TORUS KNOTS AND QUANTUM MODULAR FORMS

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ABSTRACT. In this paper we compute a q-hypergeometric expression for the cyclotomic expansion of the colored Jones polynomial for the left-handed torus knot $(2, 2t + 1)$. We use this to define a family of q-series, the simplest case of which is the generating function for strongly unimodal sequences. Special cases of these q-series are quantum modular forms, and at roots of unity these are dual to the generalized Kontsevich–Zagier series introduced by the first author. This duality generalizes a result of Bryson, Pitman, Ono, and Rhoades. We also compute Hecke-type expansions for our family of q-series.

1. Introduction and Statement of Results

Several years ago Zagier introduced quantum modular forms [30]. These are functions defined for $z \in \mathbb{Q}$ (or equivalently, for q a root of unity, where $q := e^{2\pi i z}$) which behave like modular forms. For $g(z)$ to be quantum modular means that instead of requiring that $g(z) - \chi(\gamma)(cz + d)^{-k} g\left(\frac{az+b}{cz+d}\right) = 0$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ as with classical modular forms, we only ask that $g(z) - \chi(\gamma)(cz + d)^{-k} g\left(\frac{az+b}{cz+d}\right)$ have "nice" properties such as continuity or analyticity. The word *quantum* refers to the fact that these functions have "the 'feel' of the objects in perturbative quantum field theory" [30, p. 659]. A celebrated example is the Kontsevich–Zagier series [29]

$$F(q) := \sum_{n=0}^{\infty} (q)_n, \tag{1.1}$$

where we use the standard notation,

$$(a)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}). \tag{1.2}$$

Note that $F(q)$ does not converge on any open subset of $\mathbb{C}$, but it is well-defined at any roots of unity.

Bryson et al [9] recently established a relationship between $F(q)$ and the generating function for strongly unimodal sequences,

$$U(x; q) := \sum_{n=0}^{\infty} (-x\,q)_n \, (-x^{-1}\,q)_n \, q^{n+1}. \tag{1.3}$$

Precisely, $U(-1; q)$ is dual to $F(q)$ at any root of unity $\zeta_N = e^{2\pi i/N}$,

$$F(\zeta_N^{-1}) = U(-1; \zeta_N). \tag{1.4}$$

Note that contrary to the Kontsevich–Zagier series $F(q)$, the function $U(x;q)$ converges for generic $|q| < 1$. It should be remarked that the function $U(-1;q)$ also plays a crucial role in studying a radial limit of a mock theta function of Ramanujan [11].

For other recent studies and interesting examples of quantum modular forms in number theory, combinatorics, knot theory, and Lie algebras, see *e.g.* [6, 7, 8, 10, 11, 17, 19, 26, 30].

This article is based on the observation that the identity (1.4) can be interpreted in terms of quantum topology as follows. We use the N-colored Jones polynomial $J_N(K;q)$ for a knot K, which is based on the N-dimensional representation of $U_q(s\ell_2)$ (see *e.g.* [21]). Throughout this article we use a normalization $J_N(\text{unknot}; q) = 1$. It is known that

$$J_N(K; q^{-1}) = J_N(K^*; q), \tag{1.5}$$

where K^* is a mirror image of K. The colored Jones polynomial for the right-handed trefoil $T_{(2,3)}$ and the left-handed trefoil $T^*_{(2,3)}$ may be respectively written as (see *e.g.* [12, 20, 24])

$$J_N(T_{(2,3)}; q) = q^{1-N} \sum_{n=0}^{\infty} q^{-nN}(q^{1-N})_n, \tag{1.6}$$

$$J_N(T^*_{(2,3)}; q) = \sum_{n=0}^{\infty} q^n \, (q^{1-N})_n \, (q^{1+N})_n. \tag{1.7}$$

(Note that the series defining $J_N(T_{(2,3)}; q)$ and $J_N(T^*_{(2,3)}; q)$ are in fact finite sums, terminating at $n = N$.) Letting $q = \zeta_N$ in (1.7) and $q = \zeta_N^{-1}$ in (1.6) and appealing to (1.5), one finds the duality in (1.4).

Based on the colored Jones polynomial for the torus knot $T_{(2,2t+1)}$ at roots of unity, the first author [14] introduced a family of quantum modular forms generalizing $F(q)$ (1.1),

$$F_t(q) := q^t \sum_{k_t \geq \cdots \geq k_1 \geq 0} (q)_{k_t} \prod_{i=1}^{t-1} q^{k_i(k_i+1)} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}_q. \tag{1.8}$$

Here $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the usual q-binomial coefficient,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q)_n}{(q)_{n-k}(q)_k}. \tag{1.9}$$

Note that when $t = 1$ we recover the Kontsevich–Zagier series, $F_1(q) = q \, F(q)$. Our purpose in this article is to use the perspective of quantum invariants to generalize $U(x;q)$ and (1.4). As a dual to $F_t(q)$, we make the following definition.

Definition 1.1. *The generalized U-function $U_t(x;q)$ is defined by*

$$U_t(x;q) := q^{-t} \sum_{k_t \geq \cdots \geq k_1 \geq 1} (-xq)_{k_t-1}(-x^{-1}q)_{k_t-1} \, q^{k_t} \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1} + k_i - i + 2\sum_{j=1}^{i-1} k_j \\ k_{i+1} - k_i \end{bmatrix}_q.$$

$$\tag{1.10}$$

This function is well-defined for $|q| < 1$. When $x = -1$ it is also defined when q is a root of unity. Our first main result is the following generalization of (1.4).

Theorem 1.2.

$$F_t(\zeta_N^{-1}) = U_t(-1; \zeta_N). \tag{1.11}$$

Our second main result is a Hecke-type expansion for $U_t(x; q)$. (The case $t = 1$ and $x = -1$ appears in [9].)

