OVERPARTITION THEOREMS OF THE ROGERS-RAMANUJAN TYPE

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Abstract. We give one-parameter overpartition-theoretic analogues of two classical families of partition identities: Andrews' combinatorial generalization of the Gollnitz-Gordon identities and a theorem of Andrews and Santos on partitions with attached odd parts. We also discuss geometric counterparts arising from multiple q-series identities. These involve representations of overpartitions in terms of generalized Frobenius partitions.

1. Introduction

An overpartition of n is a partition of n in which the first occurrence of a number can be overlined. A recent study of overpartitions in the context of well-poised basic hypergeometric series revealed the following theorem [16]:

Theorem. *Let $\overline{B}_k(n)$ denote the number of overpartitions of n where parts occur at most $k-1$ times, and where the total number of occurrences of j and $j+1$ together is at most k if j occurs overlined and at most $k-1$ otherwise. Let $\overline{A}_k(n)$ denote the number of overpartitions of n into parts not divisible by k. Then $\overline{A}_k(n) = \overline{B}_k(n)$.*

When there are no overlined parts in the overpartitions counted by $\overline{B}_k(n)$, then the objects are the same as those counted by $B_{k,k}(n)$ in Gordon's celebrated generalization of the Rogers-Ramanujan identities [13]:

Theorem (Gordon)**.** *Let $B_{k,i}(n)$ denote the number of partitions of n where at most $i-1$ of the parts are equal to 1 and the total number of occurrences of j and $j+1$ together is at most $k-1$. Let $A_{k,i}(n)$ denote the number of partitions of n into parts not congruent to $0, \pm i$ modulo $2k+1$. Then $A_{k,i}(n) = B_{k,i}(n)$.*

Here we will produce one-parameter analogues for overpartitions of two other well-known families of partition theorems. The first of these is due to Andrews and Santos [9] and is quite closely related to Gordon's theorem.

Theorem (Andrews-Santos)**.** *Let $C_{k,i}(n)$ denote the number of partitions of n into parts that are either even but $\not\equiv 0 \pmod{4k}$ or distinct, odd, and $\equiv \pm(2i-1) \pmod{4k}$. Let $D_{k,i}(n)$ denote the number of partitions of n wherein: (a) 2 appears as a part at most $i-1$ times, (b) the total number of occurrences of $2j$ and $2j+2$ together is at most $k-1$, and (c) $2j+1$ is allowed to appear (and may be repeated if it appears) only if the total number of appearances of $2j$ and $2j+2$ together is precisely $k-1$ (when $j=0$ we assume that 0 appears $k-i$ times). Then for $1 \le i \le k$, $C_{k,i}(n) = D_{k,i}(n)$.*

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The second is Andrews' [1] generalization of the Gollnitz-Gordon identities [12, 14] (which are the cases $k = 2$ and $i = 1$ or 2).

Theorem (Andrews)**.** *Let i and k be integers with $0 < i \leq k$. Let $E_{k,i}(n)$ denote the number of partitions of n into parts which are neither congruent to 2 modulo 4 nor congruent to $0, \pm(2i-1)$ modulo $4k$. Let $F_{k,i}(n)$ denote the number of partitions $(b_1, b_2, \ldots, b_s)$ of n in which no odd part is repeated, where $b_j - b_{j+k-1} \geq 2$ if b_j is odd and $b_j - b_{j+k-1} \geq 3$ if b_j is even, and where at most $i-1$ parts are ≤ 2. Then $E_{k,i}(n) = F_{k,i}(n)$.*

Our first theorem is the following:

Theorem 1.1. *Let $\overline{C}_k(n)$ denote the number of overpartitions of n into parts that are either even but $\not\equiv 0 \pmod{4k-2}$ or odd and $\equiv (2k-1) \pmod{4k-2}$. Let $\overline{D}_k(n)$ denote the number of overpartitions of n wherein: (a) even parts occur at most $k-1$ times, (b) the total number of appearances of $2j$ and $2j+2$ together is at most k, if $2j$ occurs overlined, and at most $k-1$ otherwise, (c) $2j+1$ is allowed to appear (non-overlined) only if the total number of appearances of $2j$ and $2j+2$ is precisely k, if $2j$ occurs overlined, and precisely $k-1$ otherwise, (d) $\overline{2j+1}$ is allowed to appear if $2j$ appears non-overlined exactly $k-1$ times. Then for $k \geq 1$, $\overline{C}_k(n) = \overline{D}_k(n)$.*

