RANK AND CONJUGATION FOR THE FROBENIUS REPRESENTATION OF AN OVERPARTITION

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Abstract. We discuss conjugation and Dyson's rank for overpartitions from the perspective of the Frobenius representation. More specifically, we translate the classical definition of Dyson's rank to the Frobenius representation of an overpartition and define a new kind of conjugation in terms of this representation. We then use q-series identities to study overpartitions that are self-conjugate with respect to this conjugation.

1. Introduction

Rank and conjugation are two of the most important notions in the theory of partitions. Since it was defined by Dyson some 60 years ago [18], the rank has played a role in studies of Ramanujan-type congruences (*e.g.* [12, 18, 21]), mock theta functions (*e.g.* [8, 24, 25]), and Rogers-Ramanujan type identities (*e.g.* [1, 5, 13, 17]), as well as in a variety of other studies related to partitions (*e.g.* [6, 11, 19, 20]). Conjugation, meanwhile, is an indispensable tool in the elementary and bijective theory of partitions (*e.g.* [4, 29]). The purpose of this paper is to begin to develop these notions as they apply to overpartitions. In particular, we define a new kind of conjugation, called F-conjugation, in terms of the Frobenius representation of an overpartition. We then employ q-series identities to study overpartitions that are F-self-conjugate.

Recall that an overpartition is a partition in which the final (or equivalently, first) occurrence of a part may be overlined [15]. The classical definition of Dyson's rank, hereafter called the D-rank, as the largest part minus the number of parts, is naturally inherited by overpartitions. The classical definition of conjugation also carries over quite nicely to the Ferrers diagram of an overpartition [15]. However, from the q-series perspective, it will be natural to introduce a different conjugation, one defined in terms of the Frobenius representation of an overpartition. This should not come a surprise, given the importance of this representation in recent works [14, 15, 16, 28]. This F-conjugation is actually a generalization of the fact that conjugation for an ordinary partition corresponds to interchanging the rows in the Frobenius symbol.

In the next section we will recall the relevant definitions, generating functions, and combinatorial algorithms from the elementary theory of overpartitions. In Section 3 we establish some basic generating functions, including the fundamental generating function for the D-rank.

Proposition 1.1. *If $\overline{p}(m,n)$ denotes the number of overpartitions of n with D-rank m, then*

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\overline{p}(m,n)z^mq^n=\sum_{n=0}^{\infty}\frac{(-1)_nq^{n(n+1)/2}}{(zq,q/z)_n}. \tag{1.1}$$

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Here we have employed the standard q-series notation

$$(a_1, ..., a_j)_n = (a_1, ..., a_j; q)_n = \prod_{k=0}^{n-1} (1 - a_1 q^k) \cdots (1 - a_j q^k). \tag{1.2}$$

Then, in Section 4, we will be able to explain what it means to conjugate the Frobenius representation of an overpartition. This conjugation is tailored to fit in naturally with the theory of basic hypergeometric series. There are numerous interesting identities for q-series that are related to (1.1), and we shall pay special attention in this paper to interpreting identities that in some way correspond to F-self-conjugate overpartitions. One simple example is the following:

Theorem 1.2. *Let $g^{\pm}(n)$ denote the number of F-self-conjugate overpartitions whose Frobenius representation has an even/odd number of columns. Then for $n \geq 1$ we have*

$$g^+(n) - g^-(n) = \begin{cases} 2(-1)^n, & \text{if } n \text{ is a square,} \\ 0, & \text{otherwise.} \end{cases}$$

A more complicated example is:

Theorem 1.3. *Let $h^{\pm}(n)$ denote the number of F-self-conjugate overpartitions whose Frobenius representation has the D-rank on the bottom row even/odd. If we write $n = 2^f m$, then for $n \geq 1$ we have*

$$h^+(n) - h^-(n) = \begin{cases} 2d(m), & \text{if } f = 0 \text{ and } m \equiv 3 \pmod 4 \\ -2d(m), & \text{if } f = 2 \text{ or } f = 0 \text{ and } m \equiv 1 \pmod 4 \\ 0, & \text{if } f = 1 \\ 2(5-f)d(m), & \text{if } f \geq 3, \end{cases}$$

where $d(m)$ is the ordinary divisor function.

We prove numerous other theorems of this type in Section 5. In Section 6, we discuss correspondences among F-self-conjugate overpartitions, odd-even overpartitions, and partitions into distinct parts whose smallest part is odd, interpreting the results of Section 5 in these new contexts. To finish, we offer some suggestions for future research.

