# Dissections of strange $q$-series

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*Dedicated to George E. Andrews on his 80th birthday*

**Abstract.** In a study of congruences for the Fishburn numbers, Andrews and Sellers observed empirically that certain polynomials appearing in the dissections of the partial sums of the Kontsevich-Zagier series are divisible by a certain $q$-factorial. This was proved by the first two authors. In this paper we extend this strong divisibility property to two generic families of $q$-hypergeometric series which, like the Kontsevich-Zagier series, agree asymptotically with partial theta functions.

**Mathematics Subject Classification (2010).** Primary 33D15.

**Keywords.** Fishburn numbers, Kontsevich-Zagier strange function, $q$-series, partial theta functions, congruences.

## 1. Introduction

Recall the usual $q$-series notation

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \tag{1.1}$$

and let $\mathcal{F}(q)$ denote the Kontsevich-Zagier "strange" function [13, 14],

$$\mathcal{F}(q) := \sum_{n\geq 0}(q;q)_n.$$

This series does not converge on any open subset of $\mathbb{C}$, but it is well-defined both at roots of unity and as a power series when $q$ is replaced by $1-q$. The coefficients $\xi(n)$ of

$$\mathcal{F}(1-q) = 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + \cdots$$

are called the Fishburn numbers, and they count a number of different combinatorial objects (see [11] for references).

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The first author was supported by a grant from the Simons Foundation (#426145 to Scott Ahlgren). Byungchan Kim was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A1A09917344).

Andrews and Sellers [4] discovered and proved a wealth of congruences for $\xi(n)$ modulo primes $p$. For example, we have

$$\begin{aligned} \xi(5n+4) \equiv \xi(5n+3) &\equiv 0 \pmod 5, \\ \xi(7n+6) &\equiv 0 \pmod 7. \end{aligned} \tag{1.2}$$

In subsequent work of the first two authors, Garvan, and Straub [1, 6, 12], similar congruences were obtained for prime powers and for generalized Fishburn numbers.

Taking a different approach, Guerzhoy, Kent, and Rolen [7] interpreted the coefficients in the asymptotic expansions of functions $P^{(1)}_{a,b,\chi}(e^{-t})$ defined in (1.8) below in terms of special values of $L$-functions, and proved congruences for these coefficients using divisibility properties of binomial coefficients. These congruences are inherited by any function whose expansion at $q = 1$ agrees with one of these expansions; these include the function $\mathcal{F}(q)$ and, more generally, the Kontsevich-Zagier functions described in Section 5 below. See [7] for details.

Although the congruences (1.2) bear a passing resemblance to Ramanujan's congruences for the partition function $p(n)$, it turns out that they arise from a divisibility property of the partial sums of $\mathcal{F}(q)$. For positive integers $N$ and $s$ consider the partial sums

$$\mathcal{F}(q; N) := \sum_{n=0}^{N} (q; q)_n$$

and the $s$-dissection

$$\mathcal{F}(q; N) = \sum_{i=0}^{s-1} q^i A_s(N, i, q^s).$$

Let $S(s) \subseteq \{0, 1, \ldots s-1\}$ denote the set of reductions modulo $s$ of the set of pentagonal numbers $m(3m+1)/2$, where $m \in \mathbb{Z}$. The key step in the proof of Andrews and Sellers is to show that if $p$ is prime and $i \notin S(p)$ then we have

$$(1-q)^n \mid A_p(pn-1, i, q). \tag{1.3}$$

This divisibility property is also important for the proof of the congruences in [6, 12].

Andrews and Sellers [4] observed empirically that $(1-q)^n$ can be strengthened to $(q; q)_n$ in (1.3). The first two authors showed that this divisibility property holds for any $s$. To be precise, define

$$\lambda(N, s) = \left\lfloor \frac{N+1}{s} \right\rfloor. \tag{1.4}$$

Then we have

**Theorem 1.1** ([1]). *Suppose that $s$ and $N$ are positive integers and that $i \notin S(s)$. Then*

$$(q; q)_{\lambda(N,s)} \mid A_s(N, i, q). \tag{1.5}$$

The proof of (1.5) relies on the fact that the Kontsevich-Zagier function satisfies the "strange identity"

$$\mathcal{F}(q)\text{``}=\text{''}-\frac{1}{2}\sum_{n\geq 1} n\left(\frac{12}{n}\right)q^{(n^2-1)/24}.$$

Here the symbol " = " means that the two sides agree to all orders at every root of unity (this is explained fully in Sections 2 and 5 of [13]). In this paper we show that a analogue of Theorem 1.1 holds for a wide class of "strange" $q$-hypergeometric series—that is, $q$-series which agree asymptotically with partial theta functions.

