RAMANUJAN-TYPE CONGRUENCES FOR THREE COLORED FROBENIUS PARTITIONS

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Abstract. Using the theory of modular forms, we show that the three colored Frobenius partition function vanishes modulo some small primes in certain arithmetic progressions.

1. Introduction

Since their discovery by Ramanujan, the partition congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5 \\ p(7n+5) &\equiv 0 \pmod 7 \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

and their generalizations have been the subject of much investigation. We establish a similar batch of congruences for $c\phi_3(n)$, the number of three colored Frobenius partitions of n. Namely, we show that for every nonnegative integer n,

$$c\phi_3(45n+23) \equiv 0 \pmod 5 \quad (1)$$

$$c\phi_3(45n+41) \equiv 0 \pmod 5 \quad (2)$$

$$c\phi_3(63n+50) \equiv 0 \pmod 7 \quad (3)$$

$$c\phi_3(99n+95) \equiv 0 \pmod{11} \quad (4)$$

$$c\phi_3(171n+50) \equiv 0 \pmod{19} \quad (5)$$

The congruence (3) was proved by K. Ono [11], whose techniques are adapted here to prove the remaining congruences. In particular, we will build modular forms whose Fourier coefficients are related to $c\phi_3(n)$ and whose congruence properties can be verified with a finite computation.

2. Preliminaries

In generalizing the Frobenius symbol for ordinary partitions, G. Andrews [1] introduced the notion of a k-colored Frobenius partition of n, a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_m \\ b_1 & b_2 & \dots & b_m \end{pmatrix} \quad (6)$$

where $\sum a_i + \sum b_i + m = n$, and where the $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^m$ are decreasing sequences of nonnegative integers in k colors arranged in order first according to size and then according

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to color. Andrews [1] also found the generating function for the number of such partitions $c\phi_k(n)$ when $k = 3$,

$$\sum_{n=0}^{\infty} c\phi_3(n)q^n = \frac{(q^6;q^6)_\infty^5 (q^2;q^2)_\infty^5}{(q;q)_\infty^5 (q^3;q^3)_\infty^2 (q^4;q^4)_\infty^2 (q^{12};q^{12})_\infty^2} + \frac{4q(q^4;q^4)_\infty^2 (q^{12};q^{12})_\infty^2}{(q;q)_\infty^3 (q^2;q^2)_\infty (q^6;q^6)_\infty}$$

Recall that $(q;q)_\infty$ is the usual notation for $\prod_{n=1}^{\infty}(1-q^n)$. Later, L. Kolitsch [8] proved a related and for our purposes more useful formula,

$$\sum_{n=0}^{\infty} \overline{c\phi_3}(n)q^n := \sum_{n=0}^{\infty} \left(c\phi_3(n) - p\left(\frac{n}{3}\right)\right) q^n = \frac{9q(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty (q;q)_\infty^3} \tag{7}$$

where $p(\alpha)$ is the ordinary partition function with the convention that $p(\alpha) = 0$ if $\alpha \notin \mathbb{Z}$. Since the product in equation (7) is essentially a product of η functions, where $\eta(z) := q^{1/24}(q;q)_\infty$ and $q := e^{2\pi i z}$, we can make use of the following fact [6, 9, 10] to construct modular forms related to the generating function for $c\phi_3(n)$:

Proposition 1 (Gordon, Hughes, Newman, Ligozat)**.** *Let*

$$f(z) = \prod_{1 \le \delta | N} \eta^{r_\delta}(\delta z)$$

be a product of eta functions which satisfies the following criteria

(*i*)

$$\sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24}$$

(*ii*)

$$\sum_{\delta | N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$$

(*iii*)

$$\prod_{\delta | N} \delta^{r_\delta} \in \mathbb{Q}^2$$

(*iv*) *For each* $d|N$,

$$\sum_{\delta | N} \frac{(d,\delta)^2 r_\delta}{\delta} \ge 0$$

Then, $f(z) \in M_k(\Gamma_0(N))$ *if* $k = \frac{1}{2}\sum r_\delta$ *is an integer.*

