)$ or $(2u+1,2v+1)$ we arrive at the following.

Proposition 5.1. *We have*

$$\sum_{n\geq 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n} = \frac{1}{(q,aq,q/a)_\infty}\left(g_{1,7,1,1,1,1}(aq^2,q^3,q,q)+q^4 g_{1,7,1,1,1,1}(aq^6,q^7,q^2,q)\right).$$

We proceed to find the three-variable indefinite theta functions for the other identities in the paper. The method is always the same, so we omit the details. Using the Bailey pairs in (2.9), (2.10), (2.13), and (2.14) we have alternative versions of (1.6) and (1.7).

Proposition 5.2. *We have*

$$\sum_{n\geq 0}\frac{(q;q^2)_n(q)_nq^n}{(aq,q/a)_n} = \frac{1}{(q,aq,q/a)_\infty}\left(g_{1,5,1,1,1,1}(aq^2,q^2,q,q)+q^3 g_{1,5,1,1,1,1}(aq^5,q^5,q^2,q)\right),$$

$$\begin{aligned}\sum_{n\geq 0}\frac{(q;q^2)_n^2q^{2n}}{(aq^2,q^2/a;q^2)_n} &= \frac{1}{(q^2,aq^2,q^2/a;q^2)_\infty}\Big(g_{2,6,2,2,2,2}(aq^3,q^3,q^2,q)\\ &\qquad+q^4 g_{2,6,2,2,2,2}(aq^7,q^7,q^4,q)\Big).\end{aligned}$$

The identities corresponding to those in Theorems 1.2 and 1.3 are contained in the following two results.

Proposition 5.3. *For $k$ a positive integer and $0 \leq \ell < k$, we have*

$$\sum_{n\geq 0} \frac{(q)_{2n}q^n}{(aq,q/a)_n} \sum_{n\geq n_{2k-1}\geq\cdots\geq n_1\geq 0} \frac{(-1)^{n_k} q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})+\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i}}{(q)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}}$$
$$= \frac{1}{(q,aq,q/a)_\infty}\Big(g_{1,4k+3,1,1,1,1}(aq^{k+1-\ell},q^{k+2+\ell},q,q)$$
$$+ q^{2+2k}g_{1,4k+3,1,1,1,1}(aq^{3k+3-\ell},q^{3k+4+\ell},q^2,q)\Big).$$

Proposition 5.4. *For $k$ a positive integer we have*

$$\sum_{n\geq 0} \frac{(q)_{2n}q^n}{(aq,q/a)_n} \sum_{n\geq n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}}{(q)_{n-n_{k-1}}\cdots(q)_{n_2-n_1}}$$
$$= \frac{1}{(q,aq,q/q)_\infty}\Big(g_{2k-1,2k+5,2k-1,1,1,1}(aq^{2k},q^{2k+1},q,q)$$
$$+ q^{2+2k}g_{2k-1,2k+5,2k-1,1,1,1}(aq^{4k+2},q^{4k+3},q^2,q)\Big).$$

Before stating the result corresponding to (2.27), note that inserting the Bailey pair in (2.19) and (2.20) into (5.1) does *not* lead to a divergent series, unlike with (1.2). We iterate (2.19) and (2.20) along the Bailey chain to obtain

$$\alpha_k(n) = \frac{q^{kn^2+(k-1)n}(1-q^{2n+1})}{1-q} \sum_{j=-n}^{n}(-1)^j q^{-j^2},$$
$$\beta_k(n) = \sum_{n\geq n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}(-1)^{n_1}}{(q)_{n-n_{k-1}}\cdots(q)_{n_2-n_1}(q^2;q^2)_{n_1}}.$$

The result is

Proposition 5.5. *For $k$ a positive integer we have*

$$\sum_{n\geq 0} \frac{(q)_{2n}q^n}{(aq,q/a)_n} \sum_{n\geq n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{(-1)^{n_1}q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}}{(q)_{n-n_{k-1}}\cdots(q)_{n_2-n_1}(q^2;q^2)_{n_1}}$$
$$= \frac{1}{(q,aq,q/a)_\infty}\Big(g_{2k-2,2k+2,2k-2,1,1,1}(aq^{2k-1},q^{2k-1},q,q)$$
$$+ q^{2k}g_{2k-2,2k+2,2k-2,1,1,1}(aq^{4k-1},q^{4k-1},q^2,q)\Big).$$

When $k = 2$ the left-hand side is the same as in (2.27).