To state our result, let $F$ and $G$ be functions of the form

$$F(q)=\sum_{n=0}^{\infty}(q;q)_n f_n(q), \tag{1.6}$$

$$G(q)=\sum_{n=0}^{\infty}(q;q^2)_n g_n(q), \tag{1.7}$$

where $f_n(q)$ and $g_n(q)$ are polynomials. (Functions of the form (1.6) are elements of the *Habiro ring*, which can be viewed as a ring of analytic functions on the set of roots of unity [8].) Note that $F(q)$ is not necessarily well-defined as a power series in $q$, but it has a power series expansion at every root of unity $\zeta$. In other words $F(\zeta e^{-t})$ has a meaningful definition as a formal power series in $t$ whose coefficients are expressed in the usual way as the "derivatives" of $F(\zeta e^{-t})$ at $t=0$. This is explained in detail in the next section. Likewise, $G(q)$ has a power series expansion at every odd-order root of unity.

We will consider partial theta functions

$$P^{(\nu)}_{a,b,\chi}(q):=\sum_{n\geq 0} n^{\nu}\chi(n)q^{\frac{n^2-a}{b}}, \tag{1.8}$$

where $\nu\in\{0,1\}$, $a\geq 0$ and $b>0$ are integers, and $\chi:\mathbb{Z}\to\mathbb{C}$ is a function satisfying the following properties:

$$\chi(n)\neq 0 \quad \text{only if} \quad \frac{n^2-a}{b}\in\mathbb{Z}, \tag{1.9}$$

and for each root of unity $\zeta$,

$$\text{the function } n\mapsto \zeta^{\frac{n^2-a}{b}}\chi(n) \text{ is periodic and has mean value zero.} \tag{1.10}$$

These assumptions are enough to ensure that for each root of unity $\zeta$, the function $P^{(\nu)}_{a,b,\chi}(\zeta e^{-t})$ has an asymptotic expansion as $t\to 0^+$ (see Section 3 below). We note that (1.10) is satisfied by any odd periodic function. To see this, suppose that $\chi$ is odd with period $T$, and let $\zeta$ be a $k$th root of unity. Set $M=\operatorname{lcm}(T,bk)$. Then we have

$$\zeta^{\frac{(M-n)^2-a}{b}}\chi(M-n)=-\zeta^{\frac{n^2-a}{b}}\chi(n),$$

and so

$$\sum_{n=0}^{M-1} \zeta^{\frac{n^2-a}{b}} \chi(n) = 0.$$

For positive integers $s$ and $N$, consider the partial sum

$$F(q;N) := \sum_{n=0}^{N} f_n(q)(q;q)_n \tag{1.11}$$

and its $s$-dissection

$$F(q;N) = \sum_{i=0}^{s-1} q^i A_{F,s}(N,i,q^s).$$

Define $S_{a,b,\chi}(s) \subseteq \{0,1,\ldots,s-1\}$ by

$$S_{a,b,\chi}(s) := \left\{ \frac{n^2-a}{b} \pmod{s} \quad : \quad \chi(n) \neq 0 \right\}.$$

Our first main result is the following.

**Theorem 1.2.** *Suppose that $F$ is a function as in* (1.6) *and that $P^{(\nu)}_{a,b,\chi}$ is a function as in* (1.8). *Suppose that for each root of unity $\zeta$ we have the asymptotic expansion*

$$P^{(\nu)}_{a,b,\chi}(\zeta e^{-t}) \sim F(\zeta e^{-t}) \quad \text{as} \quad t \to 0^+. \tag{1.12}$$

*Suppose that $s$ and $N$ are positive integers and that $i \notin S_{a,b,\chi}(s)$. Then we have*

$$(q;q)_{\lambda(N,s)} \mid A_{F,s}(N,i,q).$$

Analogously, for positive integers $s$ and $N$ with $s$ odd, consider the partial sum

$$G(q;N) := \sum_{n=0}^{N} g_n(q)(q;q^2)_n \tag{1.13}$$

and its $s$-dissection

$$G(q;N) = \sum_{i=0}^{s-1} q^i A_{G,s}(N,i,q^s).$$

Then the $A_{G,s}(N,i,q^s)$ also enjoy strong divisibility properties. Define

$$\mu(N,k,s) = \left\lfloor \frac{N}{s(2k-1)} + \frac{1}{2} \right\rfloor. \tag{1.14}$$

**Theorem 1.3.** *Suppose that $G$ is a function as in* (1.7) *and that $P^{(\nu)}_{a,b,\chi}$ is a function as in* (1.8). *Suppose that for each root of unity $\zeta$ of odd order we have*

$$P^{(\nu)}_{a,b,\chi}(\zeta e^{-t}) \sim G(\zeta e^{-t}) \qquad \text{as } t \to 0^+. \tag{1.15}$$