Here $M_k(\Gamma_0(N))$ is the $\mathbb{C}$ - vector space of holomorphic modular forms of weight k and level N. Conditions (i)-(iii) ensure that $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$ for all z with $Im(z) > 0$ and for all integers a, b, c, d such that $ad - bc = 1$ and $N|c$. Condition (iv) guarantees that "f is holomorphic on $\mathbb{Q}$." For more on the basic theory of modular forms, see Koblitz [7]. Our arguments will rely heavily on the following two facts about modular forms and their Fourier coefficients:

Proposition 2. [7] *Suppose that* $f(z) \in M_k(\Gamma_0(N))$ *with Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n.$$

(i) *For any positive integer* $t|N$,

$$f(z)|U(t) := \sum_{n=0}^{\infty} a(tn)q^n$$

is the Fourier expansion of a modular form in $M_k(\Gamma_0(N))$.

(ii) *For any positive integer* t,

$$f(z)|V(t) := \sum_{n=0}^{\infty} a(n)q^{tn}$$

is the Fourier expansion of a modular form in $M_k(\Gamma_0(tN))$.

Proposition 3. [12] *Suppose* $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N))$ *satisfies*

(i) $a(n) \in \mathbb{Z}$ *for all* n

(ii) $a(n) \equiv 0 \pmod{M}$ *for all* $n \leq 1 + \frac{kN}{12}\prod_{p|N}\left(1+\frac{1}{p}\right)$

Then, $a(n) \equiv 0 \pmod{M}$ *for all* n.

Proposition 3 is Sturm's criterion, where the quantity $\frac{kN}{12}\prod_{p|N}\left(1+\frac{1}{p}\right)$ is known as the Sturm bound. The power of the previous two propositions is that, when the Sturm bound is reasonable, a finite computation can verify congruences in certain progressions of Fourier coefficients of modular forms. Although Sturm's criterion applies only to holomorphic modular forms, it is often possible to reduce conjectured congruences for coefficients of other series $\sum b(n)q^n$ to finite computations by constructng holomorphic modular forms whose Fourier coefficients are closely related to the $b(n)$. Our primary tool will be the following simple but useful fact:

Lemma 4. *Suppose that* $f = \sum_{n=0}^{\infty} a(n)q^n$, $g = \sum_{n=0}^{\infty} d(mn)q^{mn}$ *with* $d(0) = 1$, *and*

$$fg = \sum_{n=0}^{\infty} b(n)q^n$$

Then,

(i) *If* $a(mn + r) \equiv 0 \pmod{M}$ *for* $0 \leq n \leq B$, *then* $b(mn + r) \equiv 0 \pmod{M}$ *for* $0 \leq n \leq B$.

(ii) *If* $b(mn + r) \equiv 0 \pmod{M}$ *for all* n, *then* $a(mn + r) \equiv 0 \pmod{M}$ *for all* n.

Proof: Both facts follow quickly after writing the coefficient of q^{mn+r} from fg in two ways:

$$b(mn + r) = a(mn + r) + \sum_{k\geq 1} d(mk)a(m(n - k) + r)$$

Part (i) is obvious and part (ii) comes from a simple induction argument. □

3. The congruences modulo 11 and 19

We begin with the congruence (4).