One could hardly ask for a better analogue of the theorem of Andrews and Santos. The partitions enumerated by $D_{k,i}(n)$ have the conditions of Gordon's theorem applied to their even parts, while the overpartitions enumerated by $\overline{D}_k(n)$ have the conditions of the overpartition analogue of Gordon's theorem applied to their even parts. Moreover, in both cases, odd parts are allowed provided the surrounding evens appear the maximal number of times.

In order to speak succinctly about the objects in our second theorem, we say that an odd number $2n-1$ is *unattached* in the overpartition λ if it occurs as a part, but $2n, \overline{2n}$, and $\overline{2n-1}$ do not. We also define the valuation $v_\lambda(2n)$ of an even natural number relative to an overpartition λ to be the number of occurrences of $2n, \overline{2n}$, and $\overline{2n-1}$, unless $2n-1$ occurs unattached, in which case we take $v_\lambda(2n) = 1$. We shall prove the following theorem:

Theorem 1.2. *Let $\overline{E}_k(n)$ denote the number of overpartitions of n into parts not divisible by $2k-1$. Let $\overline{F}_k(n)$ denote the number of overpartitions λ of n such that $v_\lambda(2a) \leq k-1$ for all a and such that*

$$v_\lambda(2a) + v_\lambda(2a+2) \leq \begin{cases} k+1, & \overline{2a} \text{ and } \overline{2a-1} \text{ both occur,} \\ k, & \overline{2a} \text{ or } \overline{2a-1} \text{ occurs (but not both),} \\ k, & 2a-1 \text{ occurs unattached,} \\ k-1, & \text{otherwise.} \end{cases}$$

Then $\overline{E}_k(n) = \overline{F}_k(n)$.

Notice that if there are no even overlined parts and no odd non-overlined parts, then the objects counted by $\overline{F}_k(n)$ are the same as those counted by $F_{k,k}(n)$ in Andrews' theorem.

After proving Theorems 1.1 and 1.2 in the following section, we discuss their geometric counterparts arising from multiple q-series identities. The multiple q-series shall lead us to two other functions, $\overline{\mathfrak{D}}_k(n)$ and $\overline{\mathfrak{F}}(n)$, which are also equal to $\overline{C}_k(n)$ and $\overline{E}_k(n)$, respectively. These functions will count overpartitions according to geometric decompositions associated with certain

Frobenius representations. A substantial review of the relevant combinatorics will be required, so we just state the theorems here and refer to Section 3 for a detailed description.

Theorem 1.3. *Let* $\overline{\mathfrak{D}}_k(n)$ *denote the number of Frobenius partitions of* n *whose bottom row is an overpartition into non-negative parts which occur an even number of times and whose top row is a "good" 3-regular partition whose 3-associated partition consists of a partition into even parts with at most* $k-2$ $2:1$*-rectangles together with odd parts attached below the* $k-2$*nd such rectangle. Then* $\overline{\mathfrak{D}}_k(n) = \overline{C}_k(n)$.

Theorem 1.4. *Let* $\overline{\mathfrak{F}}_k(n)$ *denote the number of overpartitions of* n *whose second Frobenius representation has a top row with at most* $k-2$ $2:1$*-rectangles. Then* $\overline{\mathfrak{F}}_k(n) = \overline{E}_k(n)$.