*Suppose that $s$ and $N$ are positive integers with $s$ odd and that $i \notin S_{a,b,\chi}(s)$. Then we have*

$$(q;q^2)_{\mu(N,1,s)} \mid A_{G,s}(N,i,q).$$

While a generic $q$-series of the form (1.6) or (1.7) is not expected to be related to a partial theta function as in (1.12) or (1.15), there are a number of examples where this is the case. For example, Hikami [9] introduced a family quantum modular forms related to torus knots, which we will discuss in Section 5. For now, we illustrate Theorem 1.3 with an example from Ramanujan's lost notebook. Consider the $q$-series

$$\mathcal{G}(q) = \sum_{n\geq 0} (q;q^2)_n q^n.$$

From [3, Entry 9.5.2] we have the identity

$$\sum_{n\geq 0} (q;q^2)_n q^n = \sum_{n\geq 0} (-1)^n q^{3n^2+2n}(1+q^{2n+1}),$$

which may be written as

$$\sum_{n\geq 0} (q;q^2)_n q^n = \sum_{n\geq 0} \chi_6(n) q^{(n^2-1)/3},$$

where

$$\chi_6(n) := \begin{cases} 1, & \text{if } n\equiv 1,2 \pmod 6,\\ -1, & \text{if } n\equiv 4,5 \pmod 6,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, for each odd-order root of unity $\zeta$ we find that

$$P^{(0)}_{1,3,\chi_6}(\zeta e^{-t}) \sim \mathcal{G}(\zeta e^{-t}) \qquad \text{as } t\to 0^+.$$

Since $\chi_6$ is odd, it satisfies conditions (1.9) and (1.10). Thus, from Theorem 1.3, we find that for $i\notin S_{1,3,\chi_6}(s)$ we have

$$(q;q^2)_{\lfloor \frac{N}{s}+\frac{1}{2}\rfloor} \mid A_{\mathcal{G},s}(N,i,q). \tag{1.16}$$

For example, when $s=5$ we have $S_{1,3,\chi_6}(5)=\{0,1,3\}$. For $N=8$ we have

$$A_{\mathcal{G},5}(8,2,q) = q^2(q;q^2)_2(1+q^2-q^3+2q^4-q^5+2q^6+q^8)$$

and

$$A_{\mathcal{G},5}(8,4,q) = -q(q;q^2)_2(1-q+q^2)(1+q+q^2+q^4+q^6),$$

as predicted by (1.16), while the factorizations of $A_{\mathcal{G},5}(8,i,q)$ into irreducible factors for $i\in\{0,1,3\}$ are

$$\begin{aligned} A_{\mathcal{G},5}(8,0,q) &= (1-q)(1+q^4-2q^5+\cdots-2q^{11}+q^{12}),\\ A_{\mathcal{G},5}(8,1,q) &= 1+2q^3-q^4+\cdots+q^{13}-q^{14},\\ A_{\mathcal{G},5}(8,3,q) &= q(-1+q^2-2q^3+2q^4-\cdots-2q^{11}+q^{12}). \end{aligned}$$

The rest of the paper is organized as follows. In the next section we discuss power series expansions of $F$ and $G$ at roots of unity, and in Section 3 we discuss the asymptotic expansions of partial theta functions. In Section 4 we prove the main theorems. In Section 5 we give two further examples—one generalizing (1.5) and one generalizing (1.16). We close with some remarks on congruences for the coefficients of $F(1-q)$ and $G(1-q)$.

## 2. Power series expansions of $F$ and $G$

Let $F(q)$ be a function as in (1.6) and $G(q)$ be a function as in (1.7). Here we collect some facts which allow us to meaningfully define $F(\zeta e^{-t})$ and $G(\zeta e^{-t})$ as formal power series.

**Lemma 2.1.** *Let $F(q;N)$ be as in* (1.11)*, and let $G(q;N)$ be as in* (1.13)*. Suppose that $\zeta$ is a $k$th root of unity.*

1. *The values $\left(q\frac{d}{dq}\right)^{\ell} F(q;N)\big|_{q=\zeta}$ are stable for $N \geq (\ell+1)k-1$.*
2. *If $k$ is odd then the values $\left(q\frac{d}{dq}\right)^{\ell} G(q;N)\big|_{q=\zeta}$ are stable for $2N \geq (2\ell+1)k$.*

*Proof.* For each positive integer $k$ we have

$$(1-q^k)^{\ell+1} \mid (q;q)_N \quad \text{for} \quad N \geq (\ell+1)k,$$
$$(1-q^{2k-1})^{\ell+1} \mid (q;q^2)_N \quad \text{for} \quad 2N \geq (2\ell+1)(2k-1)+1.$$

It follows that for $0 \leq j \leq \ell$ we have

$$\left(\frac{d}{dq}\right)^{j} (q;q)_N\big|_{q=\zeta} = 0 \quad \text{for} \quad N \geq (\ell+1)k,$$
$$\left(\frac{d}{dq}\right)^{j} (q;q^2)_N\big|_{q=\zeta} = 0 \quad \text{for odd } k \text{ and} \quad 2N \geq (2\ell+1)k+1.$$

The lemma follows since for any polynomial $f(q)$, the polynomial $\left(q\frac{d}{dq}\right)^{\ell} f(q)$ is a linear combination (with polynomial coefficients) of $\left(\frac{q}{dq}\right)^{j} f(q)$ with $0 \leq j \leq \ell$ (see for example [4, Lemma 2.2]). □