Theorem 5. *For every non-negative integer n we have*

$$c\phi_3(99n+95) \equiv 0 \pmod{11}$$

Proof: Define the η - product $g(z)$ by

$$\begin{aligned} g(z) &:= \frac{9\eta^3(9z)\eta^8(z)\eta^{16}(99z)\eta^3(297z)}{\eta(3z)\eta(11z)} \\ &= q^{103}\sum_{n=0}^{\infty}\overline{c\phi_3}(n)q^n \prod_{n=1}^{\infty}(1-q^{99n})^{16}(1-q^{297n})^3 \prod_{n=1}^{\infty}\frac{(1-q^n)^{11}}{1-q^{11n}} \\ &\equiv q^{103}\sum_{n=0}^{\infty}\overline{c\phi_3}(n)q^n \prod_{n=1}^{\infty}(1-q^{99n})^{16}(1-q^{297n})^3 \pmod{11} \\ &:= \sum_{n=103}^{\infty} r(n)q^n \end{aligned}$$

It is easily verified using Proposition 1 that $g(z) \in M_{14}(\Gamma_0(297))$. By Lemma 4, vanishing modulo 11 is shared by $r(99n)$ and $\overline{c\phi_3}(99n+95) = c\phi_3(99n+95)$. Applying Proposition 2 (i),

$$g(z)|U(99) = \sum_{n=0}^{\infty} r(99n)q^n$$

is also in $M_{14}(\Gamma_0(297))$. By Sturm's criterion and Lemma 4 (ii), if $r(99n) \equiv 0 \pmod{11}$ for all $n \leq 505$, then (4) holds for all n. Moreover, by Lemma 4 (i), $r(99n) \equiv 0 \pmod{11}$ for all $n \leq 505$ if $c\phi_3(99n+95) \equiv 0 \pmod{11}$ for all $n \leq 505$. This is verified by machine computation and hence $c\phi_3(99n+95) \equiv 0 \pmod{11}$ for all n. □

The congruence (5) is shown in a similar way.

Theorem 6. *For every non-negative integer n,*

$$c\phi_3(171n+50) \equiv 0 \pmod{19}$$

Proof: Define

$$\begin{aligned} h(z) &:= \frac{9\eta^3(9z)\eta^{16}(z)\eta^8(171z)\eta^3(513z)}{\eta(3z)\eta(19z)} \\ &= q^{121}\sum_{n=0}^{\infty}\overline{c\phi_3}(n)q^n \prod_{n=1}^{\infty}(1-q^{171n})^{8}(1-q^{513n})^3 \prod_{n=1}^{\infty}\frac{(1-q^n)^{19}}{1-q^{19n}} \\ &\equiv q^{121}\sum_{n=0}^{\infty}\overline{c\phi_3}(n)q^n \prod_{n=1}^{\infty}(1-q^{171n})^{8}(1-q^{513n})^3 \pmod{19} \\ &:= \sum_{n=121}^{\infty} r(n)q^n \end{aligned}$$

By Proposition 1, $h(z)$ is in $M_{14}(\Gamma_0(513))$ and hence so is

$$h(z)|U(171) = \sum_{n=0}^{\infty} r(171n)q^n$$

Arguing as above, if $c\phi_3(171n+50) \equiv 0 \pmod{19}$ for $n \leq 841$, then $r(171n) \equiv 0 \pmod{19}$ for $n \leq 841$, so $r(171n) \equiv 0 \pmod{19}$ for all n, and therefore (5) holds for all n. The congruence for $c\phi_3(171n+50)$ has been checked for $n \leq 841$. □

4. The congruences modulo 5

The congruences for $c\phi_3(n)$ modulo 5 are similar to the congruence modulo 7 and distinct from those modulo 11 and 19 in the sense that they are implied by congruences for

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^3(1-q^{3n})} \tag{8}$$

Theorem 7. *The coefficients $a(n)$ as defined above satisfy*

$$a(n) \equiv 0 \pmod{5} \quad \textit{if } n \equiv 13, 22, 31, 40 \pmod{45}$$

Proof: Let $f(z)$ be the η-product defined by

$$\begin{aligned} f(z) &:= \frac{\eta^{13}(45z)\eta^3(135z)\eta^7(z)}{\eta(3z)\eta^2(5z)} \\ &= q^{41}\sum_{n=0}^{\infty} a(n)q^n \prod_{n=1}^{\infty}(1-q^{45n})^{13}(1-q^{135n})^3\left(\prod_{n=1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}}\right)^2 \\ &\equiv q^{41}\sum_{n=0}^{\infty} a(n)q^n \prod_{n=1}^{\infty}(1-q^{45n})^{13}(1-q^{135n})^3 \pmod{5} \\ &:= \sum_{n=41}^{\infty} r(n)q^n \end{aligned}$$