Theorem 1.3. *We have*

$$
\begin{aligned}
&U_t(-x; q) \\
&= -q^{-t/2-1/8} \frac{(xq)_\infty (q/x)_\infty}{(q)_\infty^2} \\
&\quad \times \left(\sum_{\substack{r,s \geq 0 \\ r \not\equiv s \pmod 2}} - \sum_{\substack{r,s < 0 \\ r \not\equiv s \pmod 2}} \right) \frac{(-1)^{\frac{r-s-1}{2}} q^{\frac{1}{8}r^2 + \frac{4t+3}{4}rs + \frac{1}{8}s^2 + \frac{2+t}{2}r + \frac{t}{2}s}}{1 - xq^{\frac{r+s+1}{2}}} \\
&= -q^{-t/2-1/8} \frac{(xq)_\infty (q/x)_\infty}{(q)_\infty^2} \\
&\quad \times \left(\sum_{\substack{r,s,u \geq 0 \\ r \not\equiv s \pmod 2}} + \sum_{\substack{r,s,u < 0 \\ r \not\equiv s \pmod 2}} \right) (-1)^{\frac{r-s-1}{2}} x^u q^{\frac{1}{8}r^2 + \frac{4t+3}{4}rs + \frac{1}{8}s^2 + \frac{2+t}{2}r + \frac{t}{2}s + u\frac{r+s+1}{2}}.
\end{aligned}
$$

(1.12)

(1.13)

The paper is constructed as follows. In Section 2, we review Bailey pairs and their relation to the colored Jones polynomial. In Section 3 we study the colored Jones polynomial for the torus knot $T_{(2,2t+1)}$. In particular, we use the Bailey pair machinery to compute the coefficients of the cyclotomic expansion of $J_N(T_{(2,2t+1)}^*; q)$, which leads to Theorem 1.2. In Section 4 we prove Theorem 1.3, again using the Bailey machinery. In Section 5 we extend Theorems 1.2 and 1.3 to the vector-valued setting. We close with some suggestions for future research and an appendix containing some examples.

2. Bailey Pairs and The Colored Jones Polynomial

In this section we review facts about Bailey pairs and their relation to the colored Jones polynomial.

First recall [2] that two sequences (α_n, β_n) form a Bailey pair relative to a if

$$\beta_n = \sum_{j=0}^n \frac{\alpha_j}{(q)_{n-j}(aq)_{n+j}}, \tag{2.1}$$

or equivalently,

$$\alpha_n = \frac{1 - aq^{2n}}{1 - a} \frac{(a)_n}{(q)_n} (-1)^n q^{n(n-1)/2} \sum_{j=0}^n (q^{-n})_j (aq^n)_j q^j \beta_j. \tag{2.2}$$

The Bailey lemma [2] states that if (α_n, β_n) is a Bailey pair relative to a, then so is (α_n', β_n'), where

$$\alpha_n' = \frac{(b)_n (c)_n (aq/bc)^n}{(aq/b)_n (aq/c)_n} \alpha_n \tag{2.3}$$

and

$$\beta_n' = \sum_{k=0}^{n} \frac{(b)_k (c)_k (aq/bc)_{n-k} (aq/bc)^k}{(aq/b)_n (aq/c)_n (q)_{n-k}} \beta_k. \tag{2.4}$$

In particular, if $b, c \to \infty$ then we have

$$\alpha_n' = a^n q^{n^2} \alpha_n \tag{2.5}$$

and

$$\beta_n' = \sum_{k=0}^{n} \frac{a^k q^{k^2}}{(q)_{n-k}} \beta_k. \tag{2.6}$$

Inserting (2.3) and (2.4) back in the definition (2.1) and letting $n \to \infty$, we have

$$\sum_{n \geq 0} (b)_n (c)_n (aq/bc)^n \beta_n = \frac{(aq/b)_\infty (aq/c)_\infty}{(aq)_\infty (aq/bc)_\infty} \sum_{n \geq 0} \frac{(b)_n (c)_n (aq/bc)^n}{(aq/b)_n (aq/c)_n} \alpha_n. \tag{2.7}$$

Next recall the cyclotomic expansion of the colored Jones polynomial due to Habiro [13]

$$J_N(K; q) = \sum_{n=0}^{\infty} C_n(K; q) \, (q^{1+N})_n \, (q^{1-N})_n, \tag{2.8}$$

where we have

$$C_n(K; q) \in \mathbb{Z}[q, q^{-1}]. \tag{2.9}$$

The colored Jones polynomial $J_N(K; q)$ and the coefficients $C_n(K; q)$ defined in (2.8) can be regarded as a Bailey pair (α_n, β_n) relative to q^2. Namely, comparing equations (2.8) and (2.2) we have (see also [13, 16])

$$\begin{aligned} \alpha_n &= \frac{(1 - q^{n+1})(1 - q^{2n+2})}{(1 - q)(1 - q^2)} (-1)^n \, q^{\frac{1}{2}n(n-1)} \, J_{n+1}(K; q), \\ \beta_n &= q^{-n} \, C_n(K; q). \end{aligned} \tag{2.10}$$

Equation (2.1) gives the inverse transform

$$C_n(K; q) = -q^{n+1} \sum_{\ell=1}^{n+1} \frac{(1 - q^\ell)(1 - q^{2\ell})}{(q)_{n+1-\ell} \, (q)_{n+1+\ell}} (-1)^\ell \, q^{\frac{1}{2}\ell(\ell-3)} \, J_\ell(K; q). \tag{2.11}$$

3. The Colored Jones Polynomial for Torus Knots

For some knots K, explicit forms of $J_N(K; q)$ and/or $C_N(K; q)$ are known in the literature. For instance, when K is the right-handed torus knot $T_{(s,t)}$, where s and t are coprime positive integers, the colored Jones polynomial is given by [25, 27]

$$J_N(T_{(s,t)}; q) = \frac{q^{\frac{1}{4}st(1-N^2)}}{q^{\frac{N}{2}} - q^{-\frac{N}{2}}} \sum_{j=-\frac{N-1}{2}}^{\frac{N-1}{2}} q^{stj^2} \left(q^{-(s+t)j+\frac{1}{2}} - q^{-(s-t)j-\frac{1}{2}} \right). \tag{3.1}$$

Using difference equations, the first author [15] constructed a q-hypergeometric expression for $J_N(T_{(s,t)}; q)$ when $s = 2$,

$$J_N(T_{(2,2t+1)}; q) = q^{t(1-N)} \sum_{k_t \geq \cdots \geq k_1 \geq 0} (q^{1-N})_{k_t} \, q^{-Nk_t} \prod_{i=1}^{t-1} q^{k_i(k_i+1-2N)} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}_q. \qquad (3.2)$$