For any polynomial $f(q)$, any $\zeta$ and any $\ell \geq 0$ we have [4, Lemma 2.3]

$$\left(\frac{d}{dt}\right)^{\ell} f(\zeta e^{-t})\big|_{t=0} = (-1)^{\ell}\left(q\frac{d}{dq}\right)^{\ell} f(q)\big|_{q=\zeta}. \tag{2.1}$$

Let $F(q)$ be as in (1.6) and let $\zeta$ be a $k$th root of unity. The last fact together with Lemma 2.1 allows us to define

$$\left(\frac{d}{dt}\right)^{\ell} F(\zeta e^{-t})\big|_{t=0} := \left(\frac{d}{dt}\right)^{\ell} F(\zeta e^{-t};N)\big|_{t=0} \qquad \text{for any } N \geq k(\ell+1)-1.$$

We therefore have a formal series expansion

$$F(\zeta e^{-t}) = \sum_{\ell=0}^{\infty} \frac{\left(\frac{d}{dt}\right)^{\ell} F(\zeta e^{-t})\big|_{t=0}}{\ell!}\, t^{\ell}. \tag{2.2}$$

Similarly, if $G(q)$ is a function as in (1.7) and $\zeta$ is a $k$th root of unity with odd $k$, then we can define

$$\left(\frac{d}{dt}\right)^{\ell} G(\zeta e^{-t})\big|_{t=0} := \left(\frac{d}{dt}\right)^{\ell} G(\zeta e^{-t};N)\big|_{t=0} \qquad \text{for any } 2N \geq k(2\ell+1), \tag{2.3}$$

using (2.1) and Lemma 2.1. Thus, we have a formal series expansion

$$G(\zeta e^{-t}) = \sum_{\ell=0}^{\infty} \frac{\left(\frac{d}{dt}\right)^{\ell} G(\zeta e^{-t})\big|_{t=0}}{\ell!}\, t^{\ell}. \tag{2.4}$$

## 3. The asymptotics of $P^{(\nu)}_{a,b,\chi}$

In this section we discuss the asymptotic expansion of the partial theta functions $P^{(\nu)}_{a,b,\chi}(q)$ defined in (1.8). Recall that

$$P^{(\nu)}_{a,b,\chi}(q) := \sum_{n\geq 0} n^{\nu}\chi(n) q^{\frac{n^2-a}{b}},$$

where $\nu \in \{0,1\}$, $a \geq 0$ and $b > 0$ are integers, and $\chi : \mathbb{Z} \to \mathbb{C}$ is a function satisfying properties (1.9) and (1.10).

The properties which we describe in the next proposition are more or less standard (see for example [10, p. 98]). For convenience and completeness we sketch a proof of the following:

**Proposition 3.1.** *Suppose that $P^{(\nu)}_{a,b,\chi}(q)$ is as in (1.8). Let $\zeta$ be a root of unity and let $N$ be a period of the function $n \mapsto \zeta^{\frac{n^2-a}{b}}\chi(n)$. Then we have the asymptotic expansion*

$$P^{(\nu)}_{a,b,\chi}(\zeta e^{-t}) \sim \sum_{n=0}^{\infty} \gamma_n(\zeta) t^n, \qquad t \to 0^+,$$

*where*

$$\gamma_n(\zeta) = \sum_{\substack{1\leq m\leq N \\ \chi(m)\neq 0}} a(m,n,N)\zeta^{\frac{m^2-a}{b}} \tag{3.1}$$

*with certain complex numbers $a(m,n,N)$.*

We begin with a lemma. For $n \geq 0$ let $B_n(x)$ denote the $n$th Bernoulli polynomial. In the rest of this section we use $s$ for a complex variable since there can be no confusion with the parameter $s$ used above.

**Lemma 3.2.** *Let $C : \mathbb{Z} \to \mathbb{C}$ be a function with period $N$ and mean value zero, and let*

$$L(s,C) := \sum_{n=1}^{\infty} \frac{C(n)}{n^s}, \qquad \mathrm{Re}(s) > 0.$$

*Then $L(s,C)$ has an analytic continuation to $\mathbb{C}$, and we have*

$$L(-n,C)=\frac{-N^n}{n+1}\sum_{m=1}^{N}C(m)B_{n+1}\left(\frac{m}{N}\right)\qquad \textit{for } n\geq 0. \tag{3.2}$$

*Proof.* Let $\zeta(s,\alpha)$ denote the Hurwitz zeta function, whose properties are described for example in [5, Chapter 12]. We have

$$L(s,C)=N^{-s}\sum_{m=1}^{N}C(m)\zeta\left(s,\tfrac{m}{N}\right). \tag{3.3}$$

The lemma follows using the fact that each Hurwitz zeta function has only a simple pole with residue 1 at $s=1$ and the formula for the value of each function at $s=-n$ [5, Thm. 12.13]. □

*Proof of Proposition* 3.1. It is enough to prove the proposition for the function

$$f(t):=e^{-\frac{at}{b}}P^{(\nu)}_{a,b,\chi}(\zeta e^{-t})=\sum_{n\geq 1}n^{\nu}\chi(n)\zeta^{\frac{n^2-a}{b}}e^{-\frac{n^2t}{b}},\qquad t>0.$$