2. Combinatorial Preparation

We begin with some recollections about overpartitions, mainly from the foundations laid in [15]. First, an overpartition can be represented by an ordinary Ferrers diagram in which the corners may be colored. Conjugating, or reading the columns of the diagram, gives another overpartition. Next, those familiar with [15] will recall a certain statistic that we called the rank and defined to be the number of *eligible* integers that are less than the largest part but *do not occur overlined* in the overpartition. We use the term eligible integers here in order to define this rank at once for overpartitions into positive parts and overpartitions into non-negative parts. The rank and the D-rank of an overpartition into positive parts are generally not the same, although they are if the overpartition has no non-overlined parts. We shall often require this old rank, as it arises naturally in generating functions for overpartitions:

Proposition 2.1 ([15]). *Let $\overline{p}_{k,l,m}(n)$ denote the number of overpartitions of n into k parts with l overlined parts and rank m. Then*

$$\sum_{l,m,n=0}^{\infty} \overline{p}_{k,l,m}(n) a^l b^m z^k q^n = \frac{(-a)_k (zq)^k}{(bq)_k}. \tag{2.1}$$

It will be helpful to recall the proof of this proposition, which we now transcribe from [15] and refer to as Agorithm I. It is originally due to Joichi and Stanton [26]

Algorithm I. The generating function for overpartitions with a given number of parts. The function $(zq)^k/(bq;q)_k$ generates a partition λ into k positive (non-overlined) parts, where the exponent of z keeps track of the number of parts and the exponent of b records the largest part minus 1. Note that since there are not yet any overlined parts, this is the same as the rank. Now $(-a;q)_k$ generates a partition $\mu = \mu_1 + \cdots + \mu_j$ into distinct non-negative parts less than k, with the exponent of a tracking the number of parts. For each of these μ_i beginning with the largest, we add 1 to the first μ_i parts of λ, and then overline the $(\mu_i + 1)$th part of λ. Here the parts of λ are written in non-increasing order. This operation leaves the rank invariant and counts one overlined part for each part of μ. For example, if $k = 5$, $\lambda = 8 + 4 + 4 + 2 + 1$, and $\mu = 4 + 3 + 0$, then we have

$$\begin{aligned}(8+4+4+2+1, 4+3+0) &\iff (9+5+5+3+\overline{1}, 3+0)\\ &\iff (10+6+6+\overline{3}+\overline{1}, 0)\\ &\iff (\overline{10}+6+6+\overline{3}+\overline{1})\end{aligned}$$

The result is obviously an overpartition and the process is easily inverted.

We shall also need to refer to the co-rank of an overpartition, defined to be the number of overlined parts less than the largest part [15]. From Algorithm I, it is easy to see that if $\overline{p}_{k,m}(n)$ denotes the number of overpartitions of n into exactly k parts with co-rank m, then for $k \geq 1$ we have

$$\sum_{m,n} \overline{p}_{k,m}(n) a^m q^n = 2\frac{(-aq)_{k-1} q^k}{(q)_k}. \tag{2.2}$$

Now we turn to the Frobenius representation of an overpartition. Following [2], a Frobenius partition of n is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_k \\ b_1 & b_2 & \ldots & b_k \end{pmatrix} \tag{2.3}$$

where $\sum a_i$ is a partition taken from a set A, $\sum b_i$ is a partition taken from a set B, and $k + \sum(a_i + b_i) = n$. The number of such Frobenius partitions of n is denoted by $p_{A,B}(n)$. In [15], it is shown that if Q denotes the set of partitions into distinct non-negative parts and if $\mathcal{O}$ denotes the set of overpartitions into non-negative parts, then $p_{Q,\mathcal{O}}(n)$ is equal to the number of overpartitions of n. More specifically, we have the following:

Proposition 2.2. *There is a one-to-one correspondence between overpartitions λ of n and Frobenius partitions ν counted by $p_{Q,\mathcal{O}}(n)$ in which (i) the number of overlined parts in λ is equal to the number of non-overlined parts in the bottom row of ν, (ii) the number of parts in λ is equal to the sum of the number of columns and the rank of the bottom row of ν, and (iii) the largest part of λ is one more than the largest part of the top row of ν.*

Again it will helpful to recall the proof , so we transcribe it from [15] and refer to it as Algorithm II.