2. Proofs of Theorems 1.1 and 1.2

2.1. **Proof of Theorem 1.1.** We begin by defining

$$\begin{aligned} J_{k,i}(a,b;x;q) &= \frac{(xq/a,xq/b)_\infty}{(xq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n x^{kn} q^{n((2k-1)n+3)/2+(k-i)n}(xq)_n(a,b)_n}{(q,xq/a,xq/b)_n(ab)^n} \\ &\times \left(1-\frac{x^i q^{(2n+1)i-2n}(1-aq^n)(1-bq^n)}{(1-xq^{n+1}/a)(1-xq^{n+1}/b)(ab)}\right), \end{aligned} \tag{2.1}$$

where

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j) \tag{2.2}$$

and

$$(a_1,a_2,...,a_k)_n = (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n. \tag{2.3}$$

From now on, let us assume that $1 \le i \le k$. Define

$$R'_{k,i}(x) = \frac{J_{k,i}(-1,q;x;q^2)}{(xq;q^2)_\infty}. \tag{2.4}$$

Invoking [2, Eq.(2.1),(2.2)], we have the following q-difference equations for the $R'_{k,i}(x)$:

$$\begin{aligned} R'_{k,1}(x) &= \frac{1+xq}{1-xq}R'_{k,k}(xq^2), \\ R'_{k,2}(x) &= \frac{xq^2}{1-xq}R'_{k,k-1}(xq^2) + R'_{k,k}(xq^2) + \frac{xq^2}{1-xq}R'_{k,k}(xq^2), \\ R'_{k,i}(x) - R'_{k,i-1}(x) &= \frac{(xq^2)^{i-1}}{1-xq}R'_{k,k-i+1}(xq^2) + \frac{(xq^2)^{i-1}-x^{i-1}q^{2i-3}}{1-xq}R'_{k,k-i+2}(xq^2) \\ &- \frac{x^{i-1}q^{2i-3}}{1-xq}R'_{k,k-i+3}(xq^2) \qquad (3 \le i \le k). \end{aligned} \tag{2.5}$$

We actually require an equivalent set of difference equations, which follow easily from (2.5) by induction.

$$
\begin{aligned}
R'_{k,1}(x) &= \frac{1+xq}{1-xq} R'_{k,k}(xq^2), \\
R'_{k,i}(x) &= \frac{x^{i-1}q^{2i-2}}{1-xq} R'_{k,k-i+1}(xq^2) + \left(x^{i-2}q^{2i-4} + \frac{x^{i-1}q^{2i-2}}{1-xq} \right) R'_{k,k-i+2}(xq^2) \\
&+ \sum_{t=0}^{i-3} (x^t q^{2t} + x^{t+1}q^{2t+2}) R'_{k,k-t}(xq^2) \qquad (2 \leq i \leq k). \qquad (2.6)
\end{aligned}
$$

If we write

$$R'_{k,i}(x) = \sum_{m,n \geq 0} r'_{k,i}(m,n) x^m q^n, \tag{2.7}$$

then (2.6) implies that

$$r'_{k,1}(m,n) = r'_{k,k}(m, n-2m) + 2 \sum_{j \geq 1} r'_{k,k}(m-j, n-2m+j) \tag{2.8}$$

and, for $2 \leq i \leq k$,

$$
\begin{aligned}
r'_{k,i}(m,n) &= \sum_{j \geq 0} r'_{k,k-i+1}(m-i+1-j, n-2m+j) \\
&+ \sum_{j \geq 0} r'_{k,k-i+2}(m-i+1-j, n-2m+j) \\
&+ \sum_{t=2}^{i-1} \left(r'_{k,k-i+t}(m-i+t, n-2m) + r'_{k,k-i+t+1}(m-i+t, n-2m) \right) \\
&+ r'_{k,k}(m, n-2m). \qquad (2.9)
\end{aligned}
$$

From the definition of the $R'_{k,i}$, we have

$$r'_{k,i}(m,n) = \begin{cases} 1, & (m,n) = (0,0), \\ 0, & m \leq 0 \text{ or } n \leq 0,\ (m,n) \neq (0,0). \end{cases} \tag{2.10}$$

This fact together with the recurrences above uniquely define the $r'_{k,i}(m,n)$. Now let $s'_{k,i}(m,n)$ denote the number of overpartitions counted by $\overline{D}_k(n)$ which have m (positive) parts, $k-i$ non-overlined zeros, and at most $i-1$ occurrences of 2 (overlined or not). The $s'_{k,i}(m,n)$ clearly satisfy (2.10), and we shall show that they also satisfy the recurrences (2.8) and (2.9), allowing us to conclude that $s'_{k,i}(m,n) = r'_{k,i}(m,n)$.