Setting

$$C(n):=\zeta^{\frac{n^2-a}{b}}\chi(n), \tag{3.4}$$

we have the Mellin transform

$$\int_0^{\infty}f(t)t^{s-1}\,dt=b^s\Gamma(s)L(2s-\nu,C),\qquad \mathrm{Re}(s)>\frac{1}{2}.$$

Inverting, we find that

$$f(t)=\frac{1}{2\pi i}\int_{x=c}b^s\Gamma(s)L(2s-\nu,C)t^{-s}\,ds,$$

for $c>\frac{1}{2}$, where we write $s=x+iy$. Using (3.3), the functional equation for the Hurwitz zeta functions, and the asymptotics of the Gamma function, we find that, for fixed $x$, the function $L(s,C)$ has at most polynomial growth in $|y|$ as $|y|\to\infty$. Shifting the contour to the line $x=-R-\frac{1}{2}$ we find that for each $R\geq 0$ we have

$$f(t)=\sum_{n=0}^{R}\frac{(-1)^n}{b^n n!}L(-2n-\nu,C)t^n+O\left(t^{R+\frac{1}{2}}\right),$$

from which

$$f(t)\sim\sum_{n=0}^{\infty}\frac{(-1)^n}{b^n n!}L(-2n-\nu,C)t^n.$$

The proposition follows from (3.4) and (3.2). □

## 4. Proof of Theorems 1.2 and 1.3

We begin with a lemma. The first assertion is proved in [4, Lemma 2.4], and the second, which is basically equation (2.4) in [1], follows by extracting an arithmetic progression using orthogonality. (We note that there is an error in the published version of [1] which is corrected below; in that version the operators $\frac{d}{dq}$ and $q\frac{d}{dq}$ are conflated in the statement of (2.3) and (2.4). This does not affect the truth of the rest of the results.)

Let $C_{\ell,i,j}(s)$ be the array of integers defined recursively as follows:

1. $C_{0,0,0}(s) = 1$,
2. $C_{\ell,i,0}(s) = i^\ell$ and $C_{\ell,i,j}(s) = 0$ for $j \geq \ell+1$ or $j < 0$,
3. $C_{\ell+1,i,j}(s) = (i+js)C_{\ell,i,j}(s) + sC_{\ell,i,j-1}(s)$ for $1 \leq j \leq \ell$.

**Lemma 4.1.** *Suppose that $s$ is a positive integer and that*

$$h(q) = \sum_{i=0}^{s-1} q^i A_s(i, q^s)$$

*with polynomials $A_s(i,q)$. Then the following are true:*

1. *For all $\ell \geq 0$ we have*

$$\left(q\frac{d}{dq}\right)^\ell h(q) = \sum_{j=0}^{\ell}\sum_{i=0}^{s-1} C_{\ell,i,j}(s) q^{i+js} A_s^{(j)}(i, q^s).$$

2. *Let $\zeta_s$ be a primitive $s$th root of unity. Then for $\ell \geq 0$ and $i_0 \in \{0,\dots,s-1\}$ we have*

$$\sum_{j=0}^{\ell} C_{\ell,i_0,j}(s) q^{i_0+js} A_s^{(j)}(i_0, q^s) = \frac{1}{s}\sum_{k=0}^{s-1} \zeta_s^{-ki_0}\left(\left(q\tfrac{d}{dq}\right)^\ell h(q)\right)\Big|_{q\to\zeta_s^k q}. \tag{4.1}$$

*Proof of Theorem* 1.2. Suppose that $F(q)$ and $P_{a,b,\chi}(q)$ are as in the statement of the theorem. Suppose that $s$ and $k$ are positive integers, that $i \notin S_{a,b,\chi}(s)$ and that $\zeta_k$ is a primitive $k$th root of unity. Let $\Phi_k(q)$ be the $k$th cyclotomic polynomial. Recall the definition (1.4) of $\lambda(N,s)$ and note that since

$$(q;q)_n = \pm \prod_{k=1}^{n} \Phi_k(q)^{\lfloor \frac{n}{k} \rfloor} \tag{4.2}$$

and

$$\left\lfloor \frac{\lfloor \frac{x}{s} \rfloor}{k} \right\rfloor = \left\lfloor \frac{x}{ks} \right\rfloor,$$

we have

$$(q;q)_{\lambda(N,s)} = \pm \prod_{k=1}^{\lambda(N,s)} \Phi_k(q)^{\lambda(N,ks)}.$$

Therefore, Theorem 1.2 will follow once we show for each $\ell \geq 0$ that

$$A_{F,s}^{(\ell)}(N, i, \zeta_k) = 0 \quad \text{for} \quad N \geq (\ell+1)ks - 1,$$

since this implies that $\Phi_k(q)^{\lambda(N,ks)} \mid A_{F,s}(N,i,q)$ for $1 \leq k \leq \lambda(N,s)$.