Algorithm II. The Frobenius representation of an overpartition. We use the notion of a *hook*. Given a positive integer a and a non-negative integer b, $h(a,b)$ is the hook that corresponds to the partition $(a,1,\dots,1)$ where there are b ones. Combining a hook $h(a,b)$ and a partition α is possible if and only if $a > \alpha_1$ and $b \geq l(\alpha)$, where $l(\alpha)$ denotes the number of parts of α. The result of the union is $\beta = h(a,b) \cup \alpha$ with $\beta_1 = a$, $l(\beta) = b+1$ and $\beta_i = \alpha_{i-1}+1$ for $i > 1$.

Now take a Frobenius partition ν counted by $p_{Q,\mathcal{O}}(n)$, increase the entries on the top row by 1 and initialize α and β to the empty object, ϵ. Beginning with the rightmost column of ν, we proceed to the left, building α into an ordinary partition and β into a partition into distinct parts. At the i^{th} column, if b_i is overlined, then we add the hook $h(a_i,b_i)$ to α. Otherwise, we add the part b_i to α' (the conjugate of α) and the part a_i to β. Joining the parts of α together with the parts of β gives the overpartition λ. An example is given below starting with $\nu = \begin{pmatrix} 7 & 5 & 4 & 2 & 0 \\ 6 & \overline{4} & 4 & 3 & \overline{1} \end{pmatrix}$.

ν	α	β
$\begin{pmatrix} 8 & 6 & 5 & 3 & 1 \\ 6 & \overline{4} & 4 & 3 & \overline{1} \end{pmatrix}$	ϵ	ϵ
$\begin{pmatrix} 8 & 6 & 5 & 3 \\ 6 & \overline{4} & 4 & 3 \end{pmatrix}$	$(1,1)$	ϵ
$\begin{pmatrix} 8 & 6 & 5 \\ 6 & \overline{4} & 4 \end{pmatrix}$	$(2,2,1)$	(3)
$\begin{pmatrix} 8 & 6 \\ 6 & \overline{4} \end{pmatrix}$	$(3,3,2,1)$	$(5,3)$
$\begin{pmatrix} 8 \\ 6 \end{pmatrix}$	$(6,4,4,3,2)$	$(5,3)$
ϵ	$(7,5,5,4,3,1)$	$(8,5,3)$

We get $\lambda = (\bar{8},7,\bar{5},5,5,4,\bar{3},3,1)$. The reverse bijection is easily described. Given α and β, we set the Frobenius partition equal to ϵ. We proceed until α and β are empty, at each step adding a column to the Frobenius partition according to the following rule: If $\beta_1 \geq \alpha_1$ then add the column $\begin{pmatrix} \beta_1 \\ l(\alpha) \end{pmatrix}$ and decrease the parts of α by 1 and delete the largest part of β. Otherwise add the column $\begin{pmatrix} \alpha_1 \\ \overline{l(\alpha)-1} \end{pmatrix}$ and we delete the hook $h(\alpha_1, l(\alpha)-1)$ from α. Finally, decrease by 1 the entries of the top row. For example, using this recipe one easily traces the pair (α,β) above back to the Frobenius partition.

3. The D-rank and basic generating functions

Using the definition of the D-rank as the largest part minus the number of parts, one finds nice generating functions for the number of overpartitions of n with D-rank m. For example, summing according to the largest part of the overpartition gives

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\overline{p}(m,n)z^m q^n = 1+2z^{-1}\sum_{n=1}^{\infty}\frac{(-q/z)_{n-1}(zq)^n}{(q/z)_n}, \tag{3.1}$$

while summing according to the Durfee square size of the overpartition gives

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\overline{p}(m,n)z^m q^n = \sum_{n=0}^{\infty}\frac{(-zq,-q/z)_n q^{n^2}(1+q^{2n+1})}{(zq,q/z)_n}. \tag{3.2}$$

However, as mentioned in the introduction, we shall find it most fruitful to think in terms of the Frobenius representation. From Algorithm II we may deduce the following alternative definition of the D-rank:

Definition 3.1. *In terms of its Frobenius representation, the D-rank of an overpartititon is the largest part on the top row minus the largest part on the bottom row minus the number of non-overlined parts less than the largest on the bottom row.*

For example, the D-rank of the overpartition presented in Algorithm II is -1. We are now prepared to prove (1.1).

Proof of Proposition 1.1. Considering the series in (1.1) as a generating function for Frobenius representations of overpartitions, the factor $q^{n(n-1)/2}/(zq)_n$ generates the top row, where the exponent of z counts the rank. The factor $(-1)_n/(q/z)_n$ generates the bottom row, and the exponent of z is the negative of the rank. Hence, the exponent of z in this generating function counts the rank of the top row minus the rank of the bottom row, which is one more than the largest part on top row minus the number of parts (on top) minus the largest part on the bottom row plus the number of overlined parts less than the largest on the bottom row, which is the same as the definition of D-rank in Definition 3.1. □

Before moving on to the definition of the Frobenius conjugation, we record some of the basic generating functions for the rank and D-rank, in the spirit of Dyson [12, 18].