(See [18] for similar expressions for some other torus knots.) Comparing this with the generalized Kontsevich–Zagier series (1.8), we find that $J_N(T_{(2,2t+1)}; q)$ and $F_t(q)$ agree at roots of unity,

$$J_N(T_{(2,2t+1)}; \zeta_N) = F_t(\zeta_N). \qquad (3.3)$$

With (1.5) and (2.8) in mind, we see that to discover Definition 1.1 and prove Theorem 1.2 we need to compute the cyclotomic expansion of the colored Jones polynomial of the left-handed torus knot $T^*_{(2,2t+1)}$. Recalling that the colored Jones polynomial for the mirror image K^* is given from that for K (1.5), we find from (3.1) that

$$(1 - q^N) \, J_N(T^*_{(2,2t+1)}; q) = (-1)^N q^{-t+\frac{1}{2}N+\frac{2t+1}{2}N^2} \sum_{k=-N}^{N-1} (-1)^k q^{-\frac{2t+1}{2}k(k+1)+k}. \qquad (3.4)$$

Then the coefficients C_n in the cyclotomic expansion (2.8) are given from the inverse transform (2.11) as

$$C_{n-1}(T^*_{(2,2t+1)}; q)$$
$$= -q^{n-t} \sum_{\ell=0}^{n} \frac{1}{(q)_{n-\ell}(q)_{n+\ell}} q^{(t+1)\ell^2-\ell} (1 - q^{2\ell}) \sum_{k=-\ell}^{\ell-1} (-1)^k \, q^{-(t+\frac{1}{2})k^2-(t-\frac{1}{2})k}. \qquad (3.5)$$

In the following proposition we give a q-hypergeometric expression for the coefficients $C_n(T^*_{(2,2t+1)}; q)$. We use the usual characteristic function

$$\chi(X) := \begin{cases} 1, & \text{when } X \text{ is true,} \\ 0, & \text{when } X \text{ is false.} \end{cases}$$

Proposition 3.1. *We have*

$$- q^{t-n} C_{n-1}(T^*_{(2,2t+1)}; q) =$$
$$\sum_{n \geq n_{2t-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{t-1} n_{t+i}^2 + \binom{n_t}{2} - \sum_{i=1}^{t-1} n_i n_{i+1} - \sum_{i=1}^{t-2} n_i} (-1)^{n_t} (1 - q^{n_t - \chi(t\geq 2)n_{t-1}})}{(q)_{n-n_{2t-1}}(q)_{n_{2t-1}-n_{2t-2}} \cdots (q)_{n_2-n_1}(q)_{n_1}}. \qquad (3.6)$$

Proof. In light of equations (3.5) and (2.1), we need to find β'_n such that (α'_n, β'_n) form a Bailey pair relative to 1, where

$$\alpha'_n = q^{(t+1)n^2-n}(1 - q^{2n}) \sum_{j=-n}^{n-1} (-1)^j q^{-((2t+1)j^2+(2t-1)j)/2}. \qquad (3.7)$$

We require a result of the second author. Namely, in part (ii) of Theorem 1.1 of [23], let $k = K = t$, $\ell = t - 1$, and $m = 0$. Then (α_n, β_n) form a Bailey pair relative to 1, where

$$\beta_n = \sum_{n \geq n_{2t-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{t-1} n_{t+i}^2 + \binom{n_t+1}{2} - \sum_{i=1}^{t-1} n_i n_{i+1} - \sum_{i=1}^{t-1} n_i} (-1)^{n_t}}{(q)_{n-n_{2t-1}}(q)_{n_{2t-1}-n_{2t-2}} \cdots (q)_{n_2-n_1}(q)_{n_1}} \qquad (3.8)$$

and

$$\alpha_n = q^{(t+1)n^2+n} \sum_{j=-n}^{n} (-1)^j q^{-((2t+1)j^2+(2t-1)j)/2}$$

$$- \chi(n \neq 0)q^{(t+1)n^2-n} \sum_{j=-n+1}^{n-1} (-1)^j q^{-((2t+1)j^2+(2t-1)j)/2}$$

$$= -q^{(t+1)n^2-n}(1-q^{2n}) \sum_{j=-n}^{n-1} (-1)^j q^{-((2t+1)j^2+(2t-1)j)/2} \tag{3.9}$$

$$+ \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{\frac{n^2}{2}+\left(\frac{2t-3}{2}\right)n} + (-1)^n q^{\frac{n^2}{2}-\left(\frac{2t-3}{2}\right)n}, & \text{if } n \geq 1. \end{cases} \tag{3.10}$$

Let $\alpha_n = -\alpha_n' + \alpha_n'' = (3.9) + (3.10)$. To find β_n'', we start with equations (3.9) and (3.10) of [23], which state that

$$\alpha_n^* = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n \left(q^{(-(2k-1)n^2-(2\ell+1)n)/2} + q^{(-(2k-1)n^2+(2\ell+1)n)/2} \right), & \text{if } n > 0, \end{cases} \tag{3.11}$$

and

$$\beta_n^* = \beta_{n_k} = (-1)^{n_k} q^{-\binom{n_k+1}{2}} \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{-\sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i}}{(q)_{n_k-n_{k-1}} \cdots (q)_{n_2-n_1}(q)_{n_1}}, \tag{3.12}$$

form a Bailey pair relative to 1. Using this Bailey pair with $k = t$ and

$$\ell = \begin{cases} 0, & \text{for } t = 1, \\ t-2, & \text{for } t \geq 2, \end{cases}$$

we iterate equations (2.5) and (2.6) t times. Then α_n^* becomes α_n'' and

$$\beta_n'' = \sum_{n \geq n_{2t-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{t-1} n_{t+i}^2 + \binom{n_t}{2} - \sum_{i=1}^{t-1} n_i n_{i+1} - \sum_{i=1}^{t-1} n_i}(-1)^{n_t}}{(q)_{n-n_{2t-1}}(q)_{n_{2t-1}-n_{2t-2}} \cdots (q)_{n_2-n_1}(q)_{n_1}}. \tag{3.13}$$

Taking $\beta_n'' - \beta_n$ gives the expression for $-q^{t-n}C_{n-1}(T_{(2,2t+1)}^*)$. $\square$

While the above proposition does furnish an attractive q-hypergeometric expression for $C_n(T_{(2,2t+1)}^*; q)$, it is not apparent that these coefficients are Laurent polynomials in q, as guaranteed by (2.9). This is made clear with the next proposition.