We turn to the three results stated in Theroems 3.1–3.3 at the beginning of Section 3. We obtain the following indefinite ternary theta series versions.

Proposition 5.6. *For $k$ a positive integer and $0 \le \ell < k$, we have*

$$\sum_{n\ge 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n}\sum_{n\ge n_{2k-1}\ge\cdots\ge n_1\ge 0}\frac{(-1)^{n_k}q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i+\binom{n_1+1}{2}}(-q)_{n_k}}{(q)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q^2;q^2)_{n_1}(-q)_{n_{k+1}}}$$
$$=\frac{1}{(q,aq,q/a)_\infty}\Big(g_{1,4k+1,1,1,1,1}(aq^{k+1-\ell},q^{k+1+\ell},q,q)$$
$$+q^{1+2k}g_{1,4k+3,1,1,1,1}(aq^{3k+2-\ell},q^{3k+2+\ell},q^2,q)\Big).$$

Proposition 5.7. *For $k$ a positive integer and $0 \le \ell < k$, we have*

$$\sum_{n\ge 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n}\sum_{n\ge n_{2k-1}\ge\cdots\ge n_1\ge 0}\frac{(-1)^{n_k}q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})+\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i+\binom{n_1+1}{2}}}{(q)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q^2;q^2)_{n_1}}$$
$$=\frac{1}{(q,aq,q/a)_\infty}\Big(g_{2,4k+2,2,1,1,1}(aq^{k+2-\ell},q^{k+2+\ell},q,q)$$
$$+q^{2+2k}g_{2,4k+2,2,1,1,1}(aq^{3k+4-\ell},q^{3k+4+\ell},q^2,q)\Big).$$

Proposition 5.8. *For $k$ a positive integer and $0 \le \ell < k$, we have*

$$\sum_{n\ge 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n}\sum_{n\ge n_{2k}\ge n_{2k-1}\ge\cdots\ge n_1\ge 0}\frac{(-1)^{n_k}q^{n_{2k}^2+n_{2k}+\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i}(-q)_{n_k}}{(q)_{n-n_{2k}}(q)_{n_{2k}-n_{2k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}(-q)_{n_{k+1}}}$$
$$=\frac{1}{(q,aq,q/a)_\infty}\Big(g_{2,4k+4,2,1,1,1}(aq^{k+2-\ell},q^{k+3+\ell},q,q)$$
$$+q^{3+2k}g_{2,4k+4,2,1,1,1}(aq^{3k+5-\ell},q^{3k+6+\ell},q^2,q)\Big).$$

Finally, we have the two results stated at the beginning of Section 4. Corresponding to Theorem 4.1 and Theorem 4.2, we have the following.

Proposition 5.9. *For $k$ a positive integer we have*

$$\sum_{n\ge 0}\frac{(q)_{2n}q^n}{(aq,q/a)_n}\sum_{n\ge n_{k-1}\ge\cdots\ge n_1\ge 0}\frac{q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}}{(q)_{n-n_{k-1}}\cdots(q)_{n_2-n_1}(-q)_{n_1}}$$
$$=\frac{1}{(q,aq,q/a)_\infty}\Big(g_{2k-1,2k+3,2k-1,1,1,1}(aq^{2k},q^{2k},q,q)$$
$$+q^{1+2k}g_{2k-1,2k+3,2k-1,1,1,1}(aq^{4k+1},q^{4k+1},q^2,q)\Big).$$

Proposition 5.10. *For $k$ a positive integer we have*

$$\sum_{n\ge 0}\frac{(q^2;q^2)_{2n}q^n}{(aq^2,q^2/a;q^2)_n}\sum_{n\ge n_{k-1}\ge\cdots\ge n_1\ge 0}\frac{q^{2n_{k-1}^2+2n_{k-1}+\cdots+2n_1^2+2n_1}(q;q^2)_{n_1}}{(q^2;q^2)_{n-n_{k-1}}\cdots(q^2;q^2)_{n_2-n_1}(q^4;q^4)_{n_1}(-q;q^2)_{n_1}}$$
$$=\frac{1}{(q^2,aq^2,q^2/a;q^2)_\infty}\Big(g_{4k-2,4k+2,4k-2,2,2,2}(aq^{4k-1},q^{4k-1},q^2,q)$$
$$+q^{4k}g_{4k-2,4k+2,4k-2,2,2,2}(aq^{8k-1},q^{8k-1},q^4,q)\Big).$$

## 6. Residual identities

In this final section we compute the so-called *residual identities* corresponding to the partial theta identities in Sections 1-4. The notion of the residual identity of a partial theta identity goes back to Andrews [1], the idea being to evaluate the residue around the pole $a = q^N$ and then use analytic continuation to replace $q^N$ by $a$. This has been further discussed by Warnaar [26] and the second author [20], who both carried out a number of examples and gave some applications. Here we give details for the partial theta identity (1.5) and simply state the other results.