Proposition 3.2. *For any integer m we have*

$$\sum_{n=1}^{\infty}\overline{p}(m,n)q^n = 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2+|m|n}(1-q^n)}{(1+q^n)}. \tag{3.3}$$

Proof. We employ Watson's transformation,

$$\sum_{n=0}^{\infty}\frac{(aq/bc,d,e)_n(\frac{aq}{de})^n}{(q,aq/b,aq/c)_n} = \frac{(aq/d,aq/e)_\infty}{(aq,aq/de)_\infty}\sum_{n=0}^{\infty}\frac{(a,\sqrt{a}q,-\sqrt{a}q,b,c,d,e)_n(aq)^{2n}(-1)^n q^{n(n-1)/2}}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e)_n(bcde)^n}. \tag{3.4}$$

Therein we take $a=1$, $b=z$, $c=1/z$, $d=-1$, and let $e\to\infty$. The result is

$$\sum_{n=0}^{\infty}\frac{(-1)_n q^{n(n+1)/2}}{(zq,q/z)_n} = \frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{n=1}^{\infty}\frac{(1-z)(1-1/z)(-1)^n q^{n^2+n}}{(1-zq^n)(1-q^n/z)}\right).$$

Now it is easily verified that

$$\frac{(1-z)(1-1/z)q^n}{(1-zq^n)(1-q^n/z)} = 1-\frac{(1-q^n)}{(1+q^n)}\sum_{m=0}^{\infty}z^m q^{mn}-\frac{(1-q^n)}{(1+q^n)}\sum_{m=1}^{\infty}z^{-m}q^{mn}.$$

Substituting this into the above equation and picking off the coefficient of z^m immediately gives the desired result for $m > 0$, as well as for $m < 0$ by applying the classical conjugation of Ferrers diagrams. For $m = 0$, we get

$$\sum_{n=1}^{\infty} \overline{p}(0,n)q^n = -1 + \frac{(-q)_\infty}{(q)_\infty}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}(1-q^n)}{(1+q^n)}\right),$$

and we must observe that the first two terms inside the parentheses can be summed to $(q)_\infty/(-q)_\infty$ by Jacobi's triple product identity,

$$\sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2} = (-1/z, -zq, q)_\infty. \tag{3.5}$$

□

Proposition 3.3. *If $\overline{p}_m(n)$ denotes the number of overpartitions of n with rank m, then for $m \geq 0$ we have*

$$\sum_{n=1}^{\infty} \overline{p}_m(n)q^n = 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n(n+1)/2+mn}(1-q^n)}{(q)_n(1+q^n)}. \tag{3.6}$$

Proof. We again use (3.4), this time letting $a = -e$, $b = -e/z$, $c = z$, and $d = -1$, and then letting $e \to 0$. The result is

$$\sum_{n=0}^{\infty}\frac{(-1)_n q^n}{(zq)_n} = 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}\frac{(1-z)(-1)^n q^{n(n+3)/2}}{(q)_n(1-zq^n)(1+q^n)}.$$

Expanding $(1-z)/(1-zq^n)$ using the binomial series and then picking off the coefficient of z^m immediately gives the desired result for $m > 0$. For $m = 0$, we get

$$\sum_{n=1}^{\infty}\overline{p}_0(n)q^n = -1 + 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+3)/2}}{(q)_n(1+q^n)}, \tag{3.7}$$

but we can use the q-Gauss identity,

$$\sum_{n=0}^{\infty}\frac{(a,b)_n(c/ab)^n}{(q,c)_n} = \frac{(c/a,c/b)_\infty}{(c,c/ab)_\infty}, \tag{3.8}$$

to deduce that

$$-1 = -2\frac{(-q)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(q)_n(1+q^n)} \tag{3.9}$$

and then substitute (3.9) back into (3.7). A little simplification then settles the case $m = 0$. □

4. Conjugating the Frobenius representation of an overpartition

We turn now to the definition of conjugation for the Frobenius representation of an overpartition. The idea is to swap the roles played by the partitions generated by $1/(zq)_n$ and $1/(q/z)_n$ in (1.1).