Proposition 3.2. *We have*

$$C_n(T_{(2,2t+1)}^*; q) = q^{n+1-t} \sum_{n+1=k_t \geq k_{t-1} \geq \cdots \geq k_1 \geq 1} \prod_{a=1}^{t-1} q^{k_a^2} \frac{(q^{1-a+\sum_{i=1}^{a} 2k_i})_{k_{a+1}-k_a}}{(q)_{k_{a+1}-k_a}} \tag{3.14}$$

$$= q^{n+1-t} \sum_{n+1=k_t \geq k_{t-1} \geq \cdots \geq k_1 \geq 1} \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1} + k_i - i + 2\sum_{j=1}^{i-1} k_j \\ k_{i+1} - k_i \end{bmatrix}_q. \tag{3.15}$$

Proof. We recall the classical q-binomial identity,

$$\sum_{n=0}^{N} z^n q^{\binom{n}{2}} \begin{bmatrix} N \\ n \end{bmatrix}_q = (z)_N. \tag{3.16}$$

Letting $z = -zq^a$, $N = b - a$, and shifting n to $n - a$, we have the identity

$$\sum_{n=a}^{b} \frac{q^{\binom{n}{2}}(-z)^n}{(q)_{b-n}(q)_{n-a}} = (-z)^a q^{\binom{a}{2}} \frac{(z\,q^a)_{b-a}}{(q)_{b-a}}. \tag{3.17}$$

Using this identity, we also have, for arbitrary c,

$$\sum_{n=a}^{b} \frac{(-1)^n (1-q^{n-c}) q^{\binom{n}{2}-an}}{(q)_{b-n}(q)_{n-a}} = \begin{cases} (-1)^a q^{-\binom{a+1}{2}}(1-q^{a-c}), & \text{if } a = b, \\ (-1)^{a+1} q^{-\binom{a}{2}-c}, & \text{otherwise.} \end{cases} \tag{3.18}$$

We may use the two identities (3.17) and (3.18) to transform (3.6) into (3.14) as follows. First, if $n_{t+1} = n_{t-1}$ then the sum in (3.6) vanishes, so we may assume $n_{t+1} > n_{t-1}$. The sum over n_t is

$$\sum_{n_t=n_{t-1}}^{n_{t+1}} \frac{(-1)^{n_t} q^{\binom{n_t}{2}-n_{t-1}n_t}(1-q^{n_t-n_{t-1}})}{(q)_{n_{t+1}-n_t}(q)_{n_t-n_{t-1}}},$$

and the second part of identity (3.18) then enables us to evaluate this sum, giving

$$-q^{t-n-1}C_n(T_{(2,2t+1)}^*; q) =$$
$$\sum_{n+1 \geq n_{2t-1} \geq \cdots \geq n_{t+2} \geq n_{t+1} > n_{t-1} \geq n_{t-2} \geq \cdots \geq n_1 \geq 0} (-1)^{1+n_{t-1}} q^{\sum_{i=1}^{t-1} n_{t+i}^2 - \sum_{i=1}^{t-2} n_i n_{i+1} - \sum_{i=1}^{t-1} n_i}$$
$$\times \frac{q^{-\binom{n_{t-1}}{2}}}{(q)_{n+1-n_{2t-1}} \cdots (q)_{n_{t+2}-n_{t+1}}(q)_{n_{t-1}-n_{t-2}} \cdots (q)_{n_2-n_1}(q)_{n_1}}. \tag{3.19}$$

We then set $n_{t+1} = n_{t-1} + k_1$, with $k_1 \geq 1$. The sum over n_{t-1} is

$$\sum_{n_{t-1}=n_{t-2}}^{n_{t+2}-k_1} \frac{(-1)^{n_{t-1}} q^{\binom{n_{t-1}}{2}-n_{t-2}n_{t-1}+2k_1 n_{t-1}}}{(q)_{n_{t+2}-k_1-n_{t-1}}(q)_{n_{t-1}-n_{t-2}}},$$

and (3.17) allows us to evaluate this sum, resulting in

$$q^{t-n-1}C_n(T_{(2,2t+1)}^*; q) =$$
$$\sum_{n+1 \geq n_{2t-1} \geq \cdots \geq n_{t+2} \geq n_{t-2}+k_1 > n_{t-2} \geq \cdots \geq n_1 \geq 0} (-1)^{n_{t-2}} q^{\sum_{i=2}^{t-1} n_{t+i}^2 - \sum_{i=1}^{t-3} n_i n_{i+1} - \sum_{i=1}^{t-2} n_i}$$
$$\times \frac{(q^{2k_1})_{n_{t+2}-n_{t-2}-k_1}}{(q)_{n_{t+2}-n_{t-2}-k_1}} \cdot \frac{q^{-\binom{n_{t-2}}{2}-n_{t-2}+2k_1 n_{t-2}+k_1^2}}{(q)_{n+1-n_{2t-1}} \cdots (q)_{n_{t+3}-n_{t+2}}(q)_{n_{t-2}-n_{t-3}} \cdots (q)_{n_2-n_1}(q)_{n_1}}.$$

We continue in the same manner, next setting $n_{t+2} = n_{t-2} + k_2$, with $k_2 \geq k_1$, and we may then take the sum over n_{t-2} using (3.17). Iterating this process (taking the sum over n_{t-a} after setting $n_{t+a} = n_{t-a} + k_a$), we arrive at (3.14). The expression in equation (3.15) follows from the fact that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^{k+1})_{n-k}}{(q)_{n-k}}. \tag{3.20}$$

$\square$

We are now prepared to prove the duality in Theorem 1.2.

Proof of Theorem 1.2. Comparing equations (3.14) and (1.10), we see that

$$U_t(x;q) = \sum_{n=0}^{\infty} C_n(T^*_{(2,2t+1)};q)\,(-x\,q)_n(-x^{-1}\,q)_n. \qquad (3.21)$$

Therefore, by (2.8), $U_t(x;q)$ gives the colored Jones polynomial for $T^*_{(2,2t+1)}$ when $x = q^N$,

$$J_N(T^*_{(2,2t+1)};q) = U_t(-q^N;q). \qquad (3.22)$$

Combining (3.22) with (3.3) and (1.5) gives the statement of the theorem. $\qquad\square$

4. Hecke-Type Formulae

In this section we prove Theorem 1.3 as well as a simpler formula when $t = 1$.