From the definition we find that

$$A_{F,s}(N,i,q)=\sum_{j=0}^{k-1} q^j A_{F,ks}(N,i+js,q^k).$$

If $i\notin S_{a,b,\chi}(s)$, then $i+js\notin S_{a,b,\chi}(ks)$. It is therefore enough to show that for all $s$, $k$, and $\ell$, and for $i\notin S_{a,b,\chi}(ks)$, we have

$$A^{(\ell)}_{F,ks}(N,i,1)=0 \qquad \text{for } N\geq(\ell+1)ks-1.$$

After replacing $ks$ by $s$, it is enough to show that for all $s$ and $\ell$, and for $i\notin S_{a,b,\chi}(s)$, we have

$$A^{(\ell)}_{F,s}(N,i,1)=0 \qquad \text{for } N\geq(\ell+1)s-1. \tag{4.3}$$

We prove (4.3) by induction on $\ell$. For the base case $\ell=0$, assume that $N\geq s-1$. Using (4.1) with $q=1$ gives

$$A_{F,s}(N,i,1)=\frac{1}{s}\sum_{j=0}^{s-1}\zeta_s^{-ji}F(\zeta_s^j;N).$$

By (1.12), (2.1), Lemma 2.1, and Proposition 3.1 we find that

$$A_{F,s}(N,i,1)=\frac{1}{s}\sum_{j=1}^{s}\zeta_s^{-ji}\gamma_0(\zeta_s^j).$$

By (3.1) and orthogonality (recalling that $i\notin S_{a,b,\chi}(s)$), we find that $A_{F,s}(N,i,1)=0$.

For the induction step, suppose that $N\geq(\ell+1)s-1$, that $i\notin S_{a,b,\chi}(s)$, and that (4.3) holds with $\ell$ replaced by $j$ for $1\leq j\leq\ell-1$. By (4.1) and the induction hypothesis we have

$$C_{\ell,i,\ell}(s)A^{(\ell)}_{F,s}(N,i,1)=\frac{1}{s}\sum_{j=1}^{s}\zeta_s^{-ji}\left(q\frac{d}{dq}\right)^{\ell}F(q;N)\Big|_{q=\zeta_s^j}.$$

Using Proposition 3.1, (2.2), (3.1), and orthogonality, we find as above that

$$C_{\ell,i,\ell}(t)A^{(\ell)}_{F,s}(N,i,1)=0.$$

This establishes (4.3) since $C_{\ell,i,\ell}(s)>0$. Theorem 1.2 follows. □

*Proof of Theorem* 1.3. Suppose that $s$ and $k$ are positive integers with $s$ odd, that $i\notin S_{a,b,\chi}(s)$ and that $\zeta_{2k-1}$ is a $(2k-1)$th root of unity. Recall the definition (1.14) of $\mu(N,k,s)$. In analogy with (4.2), we have

$$(q;q^2)_n=\pm\prod_{k=1}^{n}\Phi_{2k-1}(q)^{\lfloor\frac{(2n-1)}{2(2k-1)}+\frac{1}{2}\rfloor},$$

and as above we obtain

$$(q;q^2)_{\mu(N,1,s)}=\pm\prod_{k=1}^{\mu(N,1,s)}\Phi_{2k-1}(q)^{\mu(N,k,s)}.$$

Therefore, Theorem 1.3 follows once we show for each $\ell \geq 0$ that

$$A_{G,s}^{(\ell)}(N, i, \zeta_{2k-1}) = 0 \quad \text{for} \quad 2N \geq (2\ell+1)(2k-1)s.$$

The rest of the proof is similar to that of Theorem 1.2 (we require $s$ to be odd because $G(q)$ has a series expansion only at odd-order roots of unity). Arguing as above, we show that for each odd $s$ we have

$$A_{G,s}^{(\ell)}(N, i, 1) = 0 \quad \text{for} \quad 2N \geq (2\ell+1)s,$$

and the result follows. □

## 5. Examples

In this section we illustrate Theorems 1.2 and 1.3 with two families of examples.

### 5.1. The generalized Kontsevich-Zagier functions

In a study of quantum modular forms related to torus knots and the Andrews-Gordon identities, Hikami [9] defined the functions

$$X_m^{(\alpha)}(q) := \sum_{k_1, k_2, \ldots, k_m \geq 0} (q;q)_{k_m} q^{k_1^2 + \cdots + k_{m-1}^2 + k_{\alpha+1} + \cdots + k_{m-1}} \times \left( \prod_{\substack{i=1 \\ i \neq \alpha}}^{m-1} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix} \right) \begin{bmatrix} k_{\alpha+1} + 1 \\ k_\alpha \end{bmatrix}, \tag{5.1}$$

where $m$ is a positive integer and $\alpha \in \{0, 1, \ldots, m-1\}$. Here we have used the usual $q$-binomial coefficient (or Gaussian polynomial)

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

The simplest example

$$X_1^{(0)}(q) = \sum_{n \geq 0} (q;q)_n$$

is the Kontsevich-Zagier function. From (5.1) we can write

$$X_m^{(\alpha)}(q) = \sum_{k_m \geq 0} (q;q)_{k_m} f_{k_m}^{(\alpha)}(q),$$

with polynomials $f_{k_m}^{(\alpha)}(q)$.