Algorithm III. Conjugation of the Frobenius representation of an overpartition. Starting with the top row of the Frobenius representation, remove a staircase - 0 from the smallest part, 1 from the next smallest, and so on, to get a partition λ_1 into n non-negative parts. Next take the overpartition in the bottom row and do Algorithm I in reverse to get a second partition λ_2 into n non-negative parts and a partition μ into distinct non-negative parts less than n. Now swap λ_1 and λ_2, add the staircase back onto λ_2 to form the new top row, and perform Algorithm I with μ and λ_1 to get the new bottom row.

For example, let's take the overpartition whose Frobenius representation is $\begin{pmatrix} 7 & 5 & 4 & 2 & \overline{0} \\ 6 & \overline{4} & 4 & 3 & \overline{1} \end{pmatrix}$. Then λ_1 is equal to $(3, 2, 2, 1, 0)$, μ is equal to $(4, 1)$, and λ_2 is equal to $(4, 3, 3, 2, 1)$. Interchanging λ_1 and λ_2 and reassembling gives the overpartition $\begin{pmatrix} 8 & 6 & 5 & 3 & \overline{1} \\ 5 & \overline{3} & 3 & 2 & \overline{0} \end{pmatrix}$. For another example, we leave it to the reader to verify that $\begin{pmatrix} 12 & \overline{8} & 7 & 5 & \overline{3} & 0 \\ \overline{9} & 5 & 5 & 4 & \overline{2} & 0 \end{pmatrix}$ is F-self-conjugate.

Having defined this conjugation, a number of comments are in order. First, if there are no non-overlined parts in the bottom row, then Algorithm III just interchanges the rows of the Frobenius symbol. Second, the F-conjugation of an overpartition does not correspond to the classical conjugation of Ferrers diagrams. One way to see this is to observe (computationally) that the number of F-self-conjugate overpartitions of n dominates the number of overpartitions of n that are classically self-conjugate. Finally, we point out an easy way to check if an overpartition having Frobenius representation (2.3) is F-self-conjugate: such an overpartition is F-self-conjugate if (i) $a_k - b_k = 0$ and (ii) for $1 \leq i \leq k-1, a_i - b_i = a_{i+1} - b_{i+1} + \delta(i+1)$, where $\delta(j)$ is 1 if b_j is non-overlined and 0 otherwise.

5. Theorems on self-conjugate overpartitions

Self-conjugate partitions are encountered early on in the study of partitions. Their generating function is an infinite product, making them the subject of one of the most elementary partition theorems: the number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts. This theorem is easily proved by reading the hooks of the Ferrers diagram, and this proof easily extends to the case of overpartitions. The result is the following.

Theorem 5.1. *The number of overpartitions of n with k overlined parts whose Ferrers diagram is self-conjugate is equal to the number of overpartitions of n into distinct odd parts with $\lceil k/2 \rceil$ overlined parts where two parts differ by at least 4 if the larger is overlined.*

Now the generating function for overpartitions that are F-self-conjugate is

$$\sum_{n=0}^{\infty} \frac{(-1)_n q^{n(n+1)/2}}{(q^2;q^2)_n} = 1 + 2q + 4q^3 + 2q^4 + 4q^5 + 4q^6 + 8q^7 + 8q^8 + 10q^9 + \cdots, \tag{5.1}$$

which is not infinite product. Using elementary q-series transformations, however, we can still prove some identities. These involve differences of overpartition functions. To assist the reader interested in verifying the theorems that follow for small values of n, we note that the F-self-conjugate overpartitions of 3 are $\begin{pmatrix} \overline{1} \\ 1 \end{pmatrix}$, $\begin{pmatrix} \overline{1} \\ \overline{1} \end{pmatrix}$, $\begin{pmatrix} 1 & \overline{0} \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 1 & \overline{0} \\ \overline{0} & 0 \end{pmatrix}$, while the F-self-conjugate overpartitions of 4 are $\begin{pmatrix} 1 & \overline{0} \\ 1 & \overline{0} \end{pmatrix}$ and $\begin{pmatrix} 1 & \overline{0} \\ \overline{1} & \overline{0} \end{pmatrix}$.