Proof of Theorem 1.3. In equations (2.5) and (2.6) we let $a = 1$, $b = 1/c = x$, and apply α'_n and $\beta'_n = -q^{t-n}C_{n-1}$ from the proof of Proposition 3.1. Recalling (3.21), we obtain

$$U_t(-x;q) = \frac{-q^{-t}(xq)_\infty(q/x)_\infty}{(q)_\infty^2} \sum_{n\geq 1}\sum_{k=-n}^{n-1} \frac{(-1)^k q^{(t+1)n^2-(t+\frac{1}{2})k^2-(t-\frac{1}{2})k}(1-q^{2n})}{(1-xq^n)(1-q^n/x)}$$

$$= \frac{-q^{-t}(xq)_\infty(q/x)_\infty}{(q)_\infty^2} \sum_{n\geq 1}\sum_{k=-n}^{n-1} \frac{(-1)^k q^{(t+1)n^2-(t+\frac{1}{2})k^2-(t-\frac{1}{2})k}}{1-xq^n}$$

$$+ \frac{-q^{-t}(xq)_\infty(q/x)_\infty}{(q)_\infty^2} \sum_{n\geq 1}\sum_{k=-n}^{n-1} \frac{(-1)^k q^{(t+1)n^2+n-(t+\frac{1}{2})k^2-(t-\frac{1}{2})k}}{x(1-q^n/x)}$$

$$= \frac{-q^{-t}(xq)_\infty(q/x)_\infty}{(q)_\infty^2} \sum_{n\geq 1}\sum_{k=-n}^{n-1} \frac{(-1)^k q^{(t+1)n^2-(t+\frac{1}{2})k^2-(t-\frac{1}{2})k}}{1-xq^n}$$

$$- \frac{-q^{-t}(xq)_\infty(q/x)_\infty}{(q)_\infty^2} \sum_{n\leq 1}\sum_{k=n}^{-n-1} \frac{(-1)^k q^{(t+1)n^2-(t+\frac{1}{2})k^2-(t-\frac{1}{2})k}}{1-xq^n}.$$

Letting $n = (r + s + 1)/2$ and $k = (r - s - 1)/2$ in each of the two final sums leads to the expression in (1.12). The expression in (1.13) follows upon expanding the term $1/(1 - xq^{(r+s+1)/2})$ in (1.12) as a geometric series. $\qquad\square$

When $t = 1$, we have the following Hecke-type double sum.

Theorem 4.1.

$$(1 - x)\,U_1(-x;q) = \frac{1}{(q)_\infty}\left(\sum_{r,n\geq 0} - \sum_{r,n<0}\right)(-1)^{n+r}x^{-r}q^{n(3n+5)/2+2nr+r(r+3)/2}. \qquad (4.1)$$

Proof. To see this we use the Bailey pair relative to q (Lemma 6 of [3]),

$$\alpha_n = (-x)^{-n}q^{\binom{n+1}{2}}(1 - x^{2n+1})$$

and

$$\beta_n = \frac{(x)_{n+1}(q/x)_n}{(q^2)_{2n}}$$

together with the fact that if (α_n, β_n) is a Bailey pair relative to a, then (Cor. 1.3 of [22])

$$\sum_{n\geq 0}(aq)_{2n}q^n \beta_n = \frac{1}{(q)_\infty}\sum_{r,n\geq 0}(-a)^n q^{3n(n+1)/2+(2n+1)r}\alpha_r. \qquad (4.2)$$

This gives

$$\sum_{n\geq 0}(x)_{n+1}(q/x)_n q^n = \frac{1}{(q)_\infty}\sum_{r,n\geq 0}(-1)^{n+r}x^{-r}q^{n(3n+5)/2+2nr+r(r+3)/2}(1-x^{2r+1}). \qquad (4.3)$$

Using $(1-x^{2r+1})$ to split the right-hand side into two sums and then replacing (r,n) by $(-r-1,-n-1)$ in the second sum yields the result. □

5. The vector-valued case

The first author [15] introduced and studied the modular properties of a family of q-series $F_t^{(m)}(q)$, defined for $1 \leq m \leq t$ by

$$F_t^{(m)}(q) := q^t \sum_{k_1,\ldots,k_t=0}^{\infty}(q)_{k_t}\, q^{k_1{}^2+\cdots+k_{t-1}{}^2+k_m+\cdots+k_{t-1}}\prod_{i=1}^{t-1}\begin{bmatrix}k_{i+1}+\delta_{i,m-1}\\k_i\end{bmatrix}_q. \qquad (5.1)$$

The case $m=1$ corresponds to (1.8), $F_t(q) = F_t^{(1)}(q)$.

He showed [15] that at the N-th root of unity we have

$$\zeta_N^{-t+\frac{(2t+1-2m)^2}{8(2t+1)}}F_t^{(m)}(\zeta_N) = (2t+1)\,N\sum_{k=1}^{4(2t+1)N}\chi_{8t+4}^{(m)}(k)\,\zeta_N^{\frac{k^2}{8(2t+1)}}B_2\left(\frac{k}{4(2t+1)N}\right), \qquad (5.2)$$

where the Bernoulli polynomial is $B_2(x)=x^2-x+\frac{1}{6}$, and the periodic function is

$$\chi_{8t+4}^{(m)}(k) := \begin{cases}1, & \text{when } k = \pm(2t+1-2m) \mod 8t+4,\\ -1, & \text{when } k = \pm(2t+1+2m) \mod 8t+4,\\ 0, & \text{otherwise.}\end{cases} \qquad (5.3)$$

This is a limiting value of the Eichler integral of a vector-valued modular form $\Phi_t^{(m)}(\tau)$ with weight $1/2$,

$$\Phi_t^{(m)}(\tau) := \sum_{n=0}^{\infty}\chi_{8t+4}^{(m)}(n)\,q^{\frac{1}{8(2t+1)}n^2} \qquad (5.4)$$

$$= q^{\frac{(2t+1-2m)^2}{8(2t+1)}}(q^m, q^{2t+1-m}, q^{2t+1}; q^{2t+1})_\infty,$$

where as usual $q = \mathrm{e}^{2\pi i\tau}$, and $(a, b, \cdots ; q)_\infty = (a)_\infty(b)_\infty\cdots$. Note that these q-series appeared in the Andrews–Gordon identities [1], the $t=2$ case of which corresponds to the Rogers–Ramanujan identities. The quantum modularity of $F_t^{(m)}(q)$ is given by

$$\phi_t^{(m)}(z) + \frac{1}{(i\,z)^{\frac{3}{2}}}\sum_{m'=1}^{t}\frac{2}{\sqrt{2t+1}}(-1)^{t+1+m+m'}\sin\left(\frac{2\,m\,m'}{2t+1}\pi\right)\phi_t^{(m')}(-1/z)$$

$$= \frac{\sqrt{(2t+1)i}}{2\pi}\int_0^{i\infty}\frac{\Phi_t^{(m)}(w)}{(w-z)^{\frac{3}{2}}}\,dw, \qquad (5.5)$$

where $z \in \mathbb{Q}$, and $\phi_t^{(m)}(\tau) := q^{-t+\frac{(2t+1-2m)^2}{8(2t+1)}}F_t^{(m)}(q)$. See [15] for details.