We begin with (2.8). If λ is an overpartition counted by $s'_{k,1}(m,n)$, then there are no occurrences of 2 or $\overline{2}$ in λ, but there may be ones since 0 is assumed to occur $k-1$ times. If there are no ones, then we may remove two from every part to get an overpartition counted by $s'_{k,k}(m, n-2m)$. If there are j ones ($j \geq 1$), then the first occurrence of one may be overlined or not. In each of the two cases, we can remove the j ones from λ and subtract two from each remaining part to obtain an overpartition counted by $s'_{k,k}(m-j, n-2m+j)$. Since these operations are reversible, we have (2.8).

Next we address (2.9). If λ is an overpartition counted by $s'_{k,i}(n)$ for $i \geq 2$, then λ has at most $i-1$ occurrences of 2 (overlined or not). If there are indeed $i-1$ occurrences of 2, then there may be ones, since 0 is taken to occur $k-i$ times. Notice that all such ones are non-overlined, since we have $i \geq 2$. Let us remove the ones and twos from λ and subtract two from each remaining part to get an overpartition of $n-2m+j$ into $m-i+1-j$ parts. Now, there are two possibilities: 2 occurred overlined and 2 did not occur overlined in λ.

-In the first case, since 2 was overlined in λ, there are up to $k-i+1$ twos in the new overpartition λ'. In addition, there may have been non-overlined 3's in λ if 4 occurred exactly $k-i+1$ times, so there may be non-overlined ones in λ' if 2 occurs exactly $k-i+1$ times. This is consistent with assuming that 0 occurs $i-2$ times in λ'. In other words, λ' is an overpartition counted by $s'_{k,k-i+2}(m-i+1-j, n-2m+j)$.

-In the second case, that is, if 2 does not occur overlined in λ, then there are at most $k-i$ twos in λ'. If 4 had occurred exactly $k-i$ times in λ, then there may be ones in λ', and an overlined one if $i=k$. This is consistent with assuming that 0 occurs $i-1$ times in λ'. Hence, λ' is an overpartition counted by $s'_{k,k-i+1}(m-i+1-j, n-2m+j)$.

We continue to the case where λ has $i-t$ twos, with $2 \leq t \leq i-1$. Now there cannot be any ones in λ, so we may remove the $i-t$ twos to get an overpartition λ' of $n-2m$ with $m-i+t$ parts. Again there are two possibilities: 2 occurred overlined and 2 did not occur overlined in λ.

-In the first case, there may be up to $k-i+t$ twos in λ'. If there are exactly $k-i+t$ of them, then there may have been non-overlined threes in λ and hence there may be non-overlined ones in λ'. This is consistent with the assumption that 0 occurs $i-t-1$ times in λ'. So, λ' is an overpartition counted by $s'_{k,k-i+t+1}(m-i+t, n-2m)$.

-If 2 was non-overlined in λ, then there may be as many as $k-i-1+t$ twos in λ'. If there are exactly $k-i-1+t$ of them, then there may have been threes in λ and so there may be ones in λ' (These must be non-overlined). This is consistent with the assumption that 0 occurs $i-t$ times in λ'. Hence λ' is an overpartition counted by $s'_{k,k-i+t}(m-i+t, n-2m)$.

Finally, we consider the case where there are no twos in λ, so that removing two from each part results in an overpartition λ' of $n-2m$ into m parts. There may be up to $k-1$ appearances of two in λ'. Moreover, there may be ones if there are exactly $k-1$ such appearances. This is consistent with the assumption that 0 does not occur at all in λ'. Hence, λ' is an overpartition counted by $s'_{k,k}(m, n-2m)$.