Theorem 5.2. *Let $a^{\pm}(n)$ denote the number of overpartitions of n having the largest non-overlined part minus the number of non-overlined parts even/odd. Then the number of F-self-conjugate overpartitions of n is equal to $a^+(n) - a^-(n)$.*

Proof. We employ Jackson's transformation,

$$\sum_{n=0}^{\infty} \frac{(a,b)_n z^n}{(c,q)_n} = \frac{(az)_\infty}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(a,c/b)_n (-bz)^n q^{n(n-1)/2}}{(c,az,q)_n}, \tag{5.2}$$

setting $b = -1$, $c = -q$, $z = -q/a$, and then letting $a \to \infty$. The result is

$$\sum_{n=0}^{\infty} \frac{(-1)_n q^{n(n+1)/2}}{(q^2;q^2)_n} = (-q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2}, \tag{5.3}$$

and it is easy to see that the sum on the right is the generating function for $a^+(n) - a^-(n)$. $\square$

Theorem 5.3. *Let $b^{\pm}(n)$ denote the number of overpartitions of n where (i) there is at least one non-overlined part, (ii) the smallest non-overlined part cannot also occur overlined, and (iii) the number of parts minus the number of overlined parts less than the smallest non-overlined part is even/odd. Then the number of F-self-conjugate overpartitions of n is equal to*

$$\begin{cases} 4(b^-(n) - b^+(n)) + 2(-1)^n, & \text{if } n \text{ is a square} \\ 4(b^-(n) - b^+(n)), & \text{otherwise.} \end{cases}$$

Proof. We employ the second iteration of Heine's transformation,

$$\sum_{n=0}^{\infty} \frac{(a,b)_n z^n}{(c,q)_n} = \frac{(c/b,bz)_\infty}{(c,z)_\infty} \sum_{n=0}^{\infty} \frac{(abz/c,b)_n (c/b)^n}{(q,bz)_n}, \tag{5.4}$$

setting $b = -1$, $c = -q$, $z = -q/a$, and then letting $a \to \infty$. The result is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)_n q^{n(n+1)/2}}{(q^2;q^2)_n} &= \frac{(q)_\infty}{(-q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)_n^2 q^n}{(q)_n} \\ &= \frac{(q)_\infty}{(-q)_\infty} + 4 \sum_{n=1}^{\infty} \frac{(-q)_{n-1} (q^{n+1})_\infty q^n}{(-q^n)_\infty}. \end{aligned}$$

It is straightforward to see that the sum on the right is the generating function for $4(b^-(n) - b^+(n))$, while the product on the right contributes the extra term $2(-1)^n$ as in the proof of Proposition 3.2. $\square$

We now turn to some q-series identities that link F-self-conjugate overpartitions to theta series and/or divisor functions.

Theorem 5.4. *Let $c^{\pm}(n)$ denote the number of F-self-conjugate overpartitions of n whose Frobenius representation has a bottom row with even/odd co-rank. If we write $n = 2^e m$ where m is odd, then for $n \geq 1$ we have*

$$c^+(n) - c^-(n) = 2(1-e)d(m),$$

where $d(m)$ is the classical divisor function.

Proof. From the elementary combinatorics of overpartitions described in Section 2, we have that

$$\sum_{n=1}^{\infty}(c^{+}(n)-c^{-}(n))q^{n}=2\sum_{n=1}^{\infty}\frac{(q)_{n-1}q^{n(n+1)/2}}{(q^{2};q^{2})_{n}}.$$

Now, in (3.4), let $a=-1$, $b=1$, $c,d\to\infty$, and then take the derivative with respect to e and set $e=1$. The result is

$$\sum_{n=1}^{\infty}\frac{(q)_{n-1}q^{n(n+1)/2}}{(q^{2};q^{2})_{n}}=\sum_{n=1}^{\infty}\frac{q^{n}}{1+q^{n}},$$

the right hand side being the generating function for the number of odd divisors minus the number of even divisors. The formula for $c^{+}(n)-c^{-}(n)$ follows. □

Next we treat the same generating function as above except we multiply the summand by $(-1)^{n}$. Here we will use the notation $d'(m)$ to denote the number of odd divisors of m that occur outside the interval $[\sqrt{2m/3},\sqrt{6m}]$. We note that these kinds of "middle divisor" functions have arisen in [10, 15].

Theorem 5.5. *Let $e^{\pm}(n)$ denote the number of F-self-conjugate overpartitions of n whose Frobenius representation has the number of columns plus the co-rank of the bottom row even/odd. Then for $n\geq 1$ we have*

$$e^{+}(n)-e^{-}(n)=2(2d'(n)-d(n)).$$

Proof. This proof is much like the previous one. We have

$$\sum_{n=1}^{\infty}(e^{+}(n)-e^{-}(n))q^{n}=2\sum_{n=1}^{\infty}\frac{(q)_{n-1}(-1)^{n}q^{n(n+1)/2}}{(q^{2};q^{2})_{n}}.$$