In this section we define q-series $U_t^{(m)}(q)$ so that $F_t^{(m)}(\zeta_N^{-1}) = U_t^{(m)}(-1; \zeta_N)$ (see Theorem 5.5) and we find Hecke-type formulae for $U_t^{(m)}(q)$ (see Theorem 5.6). We begin by defining an analogue of the colored Jones polynomial,

$$(1 - q^N) \, J_N^{(t,m)}(q) := (-1)^N \, q^{-t + \frac{N}{2} + \frac{2t+1}{2} N^2} \sum_{k=-N}^{N-1} (-1)^k \, q^{-\frac{2t+1}{2} k(k+1) + mk}. \qquad (5.6)$$

When $m = 1$ this coincides with the colored Jones polynomial $J_N^{(t,1)}(q) = J_N(T_{(2,2t+1)}^*; q)$.

Proposition 5.1.

$$J_N^{(t,m)}(\zeta_N) = F_t^{(m)}(\zeta_N^{-1}). \qquad (5.7)$$

Proof. The function $J_N^{(t,m)}(q)$ is also written by use of the periodic function (5.3) as

$$(1 - q^N) \, J_N^{(t,m)}(q) = (-1)^N \, q^{-t + \frac{1}{2} N + \frac{2t+1}{2} N^2} \sum_{k=1}^{2(2t+1)N} \chi_{8t+4}^{(m)}(k) \, q^{-\frac{k^2 - (2t+1-2m)^2}{8(2t+1)}}.$$

At $q \to \zeta_N$, we have

$$\zeta_N^{t - \frac{(2t+1-2m)^2}{8(2t+1)}} \, J_N^{(t,m)}(\zeta_N) = - \lim_{q \to \zeta_N} \frac{1}{1 - q^N} \sum_{k=1}^{2(2t+1)N} \chi_{8t+4}^{(m)}(k) \, q^{-\frac{k^2}{8(2t+1)}}$$

$$= -\frac{1}{8(2t+1)N} \sum_{k=1}^{2(2t+1)N} k^2 \, \chi_{8t+4}^{(m)}(k) \, \zeta_N^{-\frac{k^2}{8(2t+1)}}.$$

Using (5.3), the sum $\sum_{k=0}^{2(2t+1)N}$ may be replaced with $\frac{1}{2} \sum_{k=-2(2t+1)N}^{2(2t+1)N}$. Then the right hand side is written as

$$-\frac{1}{16(2t+1)N} \sum_{\ell=0}^{4(2t+1)N} (\ell - 2(2t+1)N)^2 \, \chi_{8t+4}^{(m)}(\ell - 2(2t+1)N) \, \zeta_N^{-\frac{(\ell - 2(2t+1)N)^2}{8(2t+1)}}.$$

Since $\chi_{8t+4}^{(m)}(k - 2(2t+1)) = -\chi_{8t+4}^{(m)}(k)$, we obtain

$$\zeta_N^{t - \frac{(2t+1-2m)^2}{8(2t+1)}} \, J_N^{(t,m)}(\zeta_N) = (2t+1) \, N \sum_{k=1}^{4(2t+1)N} \chi_{8t+4}^{(m)}(k) \, \zeta_N^{-\frac{k^2}{8(2t+1)}} \, B_2 \left(\frac{k}{4(2t+1)N} \right). \qquad (5.8)$$

Recalling (5.2), the statement follows. $\qquad \square$

Next we study a cyclotomic expansion for $J_N^{(t,m)}(q)$,

$$J_N^{(t,m)}(q) = \sum_{n=0}^{\infty} C_n^{(t,m)}(q) \, (q^{1+N})_n \, (q^{1-N})_n. \qquad (5.9)$$

Equation (2.11) shows that we have

$$C_{n-1}^{(t,m)}(q) = -q^{n-t} \sum_{\ell=0}^{n} \frac{1}{(q)_{n-\ell}(q)_{n+\ell}} \, q^{(t+1)\ell^2 - \ell} \, (1 - q^{2\ell}) \sum_{k=-\ell}^{\ell-1} (-1)^k q^{-\frac{2t+1}{2} k(k+1) + mk}. \qquad (5.10)$$

We note that $C_n^{(t,1)}(q) = C_n(T_{(2,2t+1)}^*; q)$.

The next two propositions are generalizations of Propositions 3.1 and 3.2, respectively.

Proposition 5.2. *We have*

$$-q^{t-n}C_{n-1}^{(t,m)} = \sum_{n \geq n_{2t-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{t-1} n_{t+i}^2 + \binom{n_t}{2} - \sum_{i=1}^{t-1} n_i n_{i+1} - \sum_{i=1}^{t-m-1} n_i}(-1)^{n_t}(1 - q^{n_t - \chi(t>m)n_{t-m}})}{(q)_{n-n_{2t-1}} \cdots (q)_{n_2-n_1}(q)_{n_1}}.$$

(5.11)

Proof. In light of equation (5.10) and the definition of a Bailey pair (2.1), we need to find β_n' such that

$$\alpha_n' = q^{(t+1)n^2 - n}(1 - q^{2n}) \sum_{k=-n}^{n-1} (-1)^k q^{-(2t+1)k^2/2 - (2t - (2m-1))k/2}.$$

(5.12)

The proof of this Bailey pair is exactly as in the proof of Proposition 3.1 but with $\ell = t - m$ instead of $t - 1$. □

Proposition 5.3. *We have*

$$C_n^{(t,m)}(q) = q^{n+1-t} \sum_{\substack{n+1=k_t \geq k_{t-1} \geq \cdots \geq k_1 \geq 0 \\ k_m \geq 1}} \prod_{a=1}^{t-1} q^{k_a^2} \frac{(q^{1-a+\sum_{i=1}^a (2k_i + \chi(m>i))})_{k_{a+1}-k_a}}{(q)_{k_{a+1}-k_a}}$$

(5.13)

$$= q^{n+1-t} \sum_{\substack{n+1=k_t \geq k_{t-1} \geq \cdots \geq k_1 \geq 0 \\ k_m \geq 1}} \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1} - k_i - i + \sum_{j=1}^i (2k_j + \chi(m>j)) \\ k_{i+1} - k_i \end{bmatrix}_q.$$