In (1.5), we take the limit as $a \to q^N$ after multiplying both sides by $1 - q^N/a$ and shifting the summation variable on the left-hand side by $N$. We find that

$$\sum_{n\geq 0} \frac{(q)_{2n+2N} q^{n+N}}{(q^{N+1})_{n+N}(q)_n} = \frac{1}{(q^{N+1})_\infty} \sum_{r\geq 0} (-1)^r q^{(2r+1)N} q^{3r(r+1)/2},$$

and so

$$\sum_{n\geq 0} \frac{(q)_{2N}(q^{2N+1})_{2n} q^n}{(q^{N+1})_N (q^{2N+1})_n (q)_n} = \frac{1}{(q^{N+1})_\infty} \sum_{r\geq 0} (-1)^r q^{2rN} q^{3r(r+1)/2},$$

or

$$\sum_{n\geq 0} \frac{(q^{2N+1})_{2n} q^n}{(q^{2N+1})_n (q)_n} = \frac{1}{(q)_\infty} \sum_{r\geq 0} (-1)^r q^{2rN} q^{3r(r+1)/2}.$$

Since this identity holds for any $N \in \mathbb{N}$, analytic continuation enables us to set $a = q^{2N}$. As a result, we have proven that

$$\sum_{n\geq 0} \frac{(aq)_{2n} q^n}{(q, aq)_n} = \frac{1}{(q)_\infty} \sum_{n\geq 0} (-a)^n q^{n(3n+3)/2}.$$

This is a special case of a transformation of Fine [8, Eq. (25.96), $b = 0$, $t = z = q$].

If we compute the residual identity of (1.6), we get

$$\sum_{n\geq 0} \frac{(a^2q; q^2)_n (aq)_n q^n}{(a^2q, q)_n} = \frac{1}{(q, -aq)_\infty} \sum_{n\geq 0} (-1)^n a^{3n} q^{3n^2+2n} (1 + aq^{2n+1}).$$

This is equivalent to a known identity (see (2.12) in [20]).

The residual identity of (1.7) is

$$\sum_{n\geq 0} \frac{(aq)_n^2 q^n}{(a^2q^2, q)_n} = \frac{(aq)_\infty^2}{(a^2q^2, q)_\infty} \sum_{r\geq 0} a^{2r} q^{r^2+2r} \frac{1 + aq^{r+1}}{1 - aq^{r+1}},$$

which is a special case of an identity of Watson [9, Eq. (3.2.11), $a = a^2q^2$, $b \to \infty$, $c = q$, $d = e = aq$].

We now state the residual identities of Theorems 1.2, 3.1, 3.2, and 3.3, using the same notation $H_{k,\ell}^j(i)$ as in these Theorems. We extend the definition of $(a)_n$ to all integers in the usual way, by

$$(a)_n := \frac{(a)_\infty}{(aq^n)_\infty}.$$

Theorem 6.1. *For $k$ a positive integer and $0 \le \ell < k$, we have*

$$\sum_{n\ge 0}\frac{(a^2q^{n+1})_nq^n}{(q)_n}\sum_{n_{2k-1}\ge\cdots\ge n_1\ge 0}\frac{(-1)^{n_k}q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})+\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i}}{(aq)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}}$$
$$=\frac{1}{(q,q,aq)_\infty}\sum_{i=1}^{k}(-1)^ia^{i-1}q^{\binom{i+1}{2}}H^1_{k,\ell}(i)$$
$$\times\sum_{r\ge 0}a^{(2k+2)r}q^{(2k^2+3k+1)r^2+(2k+1)ir}(1-a^{2k+2-2i}q^{(2k+1)(2r+1)(k+1-i)}).$$