Since all the operations described above are bijective, taking all the cases together establishes (2.9). Now we may conclude that $s'_{k,i}(m,n) = r'_{k,i}(m,n)$ and observe that

$$\begin{aligned}
\sum_{n\geq 0} \overline{D}_k(n)q^n &= \sum_{m,n\geq 0} s'_{k,k}(m,n)q^n \\
&= \sum_{m,n\geq 0} r'_{k,k}(m,n)q^n \\
&= R'_{k,k}(1) \\
&= \frac{(-1;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{(2k-1)n^2+2n}}{1+q^{2n}} \left(1 + \frac{q^{2k(2n+1)-4n-1}(1+q^{2n})}{1+q^{2n+2}}\right) \\
&= \frac{(-1;q^2)_\infty}{(q^2;q^2)_\infty} \left(\sum_{n=0}^{\infty} \frac{q^{(2k-1)n^2+2n}}{1+q^{2n}} + \sum_{n=1}^{\infty} \frac{q^{(2k-1)(n^2-2n+1)+(2k-1)(2n-1)}}{1+q^{2n}}\right) \\
&= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty} \left(1 + 2\sum_{n=1}^{\infty} \frac{(1+q^{2n})q^{(2k-1)n^2}}{1+q^{2n}}\right) \\
&= \frac{(-q^2;q^2)_\infty(-q^{2k-1};q^{4k-2})^2_\infty(q^{4k-2};q^{4k-2})_\infty}{(q^2;q^2)_\infty} \\
&= \frac{(-q^2;q^2)_\infty(q^{4k-2};q^{4k-2})_\infty(-q^{2k-1};q^{4k-2})_\infty}{(q^2;q^2)_\infty(-q^{4k-2};q^{4k-2})_\infty(q^{2k-1};q^{4k-2})_\infty} \\
&= \sum_{n\geq 0} \overline{C}_k(n)q^n,
\end{aligned}$$

where the antepenultimate equality follows from Jacobi's triple product identity [11, p. 239, Eq. (II.28)]. ☐

2.2. **Proof of Theorem 1.2.** :

Define

$$R_{k,i}(x) = \frac{J_{k,i}(-1,-q;x;q^2)}{(xq;q^2)_\infty}. \tag{2.11}$$

From [2, Eq.(2.1),(2.2)], we find the following q-difference equations for the $R_{k,i}(x)$:

$$\begin{aligned}
R_{k,1}(x) &= R_{k,k}(xq^2), \\
R_{k,2}(x) - R_{k,1}(x) &= \frac{xq^2}{1-xq} R_{k,k-1}(xq^2) + \frac{xq+xq+xq^2}{1-xq} R_{k,k}(xq^2), \\
R_{k,i}(x) - R_{k,i-1}(x) &= \frac{(xq^2)^{i-1}}{1-xq} R_{k,k-i+1}(xq^2) + \frac{(xq^2)^{i-1} + x^{i-1}q^{2i-3}}{1-xq} R_{k,k-i+2}(xq^2) \\
&+ \frac{x^{i-1}q^{2i-3}}{1-xq} R_{k,k-i+3}(xq^2) \qquad (3 \leq i \leq k).
\end{aligned} \tag{2.12}$$

If we write

$$R_{k,i}(x) = \sum_{m,n\geq 0} r_{k,i}(m,n)x^m q^n, \tag{2.13}$$

then (2.12) implies that

$$\begin{aligned} r_{k,1}(m,n) &= r_{k,k}(m,n-2m), && (2.14)\\ r_{k,2}(m,n)-r_{k,1}(m,n) &= \sum_{j\geq 0} r_{k,k-1}(m-1-j,n-2m+j)\\ &+ \sum_{j\geq 1} r_{k,k}(m-j,n-2m+j)\\ &+ \sum_{j\geq 0} r_{k,k}(m-1-j,n-2m+1+j)\\ &+ \sum_{j\geq 0} r_{k,k}(m-1-j,n-2m+j), && (2.15) \end{aligned}$$

and, for $3 \leq i \leq k$,

$$\begin{aligned} r_{k,i}(m,n)-r_{k,i-1}(m,n) &= \sum_{j\geq 0} r_{k,k-i+1}(m-i+1-j,n-2m+j)\\ &+ \sum_{j\geq 0} r_{k,k-i+2}(m-i+1-j,n-2m+j)\\ &+ \sum_{j\geq 0} r_{k,k-i+2}(m-i+1-j,n-2m+1+j)\\ &+ \sum_{j\geq 0} r_{k,k-i+3}(m-i+1-j,n-2m+1+j). && (2.16) \end{aligned}$$

From the definition of the $R_{k,i}$, we have

$$r_{k,i}(m,n) = \begin{cases} 1, & (m,n)=(0,0),\\ 0, & m\leq 0 \text{ or } n \leq 0,\ (m,n)\neq (0,0). \end{cases} \tag{2.17}$$

This fact together with the recurrences above uniquely define the $r_{k,i}(m,n)$.