(5.14)

Proof. The proof is similar to that of Proposition 3.2. We begin by treating the sum over n_t, which is

$$\sum_{n_t=n_{t-1}}^{n_{t+1}} \frac{(-1)^{n_t} q^{\binom{n_t}{2} - n_{t-1}n_t}(1 - q^{n_t - \chi(t>m)n_{t-m}})}{(q)_{n_{t+1}-n_t}(q)_{n_t-n_{t-1}}}.$$

(5.15)

Assuming for the moment that $n_{t+1} > n_{t-1}$, the second part of identity (3.18) enables us to evaluate this sum, giving

$$-q^{t-n-1}C_n^{(t,m)}(q) =$$

$$\sum_{n+1 \geq n_{2t-1} \geq \cdots \geq n_{t+2} \geq n_{t+1} > n_{t-1} \geq n_{t-2} \geq \cdots \geq n_1 \geq 0} (-1)^{1+n_{t-1}} q^{\sum_{i=1}^{t-1} n_{t+i}^2 - \sum_{i=1}^{t-2} n_i n_{i+1} - \sum_{i=1}^{t-m} n_i}$$

$$\times \frac{q^{-\binom{n_{t-1}}{2}}}{(q)_{n+1-n_{2t-1}} \cdots (q)_{n_{t+2}-n_{t+1}}(q)_{n_{t-1}-n_{t-2}} \cdots (q)_{n_2-n_1}(q)_{n_1}}.$$

(5.16)

The rest of the proof is the same as the proof of Proposition 3.2. The only difference between equations (3.19) and (5.16) is that the latter contains the term $q^{-\sum_{i=1}^{t-m} n_i}$ instead of the term $q^{-\sum_{i=1}^{t-1} n_i}$, which results in (5.13) instead of (3.14).

Now suppose that $n_{t+1} = n_{t-1}$. The sum (5.15) on n_t is trivial and reduces to

$$(-1)^{n_{t-1}} q^{-\binom{n_{t-1}+1}{2}}(1 - q^{n_{t-1} - \chi(t>m)n_{t-m}}).$$

(5.17)

This corresponds to $k_1 = 0$, and the sum on n_{t-1} is then

$$\sum_{n_{t-1}=n_{t-2}}^{n_{t+2}} \frac{(-1)^{n_{t-1}} q^{\binom{n_{t-1}}{2} - n_{t-2}n_{t-1}}(1 - q^{n_{t-1} - \chi(t>m)n_{t-m}})}{(q)_{n_{t+2}-n_{t-1}}(q)_{n_{t-1}-n_{t-2}}}.$$

(5.18)

If $n_{t-2} = n_{t+2}$ then we collapse the sum again and obtain $k_2 = 0$, continuing in this way until $n_{t+a} > n_{t-a}$, and then applying (3.17) and arguing as usual. Note that if $n_{t+m} = n_{t-m}$ then the sum vanishes, so we have $k_m \geq 1$. □

Using the expression for $C_n^{(t,m)}(q)$, we are now prepared to generalize Definition 1.1.

Definition 5.4. *The generalized U-function $U_t^{(m)}(x;q)$ is defined by*

$$U_t^{(m)}(x;q) := \sum_{n=0}^{\infty} C_n^{(t,m)}(q)\,(-x\,q)_n\,(-x^{-1}q)_n \tag{5.19}$$

$$= q^{-t} \sum_{\substack{k_t \geq \cdots \geq k_1 \geq 0 \\ k_m \geq 1}} (-xq)_{k_t-1}\,(-x^{-1}q)_{k_t-1}\,q^{k_t}$$

$$\times \prod_{i=1}^{t-1} q^{k_i{}^2} \begin{bmatrix} k_{i+1} - k_i - i + \sum_{j=1}^{i}(2k_j + \chi(m>j)) \\ k_{i+1} - k_i \end{bmatrix}_q .$$

By construction, $U_t^{(m)}(-1;q)$ is dual to $F_t^{(m)}(q)$ as follows.

Theorem 5.5.

$$F_t^{(m)}(\zeta_N^{-1}) = U_t^{(m)}(-1;\zeta_N). \tag{5.20}$$

Proof. We have $J_N^{(t,m)}(q) = U_t^{(m)}(-q^N;q)$ from (5.9), thus we get

$$J_N^{(t,m)}(\zeta_N) = U_t^{(m)}(-1;\zeta_N). \tag{5.21}$$

With the help of (5.7), we get (5.20). □

We end this section with the Hecke-type formula for $U_t^{(m)}(x;q)$. These follow just as those for $U_t(x;q)$ in Theorem 1.3, using the Bailey pair in Proposition 5.2 in place of the Bailey pair in Proposition 3.1

Theorem 5.6.

$$U_t^{(m)}(-x;q) = -q^{-\frac{t}{2}-\frac{m}{2}+\frac{3}{8}}\frac{(xq)_\infty(q/x)_\infty}{(q)_\infty^2} \tag{5.22}$$

$$\times \left(\sum_{\substack{r,s\geq 0 \\ r\not\equiv s \pmod 2}} - \sum_{\substack{r,s<0 \\ r\not\equiv s \pmod 2}} \right) \frac{(-1)^{\frac{r-s-1}{2}}q^{\frac{1}{8}r^2+\frac{4t+3}{4}rs+\frac{1}{8}s^2+\frac{1+m+t}{2}r+\frac{1-m+t}{2}s}}{1-xq^{\frac{r+s+1}{2}}}$$

$$= -q^{-\frac{t}{2}-\frac{m}{2}+\frac{3}{8}}\frac{(xq)_\infty(q/x)_\infty}{(q)_\infty^2} \tag{5.23}$$

$$\times \left(\sum_{\substack{r,s,u\geq 0 \\ r\not\equiv s \pmod 2}} + \sum_{\substack{r,s,u<0 \\ r\not\equiv s \pmod 2}} \right) (-1)^{\frac{r-s-1}{2}}x^u q^{\frac{1}{8}r^2+\frac{4t+3}{4}rs+\frac{1}{8}s^2+\frac{1+m+t}{2}r+\frac{1-m+t}{2}s+u\frac{r+s+1}{2}} .$$

6. Concluding Remarks

In this paper we have studied a family of quantum modular forms $U_t(-1; q)$ based on the colored Jones polynomial for the torus knot $T_{(2,2t+1)}$. We have extended the duality between $U_1(-1; q)$ and $F_1(q)$ to general t and determined a Hecke-type expansion for $U_t(x; q)$. We have further generalized these results to the vector-valued setting.