Theorem 6.2. *For $k$ a positive integer and $0 \le \ell < k$, we have*

$$\sum_{n\ge 0}\frac{(a^2q^{n+1})_nq^n}{(q)_n}\sum_{n_{2k-1}\ge\cdots\ge n_1\ge 0}\frac{(-1)^{n_k}q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i+\binom{n_1+1}{2}}(-q)_{n_k}}{(aq)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q^2;q^2)_{n_1}(-q)_{n_{k+1}}}$$
$$=\frac{1}{(q,q,aq)_\infty}\sum_{i=1}^{k}(-1)^ia^{i-1}q^{\binom{i+1}{2}}H^2_{k,\ell}(i)$$
$$\times\sum_{r\ge 0}(-1)^ra^{(2k+1)r}q^{(2k^2+k)r^2+2kir}(1+a^{2k+1-2i}q^{(2k^2+k-2ki)(2r+1)}).$$

Theorem 6.3. *For $k$ a positive integer and $0 \le \ell < k$, we have*

$$\sum_{n\ge 0}\frac{(a^2q^{n+1})_nq^n}{(q)_n}\sum_{n_{2k-1}\ge\cdots\ge n_1\ge 0}\frac{(-1)^{n_k}q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})+\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i+\binom{n_1+1}{2}}}{(aq)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q^2;q^2)_{n_1}}$$
$$=\frac{1}{(q,q,aq)_\infty}\sum_{i=1}^{2k+2}(-1)^ia^{i-1}q^{\binom{i+1}{2}}H^3_{k,\ell}(i)\sum_{r=0}^{\infty}a^{(2k+2)r}q^{(2k+1)(k+1)r^2+(2k+1)ir}$$
$$+\frac{1}{(-q,q,aq)_\infty}\sum_{\substack{i=1\\ i\not\equiv k+\ell \pmod 2}}^{2k+2}(-1)^{\frac{k+3i+\ell-1}{2}}a^{i-1}q^{\binom{i+1}{2}-\left(\frac{k+i+\ell+1}{2}\right)^2}$$
$$\times\sum_{r\ge 0,m\ge 1}(-1)^{r(k+1)}a^{(2k+2)(r+m)}q^{(k+1)^2r^2+2(k+1)(2k+1)rm+(k+1)(2k+1)m^2}$$
$$\times q^{i(2k+1)m+(k+1)(i-k-\ell)r}\left(1+(-1)^{\ell}q^{\ell((2r+1)(k+1)r+i)}\right).$$

Theorem 6.4. *For $k$ a positive integer and $0 \leq \ell < k$, we have*

$$\sum_{n\geq 0}\frac{(a^2q^{n+1})_nq^n}{(q)_n}\sum_{n_{2k}\geq n_{2k-1}\geq\cdots\geq n_1\geq 0}\frac{(-1)^{n_k}q^{\sum_{i=1}^{k}(n_{k+i}^2+n_{k+i})-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i}(-q)_{n_k}}{(aq)_{n-n_{2k}}(q)_{n_{2k}-n_{2k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}(-q)_{n_{k+1}}}$$
$$= \frac{1}{(q,q,aq)_\infty}\sum_{i=1}^{2k+3}(-1)^ia^{i-1}q^{\binom{i+1}{2}}H_{k,\ell}^4(i)\sum_{r=0}^{\infty}(-1)^ra^{(2k+3)r}q^{(2k^2+5k+3)r^2+2(k+1)ir}$$
$$+\frac{1}{(-q,q,aq)_\infty}\sum_{i=1}^{2k+3}(-1)^{\frac{k+3i-\ell-1}{2}}a^{i-1}q^{\binom{i+1}{2}-\left(\frac{k+i-\ell+1}{2}\right)^2}$$
$$\times\sum_{\substack{r\geq 0,m\geq 1\\ r\not\equiv k+i-\ell \pmod 2}}(-1)^{(2k+1)r/2+m}a^{(2k+3)(r+m)}q^{(k^2+3k+9/4)r^2+(2k^2+5k+3)m^2}$$
$$\times q^{(4k^2+10k+6)rm+(2k+3)(\ell+i-k)r/2+2(k+1)im}$$
$$+\frac{1}{(-q,q,aq)_\infty}\sum_{i=1}^{2k+3}(-1)^{\frac{k+3i+\ell}{2}}a^{i-1}q^{\binom{i+1}{2}-\left(\frac{k+i+\ell+2}{2}\right)^2}$$
$$\times\sum_{\substack{r\geq 0,m\geq 1\\ r\equiv k+i+\ell \pmod 2}}(-1)^{(2k+1)r/2+m}a^{(2k+3)(r+m)}q^{(k^2+3k+9/4)r^2+(2k^2+5k+3)m^2}$$
$$\times q^{(4k^2+10k+6)rm-(2k+3)(\ell-i+k+1)r/2+2(k+1)im}.$$

Finally, we state the three residual identities corresponding to Theorems 1.3, 4.1, and 4.2. Recall the notation from the statements of these theorems.