Now let $s_{k,i}(m,n)$ denote the number of overpartitions λ of n into m parts where $v_\lambda(2a) \leq k-1$ for all n, $v_\lambda(2) \leq i-1$, and

$$v_\lambda(2a)+v_\lambda(2a+2) \leq \begin{cases} k+1, & \overline{2a} \text{ and } \overline{2a-1} \text{ both occur},\\ k, & \overline{2a} \text{ or } \overline{2a-1} \text{ occurs (but not both)},\\ k, & 2a-1 \text{ occurs unattached},\\ k-1, & \text{otherwise}. \end{cases} \tag{2.18}$$

The $s_{k,i}(m,n)$ certainly satisfy (2.17). We shall show that they also satisfy (2.14), (2.15), and (2.16), and so conclude that they are equal to the $r_{k,i}(m,n)$. Before continuing, we make the important observation that removing all ones and twos from an overpartition which satisfies (2.18) and then subtracting 2 from each remaining part preserves the conditions (2.18).

We begin by addressing (2.14). If λ is an overpartition counted by $r_{k,1}(m,n)$, then λ has no occurrences of 1 or 2, overlined or not. Hence we can remove 2 from each part to get an overpartition λ' of $n-2m$ with m parts. We have $v_{\lambda'}(2) \leq k-1$ and the conditions (2.18) still hold. Since this operation is easily inverted, we have $s_{k,1}(m,n) = s_{k,k}(m,n-2m)$.

Next we treat (2.15). Notice that $s_{k,2}(m,n) - s_{k,1}(m,n)$ enumerates those overpartitions counted by $s_{k,2}(m,n)$ which have $v_\lambda(2) = 1$. If λ is such an overpartition, then $v_\lambda(2) = 1$ for one of four possible reasons: 2 occurs, $\overline{2}$ occurs, $\overline{1}$ occurs, or 1 occurs unattached. We consider the four cases separately.

-If 2 occurs, we may remove it, remove the j non-overlined ones ($j \geq 0$), and subtract two from each remaining part. The resulting overpartition λ' of $n - 2m + j$ has $m - 1 - j$ parts, satisfies the conditions (2.18), and has $v_{\lambda'}(2) \leq k - 2$. In other words, λ' is an overpartition counted by $r_{k,k-1}(m - 1 - j, n - 2m + j)$.

-If $\overline{2}$ occurs, then we may remove it, remove the j non-overlined ones ($j \geq 0$), and subtract two from each remaining part. The resulting overpartition λ' of $n-2m+j$ has $m-1-j$ parts, satisfies the conditions (2.18), and has $v_{\lambda'}(2) \leq k - 1$. In other words, λ' is an overpartition counted by $r_{k,k}(m - 1 - j, n - 2m + j)$.

-If $\overline{1}$ occurs, then we may remove it, remove the j non-overlined ones ($j \geq 0$), and subtract two from each remaining part. The resulting overpartition λ' of $n - 2m + 1 + j$ has $m - 1 - j$ parts, satisfies the conditions (2.18), and has $v_{\lambda'}(2) \leq k - 1$. In other words, λ' is an overpartition counted by $r_{k,k}(m - 1 - j, n - 2m + 1 + j)$.

-If 1 occurs unattached in λ then we may remove the j non-overlined ones ($j \geq 1$) and subtract two from each remaining part. The resulting overpartition λ' of $n - 2m + j$ has $m - j$ parts, satisfies the conditions (2.18), and has $v_{\lambda'}(2) \leq k - 1$. In other words, λ' is an overpartition counted by $r_{k,k}(m - j, n - 2m + j)$.

Since the operations described are easily inverted, the four cases taken together imply (2.15).