One easily verifies using Proposition 1 that $f(z) \in M_{10}(\Gamma_0(135))$. By Proposition 2 the functions $f(z)|U(9)$ and $f(z)|U(45)$ are in $M_{10}(\Gamma_0(135))$ and hence the function

$$\begin{aligned} f(z)|U(9) - f(z)|U(45)|V(5) &= \sum r(9n)q^n - \sum r(45n)q^{5n} \\ &= \sum_{n \not\equiv 0 \pmod{5}} r(9n)q^n \end{aligned}$$

is in $M_{10}(\Gamma_0(675))$. By Sturm's criterion, if it can be shown that $r(9n) \equiv 0 \pmod{5}$ when $n \leq 901$ and $n \not\equiv 0 \pmod{5}$ then $r(45n+9,18,27,36) \equiv 0 \pmod{5}$ for all n. By Lemma 4 (ii), this will imply that $a(45n+13,22,31,40) \equiv 0 \pmod{5}$ for all n. It has been verified by machine computation that $a(45n+13,22,31,40) \equiv 0 \pmod{5}$ when $n \leq 181$ and so by Lemma 4 (i), $r(45n+9,18,27,36) \equiv 0 \pmod{5}$ for $n \leq 181$ which proves the theorem. □

The proof of the congruences modulo 5 relies on the following well-known corollary of Jacobi's triple product identity:

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n^2+n}{2}}=\prod_{n=1}^{\infty}(1-q^n)^3$$

Theorem 8. *For every non-negative integer n we have*

$$\begin{aligned} c\phi_3(45n+23) &\equiv 0 \pmod 5 \\ c\phi_3(45n+41) &\equiv 0 \pmod 5 \end{aligned}$$

Proof: With the notation from (8),

$$\sum \overline{c\phi_3}(n)q^n = 9\sum a(n)q^n \sum(-1)^n(2n+1)q^{1+\frac{9n^2+9n}{2}},$$

and therefore

$$c\phi_3(45n+23)=\overline{c\phi_3}(45n+23)=9\sum_{k\geq 0}(-1)^k(2k+1)a\left(45n+23-\left(1+\frac{9k^2+9k}{2}\right)\right)$$

$$c\phi_3(45n+41)=\overline{c\phi_3}(45n+41)=9\sum_{k\geq 0}(-1)^k(2k+1)a\left(45n+41-\left(1+\frac{9k^2+9k}{2}\right)\right)$$

Modulo 45, $1+\frac{9k^2+9k}{2}$ is $1, 10$, or 28 and therefore $45n+23-(1+\frac{9k^2+9k}{2})$ and $45n+41-(1+\frac{9k^2+9k}{2})$ are always congruent to $13, 22, 31, 40 \pmod{45}$. Hence, by Theorem 7, in the above sums the right hand side is always $0 \pmod 5$. □

5. Conclusion

There are some natural questions which arise from the study of congruences of this type. First, are there generalizations of the congruences (1) - (5) to powers of 5,7,11, and 19 analogous to the generalizations of Ramanujan's congruences for the partition function? Second, to what extent can the congruence properties of $c\phi_3(n)$ be understood combinatorially by the use of ranks and cranks, as in the study of $p(n)$ mod 5,7, and 11 [2, 3, 4, 5]? This type of argument has been particularly elusive in the study of congruence properties of other objects similar to (6). In fact, Andrews [1] conjectured such a rank, $a_1 - b_1$, which might explain some congruences for generalized Frobenius partitions, but this was shown to be untrue.

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