Hikami's identity [9, eqn (70)] implies that for each root of unity $\zeta$ we have

$$P^{(1)}_{(2m-2\alpha-1)^2, 8(2m+1), \chi_{8m+4}^{(\alpha)}}(\zeta e^{-t}) \sim X_m^{(\alpha)}(\zeta e^{-t})$$

as $t \to 0^+$, where $\chi_{8m+4}^{(\alpha)}(n)$ is defined by

$$\chi_{8m+4}^{(\alpha)}(n) = \begin{cases} -1/2, & \text{if } n \equiv 2m-2\alpha-1 \text{ or } 6m+2\alpha+5 \pmod{8m+4}, \\ 1/2, & \text{if } n \equiv 2m+2\alpha+3 \text{ or } 6m-2\alpha+1 \pmod{8m+4}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

The function $\chi_{8m+4}^{(\alpha)}(n)$ satisfies condition (1.9). For (1.10) we record a short lemma.

**Lemma 5.1.** *Suppose that $\chi_{8m+4}^{(\alpha)}(n)$ is as defined in (5.2) and that $\zeta$ is a root of unity of order $M$. Define*

$$\psi(n) = \zeta^{\frac{n^2-(2m-2\alpha-1)^2}{8(2m+1)}} \chi_{8m+4}^{(\alpha)}(n).$$

*Then*

$$\sum_{n=1}^{M(8m+4)} \psi(n) = 0.$$

*Proof.* Note that $\psi$ is supported on odd integers, so we assume in what follows that $n$ is odd. From the definition, we have

$$\chi_{8m+4}^{(\alpha)}(n + M(4m+2)) = (-1)^M \chi_{8m+4}^{(\alpha)}(n). \tag{5.3}$$

The exponent in the ratio of the corresponding powers of $\zeta$ is $mM^2 + \frac{M^2+Mn}{2}$. So the ratio of these powers of $\zeta$ is

$$\zeta^{\frac{M^2+Mn}{2}}.$$

If $M$ is odd then this becomes $\zeta^{M\left(\frac{M+n}{2}\right)} = 1$, while if $M$ is even then this becomes $\zeta^{\frac{M^2}{2}}\zeta^{\frac{M}{2}n} = -1$ (since $M$ is the order of $\zeta$ and $n$ is odd). Therefore the ratio in either case is $(-1)^{M+1}$. Combining this with (5.3) gives

$$\psi(n + M(4m+2)) = -\psi(n),$$

from which the lemma follows. □

Therefore $X_m^{(\alpha)}(q)$ satisfies the conditions of Theorem 1.2, and we obtain the following.

**Corollary 5.2.** *If $s$ is a positive integer and $i \notin S_{(2m-2\alpha-1)^2, 8(2m+1), \chi_{8m+4}^{(\alpha)}}(s)$, then*

$$(q;q)_{\lambda(N,s)} \big| A_{X_m^{(\alpha)}, s}(N, i, q),$$

*where $A_{X_m^{(\alpha)}, s}(N, i, q)$ are the coefficients in the $s$-dissection of the partial sums (in $k_m$) of $X_m^{(\alpha)}(q)$.*

For example, when $s = 3$ we have $S_{9,40,\chi_{20}^{(0)}}(3) = \{0, 1\}$ and $S_{1,40,\chi_{20}^{(1)}}(3) = \{0, 2\}$. For $N = 8$ we have

$$A_{X_2^{(0)}, 3}(8, 2, q) = (q;q)_3(1+q)(1+q+q^2)(1-q+\cdots-q^{25}+q^{26})$$

and

$$A_{X_2^{(1)},3}(8,1,q) = (q;q)_3(1+q)(1-q+q^2)(1+q+q^2)(1+2q+\cdots-q^{26}+q^{27}),$$

as predicted by Corollary 5.2, while

$$\begin{aligned} A_{X_2^{(0)},3}(8,0,q) &= (1-q+q^2)(9+9q+\cdots+q^{33}+q^{34}), \\ A_{X_2^{(0)},3}(8,1,q) &= -8-7q+\cdots+q^{34}-q^{35}, \\ A_{X_2^{(1)},3}(8,0,q) &= 9-7q+\cdots+2q^{36}+q^{39}, \end{aligned}$$

and

$$A_{X_2^{(1)},3}(8,2,q) = -7+3q^3-\cdots+q^{36}-q^{38}$$

are not divisible by $(q;q)_3$.