We close with two remarks.

First, in [4, 9] the modular transformation formula was given for $U_1(-1; q)$ for generic q, based on an expression for $U_1(x; q)$ in terms of Appell–Lerch series in [4]. The expression in terms of Appell–Lerch series also shows that $U_1(x; q)$ is a mixed mock modular form for generic roots of unity $x \neq -1$ and a mock theta function when $x = \pm i$. Similar results may hold in the general case, and it is to be hoped that the Hecke series expansions established in this paper will turn out to be useful for determining modular transformation formulae for $U_t^{(m)}(x; q)$ for generic q and for x a root of unity. For now, we only know that by Theorem 5.5, $U_t^{(m)}(-1; q)$ fulfills (5.5) when q is a root of unity.

Second, both $U_1(x; q)$ and $F_1(q)$ are interesting combinatorial generating functions. The two-variable function $U_1(x; q)$ can be interpreted in terms of strongly unimodal sequences and their ranks [7, 9], while $F_1(1 - q)$ is the generating function for certain linearized chord diagrams [29] (as well as a number of other objects - see [28, A022493] for an overview with references.) Moreover, there are many nice congruences for the coefficients of $F_1(1 - q)$ [5] and $U_1(1; q)$ [9]. The generalized $U_t^{(m)}(x; q)$ and $F_t^{(m)}(1 - q)$ may have interesting combinatorial interpretations and congruence properties as well.

Acknowledgements

A. Examples: $t = 2$ and 3

Here we record the set of generalized Kontsevich–Zagier series $F_t^{(m)}(q)$ and generalized U-functions $U_t^{(m)}(x; q)$ for $t = 2$ and 3.

When $t = 2$, the set of generalized Kontsevich–Zagier series is

$$F_2^{(1)}(q) = q^2 \sum_{n=0}^{\infty} (q)_n \sum_{k=0}^{n} q^{k(k+1)} \begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{A.1}$$

$$F_2^{(2)}(q) = q^2 \sum_{n=1}^{\infty} (q)_{n-1} \sum_{k=0}^{n} q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{A.2}$$

The dual U-functions, satisfying $F_2^{(m)}(\zeta_N^{-1}) = U_2^{(m)}(-1; \zeta_N)$, are given by

$$U_2^{(1)}(x; q) = \sum_{n=0}^{\infty} (-xq)_n \, (-x^{-1}q)_n q^{n-1} \sum_{k=1}^{n+1} q^{k^2} \begin{bmatrix} n+k \\ 2k-1 \end{bmatrix}_q$$
$$= 1 + q + (x + 2 + x^{-1})q^2 + (2x + 3 + 2x^{-1})q^3 + (3x + 6 + 3x^{-1})q^4 + \cdots$$
(A.3)

$$U_2^{(2)}(x; q) = \sum_{n=0}^{\infty} (-xq)_n (-x^{-1}q)_n \, q^{n-1} \sum_{k=0}^{n+1} q^{k^2} \begin{bmatrix} n+k+1 \\ 2k \end{bmatrix}_q$$
$$= q^{-1} + 2 + (x + 2 + x^{-1})q + (2x + 4 + 2x^{-1})q^2 + (4x + 6 + 4x^{-1})q^3 + \cdots$$
(A.4)

When $t = 3$ the set of generalized Kontsevich–Zagier series is

$$F_3^{(1)}(q) = q^3 \sum_{n=0}^{\infty} (q)_n \sum_{k=0}^{n} q^{k(k+1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{j=0}^{k} q^{j(j+1)} \begin{bmatrix} k \\ j \end{bmatrix}_q, \qquad (A.5)$$

$$F_3^{(2)}(q) = q^3 \sum_{n=0}^{\infty} (q)_n \sum_{k=1}^{n+1} q^{k(k-1)} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \sum_{j=0}^{k} q^{j^2} \begin{bmatrix} k \\ j \end{bmatrix}_q, \qquad (A.6)$$

$$F_3^{(3)}(q) = q^3 \sum_{n=1}^{\infty} (q)_{n-1} \sum_{k=0}^{n} q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{j=0}^{k} q^{j^2} \begin{bmatrix} k \\ j \end{bmatrix}_q. \qquad (A.7)$$

The dual U-functions, satisfying $F_3^{(m)}(\zeta_N^{-1}) = U_3^{(m)}(-1; \zeta_N)$, are given by

$$U_3^{(1)}(x; q) = \sum_{n=0}^{\infty} \sum_{k=1}^{n+1} \sum_{j=1}^{k} (-xq)_n \, (-x^{-1}q)_n q^{n-2+k^2+j^2} \begin{bmatrix} k+j-1 \\ 2j-1 \end{bmatrix}_q \begin{bmatrix} n+k+2j-1 \\ 2k+2j-2 \end{bmatrix}_q$$
$$= 1 + q + (x + 2 + x^{-1})q^2 + (2x + 4 + 2x^{-1})q^3 + (4x + 7 + 4x^{-1})q^4 + \cdots$$
(A.8)

$$U_3^{(2)}(x; q) = \sum_{n=0}^{\infty} \sum_{k=1}^{n+1} \sum_{j=0}^{k} (-xq)_n \, (-x^{-1}q)_n q^{n-2+k^2+j^2} \begin{bmatrix} k+j \\ 2j \end{bmatrix}_q \begin{bmatrix} n+k+2j \\ 2k+2j-1 \end{bmatrix}_q$$
$$= q^{-1} + 2 + (x + 3 + x^{-1})q + (3x + 5 + 3x^{-1})q^2 + (5x + 10 + 5x^{-1})q^3 + \cdots$$
(A.9)

$$U_3^{(3)}(x; q) = \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} \sum_{j=0}^{k} (-xq)_n \, (-x^{-1}q)_n q^{n-2+k^2+j^2} \begin{bmatrix} k+j \\ 2j \end{bmatrix}_q \begin{bmatrix} n+k+2j+1 \\ 2k+2j \end{bmatrix}_q$$
$$= q^{-2} + 2q^{-1} + (x + 3 + x^{-1}) + (2x + 5 + 2x^{-1})q + (5x + 8 + 5x^{-1})q^2 + \cdots$$
(A.10)

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