In (3.4), let $a=1$, $b=-1$, $c,d\to\infty$, and then take the derivative with respect to e and set $e=1$. The result is

$$\sum_{n=1}^{\infty}\frac{(q)_{n-1}(-1)^{n}q^{n(n+1)/2}}{(q^{2};q^{2})_{n}}=2\sum_{n=1}^{\infty}\frac{q^{n(3n+1)/2}}{(1-q^{n})}-\sum_{n=1}^{\infty}q^{n^{2}}-2\sum_{n=1}^{\infty}\frac{q^{n^{2}+n}}{(1-q^{n})}.$$

As before, the last two terms contribute the ordinary divisor function, while expanding $1/(1-q^{n})$ reveals the function $d'(n)$. □

Theorem 5.6. *Let $f^{\pm}(n)$ denote the number of F-self-conjugate overpartitions of n whose Frobenius representation has the rank of the top row even/odd. Then for $n\geq 1$ we have*

$$f^{+}(n)-f^{-}(n)=\begin{cases}2, & \text{if } n \text{ is a square},\\ 0, & \text{otherwise}.\end{cases}$$

Proof. We let $a=-q$, $b=iq$, $c=-iq$, $d=q$, and $e\to\infty$ in Watson's transformation. The result is

$$\sum_{n=0}^{\infty}\frac{(-1)_{n}q^{n(n+1)/2}}{(-q^{2};q^{2})_{n}}=\sum_{n=0}^{\infty}q^{n^{2}}(1+q^{2n+1}),$$

the left hand side is easily recognizable as the generating function for $f^{+}(n)-f^{-}(n)$. □

We conclude this section by proving the two theorems mentioned in the introduction.

Proof of Theorem 1.2. We have

$$\begin{aligned}\sum_{n=0}^{\infty}(g^{+}(n)-g^{-}(n))q^n &= \sum_{n=0}^{\infty}\frac{(-1)_n(-1)^n q^{n(n+1)/2}}{(q^2;q^2)_n} \\ &= \frac{(q)_\infty}{(-q)_\infty} \\ &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2},\end{aligned}$$

the last two equalities following from (3.8) and (3.5), respectively. □

Proof of Theorem 1.3. We have

$$\sum_{n=1}^{\infty}(h^{+}(n)-h^{-}(n))q^n = 2\sum_{n=1}^{\infty}\frac{(q)_{n-1}(-1)^n q^{n(n+1)/2}}{(-q^2;q^2)_n}.$$

Taking $a=1$, $b=i$, $c=-i$, $d\to\infty$ in (3.4), and then taking the derivative with respect to e and setting $e=1$ results in

$$\sum_{n=1}^{\infty}\frac{(q)_{n-1}(-1)^n q^{n(n+1)/2}}{(-q^2;q^2)_n} = 2\sum_{n=1}^{\infty}\frac{(1+q^n)q^{n^2+n}}{(1+q^{2n})(1-q^n)} - \sum_{n=1}^{\infty}\frac{q^n}{1-q^n}.$$

Replacing the right most sum in the above equation by

$$\sum_{n=1}^{\infty}\frac{q^{n^2}(1+q^n)}{(1-q^n)}$$

and simplifying gives

$$\sum_{n=1}^{\infty}(h^{+}(n)-h^{-}(n))q^n = 2\sum_{n=1}^{\infty}\frac{-q^{n^2}(1-q^{2n})}{1+q^{2n}}.$$

Now the coefficient of q^n in the above sum is the number of divisors x of n such that x and n have the same parity, counted negatively if $x\equiv n \pmod 4$ and counted positively otherwise. The formula follows. □

6. Odd-even overpartitions and partitions into distinct parts whose smallest part is odd

In Theorem 5.1 we saw a generalization of the fact that self-conjugate partitions of n are in one-to-one correspondence with partitions of n into distinct odd parts. It turns out that this fact can also be generalized by using F-self-conjugate overpartitions, but in a different way. The F-self-conjugate partitions are in one-to-one correspondence with odd-even overpartitions.

Definition 6.1. *An odd-even overpartition is an overpartition into distinct parts with the smallest part odd and such that the difference between successive parts is odd if the smaller is non-overlined and even otherwise.*

For example, there are no odd-even overpartitions of 2, the odd-even overpartitions of 3 are $(3), (\overline{3}), (2,1)$, and $(\overline{2},1)$, and the odd-even overpartitions of 4 are $(\overline{3},\overline{1})$ and $(3,\overline{1})$. Notice that if all parts are overlined, then an odd-even overpartition is just a partition into distinct odd parts. On the other hand, if all parts are non-overlined, then we have the odd-even partitions studied by Andrews [3] in conjunction with Ramanujan's "lost" notebook. It will be noted that odd-even overpartitions are in two-to-one correspondence with partitions into distinct parts whose smallest part is odd, since we can un-overline all of the overlined parts in an odd-even overpartition without losing any information except whether the largest part had been overlined. It will not be hard to argue that the following is true:

Proposition 6.2. *The number of odd-even overpartitions of m with n parts, k of which are overlined, is equal to the number of F-self-conjugate overpartitions of m whose Frobenius representation has n columns and a bottom row with k overlined parts.*

Proof. We will show that the nth term in the series (5.1) is the generating function for odd-even overpartitions into exactly n parts. First, the factor $1/(q^2;q^2)_n$ generates a partition into n non-negative even parts. The factor $q^{n(n+1)/2}$ adds 1 to the smallest part, 2 to the next smallest, and so on, to give a partition λ whose parts are distinct, whose smallest part is odd, and whose parts alternate in parity. Now the term $(-1)_n$ generates a partition μ into distinct non-negative parts. Performing Algorithm I with λ and μ gives the odd-even overpartition. □

Having proven Proposition 6.2, Theorems 5.2 and 5.3 now also apply to odd-even overpartitions. Moreover, the above argument is sufficiently simple that we can interpret the rest of the theorems on F-self-conjugate overpartitions from the previous section in terms of odd-even overpartitions. In particular, one easily verifies that $c^{\pm}(n)$ is the number of odd-even overpartitions of n with co-rank even/odd, $e^{\pm}(n)$ is the number of odd-even overpartitions of n with the co-rank plus the number of parts even/odd, $f^{\pm}(n)$ is the number of odd-even overpartitions of n with half of the D-rank minus half of the co-rank even/odd, $g^{\pm}(n)$ is the number of odd-even overpartitions of n with an even/odd number of parts, and $h^{\pm}(n)$ is the number of overpartitions of n with half of the largest part plus half of the number of parts plus half of the co-rank even/odd.

In light of the two-to-one correspondence between odd-even overpartitions and partitions into distinct parts whose smallest part is odd, one may also interpret the results of Section 5 in the terms of the latter. In doing so, we shall speak of the number of parity changes in a partition into distinct parts, a parity change meaning that two consecutive parts have different parity. Using the notation $Q_o(n)$ for the number of partitions of n into distinct parts whose smallest part is odd, one easily verifies that $c^{\pm}(n)$ is twice the number of partitions counted by $Q_o(n)$ having an even/odd number of parity non-changes, that $e^{\pm}(n)$ is twice the number of partitions counted by $Q_o(n)$ having the number of parts plus the number of parity non-changes even/odd, that $f^{\pm}(n)$ is twice the number of partitions counted by $Q_o(n)$ having half of the D-rank minus half of the number of parity non-changes even/odd, that $g^{\pm}(n)$ is twice the number of partitions counted by $Q_o(n)$ having an even/odd number of parts, and that $h^{\pm}(n)$ is twice the number of partitions counted by $Q_o(n)$ having half of the largest part plus half of the number of parts plus half of the number of parity non-changes even/odd.

7. Suggestions for further study

First, is there a theory of successive ranks for overpartitions in the spirit of Atkin [11] or Garvan [22]? Second, is there is a "crank-type" statistic [7] for overpartitions? At first glance, the infinite product

$$\prod_{n=1}^{\infty} \frac{(-zq,-q/z,q)_\infty}{(-q,zq,q/z)_\infty} \tag{7.1}$$

seems like an excellent candidate. Third, there is a second Frobenius representation of overpartitions that arises naturally in the study of Rogers-Ramanujan type identities for overpartitions [28]. Is there a theory of ranks and conjugation for this representation? Fourth, the study of Dyson's rank for ordinary partitions has as its focus the generating function

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq,q/z)_n}, \tag{7.2}$$

while the study of Dyson's rank for overpartitions is concerned with the generating function

$$\sum_{n=0}^{\infty} \frac{(-1)_n q^{n(n+1)/2}}{(zq,q/z)_n}. \tag{7.3}$$

One could go a step further and consider the generating function

$$\sum_{n=0}^{\infty} \frac{(-1)_n^2 q^n}{(zq,q/z)_n}, \tag{7.4}$$

which is related to Frobenius partitions where we allow overpartitions in both rows [14, 15]. Finally, it would be worthwhile to make a thorough exploration of the link between series related to (7.3) and (would-be) mock theta functions. Indeed, equation (5.3) shows that the generating function for F-self-conjugate overpartitions is an infinite product times one of Ramanujan's third order mock theta functions [31].

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