**5.2. An example with $\nu = 0$**

For $k \geq 1$ let $\mathcal{G}_k(q)$ denote the $q$-series

$$\mathcal{G}_k(q) = \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} q^{n_k+2n_{k-1}^2+2n_{k-1}+\cdots+2n_1^2+2n_1} \times (q;q^2)_{n_k} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix}_{q^2} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_{q^2}.$$

Then we have the identity

$$\mathcal{G}_k(q) = \sum_{n \geq 0} (-1)^n q^{(2k+1)n^2+2kn}(1+q^{2n+1}), \tag{5.4}$$

which follows from Andrews' generalization [2] of the Watson-Whipple transformation

$$\begin{aligned} \sum_{m=0}^{N} \frac{(1-aq^{2m})}{(1-a)} &\frac{(a,b_1,c_1,\ldots,b_k,c_k,q^{-N})_m}{(q,aq/b_1,aq/c_1,\ldots,aq/b_k,aq/c_k,aq^{N+1})_m} \left(\frac{a^k q^{k+N}}{b_1c_1\cdots b_kc_k}\right)^m \\ = \frac{(aq,aq/b_kc_k)_N}{(aq/b_k,aq/c_k)_N} &\sum_{N \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{(b_k,c_k)_{n_{k-1}} \cdots (b_2,c_2)_{n_1}}{(q;q)_{n_{k-1}-n_{k-2}} \cdots (q;q)_{n_2-n_1}(q;q)_{n_1}} \\ &\times \frac{(aq/b_{k-1}c_{k-1})_{n_{k-1}-n_{k-2}} \cdots (aq/b_2c_2)_{n_2-n_1}(aq/b_1c_1)_{n_1}}{(aq/b_{k-1},aq/c_{k-1})_{n_{k-1}} \cdots (aq/b_1,aq/c_1)_{n_1}} \\ &\times \frac{(q^{-N})_{n_{k-1}}(aq)^{n_{k-2}+\cdots+n_1}q^{n_{k-1}}}{(b_kc_kq^{-N}/a)_{n_{k-1}}(b_{k-1}c_{k-1})^{n_{k-2}}\cdots(b_2c_2)^{n_1}}. \end{aligned}$$

Here we have extended the notation in (1.1) to

$$(a_1,a_2,\ldots,a_k)_n := (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n.$$

To deduce (5.4), we set $q = q^2, a = q^2, b_k = q$, and $c_k = q^2$ and then let $N \to \infty$ along with all other $b_i, c_i$.

The identity (5.4) may be written as

$$\mathcal{G}_k(q) = \sum_{n\geq 0} \chi_{4k+2}(n) q^{\frac{n^2-k^2}{2k+1}},$$

where

$$\chi_{4k+2}(n) := \begin{cases} 1, & \text{if } n \equiv k, k+1 \pmod{4k+2}, \\ -1, & \text{if } n \equiv -k, -k-1 \pmod{4k+2}, \\ 0, & \text{otherwise.} \end{cases}$$

This implies that for each odd-order root of unity $\zeta$, we have

$$P^{(0)}_{k^2,2k+1,\chi_{4k+2}}(\zeta e^{-t}) \sim G_k(\zeta e^{-t}) \qquad \text{as } t \to 0^+.$$

The function $\chi_{4k+2}(n)$ satisfies conditions (1.9) and (1.10) (see the remark following (1.10)), so Theorem 1.3 gives

**Corollary 5.3.** *Suppose that $k$ and $N$ are positive integers, that $s$ is a positive odd integer, and that $i \notin S_{k^2,2k+1,\chi_{4k+2}}(s)$. Then*

$$(q;q^2)_{\lfloor \frac{N}{s}+\frac{1}{2}\rfloor} \mid A_{\mathcal{G}_k,s}(N,i,q).$$

## 6. Remarks on congruences

Congruences for the coefficients of the functions $F(q)$ and $G(q)$ in Theorems 1.2 and 1.3 can be deduced from the results of [7]. In closing we mention another approach. Theorems 1.2 and 1.3 guarantee that many of the coefficients in the $s$-dissection are divisible by high powers of $1-q$, and the congruences follow from this fact when $s = p^r$ together with an argument as in [1, Section 3].

For example, let $\mathcal{G}_k$ be the function defined in the last section and define $\xi_{\mathcal{G}_k}(n)$ by

$$\mathcal{G}_k(1-q) = \sum_{n\geq 0} \xi_{\mathcal{G}_k}(n) q^n.$$

Consider the expansions

$$\mathcal{G}_1(1-q) = \sum_{n\geq 0} \xi_{\mathcal{G}_1}(n) q^n = 1 + q + 2q^2 + 6q^3 + 25q^4 + 135q^5 + \cdots,$$

$$\mathcal{G}_2(1-q) = \sum_{n\geq 0} \xi_{\mathcal{G}_2}(n) q^n = 1 + 2q + 6q^2 + 28q^3 + 189q^4 + 1680q^5 + \cdots.$$

Then we have such congruences as

$$\xi_{\mathcal{G}_1}(5^r n - 1) \equiv 0 \pmod{5^r},$$
$$\xi_{\mathcal{G}_1}(7^r n - 1) \equiv 0 \pmod{7^r},$$
$$\xi_{\mathcal{G}_1}(13^r n - \beta) \equiv 0 \pmod{13^r}$$

for $\beta \in \{1, 2, 3, 4\}$, and

$$\xi_{\mathcal{G}_2}(7^r n - 1) \equiv 0 \pmod{7^r},$$
$$\xi_{\mathcal{G}_2}(11^r n - 1) \equiv 0 \pmod{11^r}.$$

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