Theorem 6.5. *For $k$ a positive integer, we have*

$$\sum_{n\geq 0}\frac{(a^2q^{n+1})_nq^n}{(q)_n}\sum_{n_{k-1}\geq\cdots\geq n_1\geq 0}\frac{q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}}{(aq)_{n-n_{k-1}}\cdots(q)_{n_2-n_1}}$$
$$=\frac{1}{(q,q,aq)_\infty}\left(\sum_{i=1}^{2k+2}(-1)^ia^{i-1}q^{\binom{i+1}{2}}H_k^1(i)\sum_{r=0}^{\infty}a^{(2k+2)r}q^{(2k^2+3k+1)r^2+(2k+1)ir}\right.$$
$$+\sum_{\substack{1\leq i\leq 2k+2\\ 0\leq n\leq 2k-2\\ 1\leq b\leq 2}}(-1)^{i+n}a^{i+(2k+2)n-1}q^{Q_1(k,i,n,b)}j(q^{3n+i+b},q^{2k-1})$$
$$\left.\times\sum_{\substack{r\geq 0\\ m\geq 1}}a^{(2k+2)((2k-1)r+m)}q^{R_1(k,i,n,b,r,m)}\right).$$

THEOREM 6.6. *For $k$ a positive integer, we have*

$$\sum_{n\geq 0}\frac{(a^2q^{n+1})_n q^n}{(q)_n}\sum_{n_{k-1}\geq\cdots\geq n_1\geq 0}\frac{q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}}{(aq)_{n-n_{k-1}}\cdots(q)_{n_2-n_1}(-q)_{n_1}}$$
$$=\frac{1}{(q,q,aq)_\infty}\Bigg(\sum_{i=1}^{2k+1}(-1)^i a^{i-1}q^{\binom{i+1}{2}}H_k^2(i)\sum_{r=0}^{\infty}(-1)^r a^{(2k+1)r}q^{(2k^2+k)r^2+2kir}$$
$$+2\sum_{\substack{1\leq i\leq 2k+1\\0\leq n\leq 2k-2}}(-1)^i a^{i+(2k+1)n-1}q^{Q_2(k,i,n)}j(q^{2n+i+1},q^{2k-1})$$
$$\times\sum_{\substack{r\geq 0\\m\geq 1}}(-1)^m a^{(2k+1)((2k-1)r+m)}q^{R_2(k,i,n,r,m)}\Bigg).$$

THEOREM 6.7. *For $k$ a positive integer, we have*

$$\sum_{n\geq 0}\frac{(a^2q^{2n+2};q^2)_n q^{2n}}{(q^2;q^2)_n}\sum_{n_{k-1}\geq\cdots\geq n_1\geq 0}\frac{q^{2n_{k-1}^2+2n_{k-1}+\cdots+2n_1^2+2n_1}(q;q^2)_{n_1}}{(q^2;q^2)_{n-n_{k-1}}\cdots(q^2;q^2)_{n_2-n_1}(q^4;q^4)_{n_1}(-q;q^2)_{n_1}}$$
$$=\frac{1}{(q^2,q^2,aq^2;q^2)_\infty}\Bigg(\sum_{i=1}^{2k}(-1)^i a^{i-1}q^{i^2+i}H_k^3(i)\sum_{r\geq 0}a^{2kr}q^{(4k^2-2k)r^2+(4k-2)ir}$$
$$+2\sum_{\substack{1\leq i\leq 2k\\0\leq n\leq 2k-2}}(-1)^{i+n}a^{2kn+i-1}q^{Q_3(k,i,n)}j(q^{2n+2i+1},q^{4k-2})$$
$$\times\sum_{\substack{r\geq 0\\m\geq 1}}a^{2k((2k-1)r+m)}q^{R_3(k,i,n,r,m)}\Bigg).$$


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