Finally, we tackle (2.16). Notice that $s_{k,i}(m,n) - s_{k,i-1}(m,n)$ enumerates those overpartitions counted by $s_{k,i}(m,n)$ which have $v_\lambda(2) = i-1$. If λ is such an overpartition, then $v_\lambda(2) = i-1$ for one of four possible reasons: 2 occurs $i-1$ times, 2 occurs $i-2$ times and $\overline{2}$ occurs, 2 occurs $i-2$ times and $\overline{1}$ occurs, or 2 occurs $i-3$ times and both $\overline{2}$ and $\overline{1}$ occur. Observe that here $i - 1 \geq 2$ so that λ cannot have an unattached occurrence of 1. We again consider the four cases separately.

-If 2 occurs $i - 1$ times, we may remove them, remove the j non-overlined ones ($j \geq 0$), and subtract two from each remaining part. The resulting overpartition λ' of $n - 2m + j$ has $m - i + 1 - j$ parts, satisfies the conditions (2.18), and has $v_{\lambda'}(2) \leq k - i$. In other words, λ' is an overpartition counted by $r_{k,k-i+1}(m - i + 1 - j, n - 2m + j)$.

-If 2 occurs $i - 2$ times and $\overline{2}$ occurs, we may remove them, remove the j non-overlined ones ($j \geq 0$), and subtract two from each remaining part. The resulting overpartition λ' of $n-2m+j$ has $m-i+1-j$ parts, satisfies the conditions (2.18), and has $v_{\lambda'}(2) \leq k-i+1$. In other words, λ' is an overpartition counted by $r_{k,k-i+2}(m - i + 1 - j, n - 2m + j)$.

-If 2 occurs $i - 2$ times and $\overline{1}$ occurs, we may remove them, remove the j non-overlined ones ($j \geq 0$), and subtract two from each remaining part. The resulting overpartition λ' of $n - 2m + 1 + j$ has $m - i + 1 - j$ parts, satisfies the conditions (2.18), and has $v_{\lambda'}(2) \leq k - i + 1$. In other words, λ' is an overpartition counted by $r_{k,k-i+2}(m - i + 1 - j, n - 2m + 1 + j)$.

-If 2 occurs $i - 3$ times and both $\overline{2}$ and $\overline{1}$ occur, we may remove them, remove the j non-overlined ones ($j \geq 0$), and subtract two from each remaining part. The resulting overpartition λ' of $n-2m+1+j$ has $m-i+1-j$ parts, satisfies the conditions (2.18), and has $v_{\lambda'}(2) \leq k-i+2$. In other words, λ' is an overpartition counted by $r_{k,k-i+3}(m - i + 1 - j, n - 2m + 1 + j)$.

Again the above operations are invertible, so the four cases taken together imply (2.16). We may now conclude that $s_{k,i}(m,n) = r_{k,i}(m,n)$. Hence we have

$$\begin{aligned}
\sum_{n\geq 0} \overline{F}_k(n)q^n &= \sum_{m,n\geq 0} s_{k,k}(m,n)q^n \\
&= \sum_{m,n\geq 0} r_{k,k}(m,n)q^n \\
&= R_{k,k}(1) \\
&= \frac{(-1)_\infty}{(q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{(2k-1)n^2+2n}}{1+q^{2n}} \left(1 - \frac{q^{2k(2n+1)-4n-1}(1+q^{2n})}{1+q^{2n+2}}\right) \\
&= \frac{(-1)_\infty}{(q)_\infty} \left(\sum_{n=0}^{\infty} \frac{(-1)^n q^{(2k-1)n^2+2n}}{1+q^{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{(2k-1)(n^2-2n+1)+(2k-1)(2n-1)}}{1+q^{2n}}\right) \\
&= \frac{(-q)_\infty}{(q)_\infty} \left(1 + 2\sum_{n=1}^{\infty} \frac{(1+q^{2n})(-1)^n q^{(2k-1)n^2}}{1+q^{2n}}\right) \\
&= \frac{(-q)_\infty (q^{2k-1};q^{2k-1})_\infty}{(q)_\infty(-q^{2k-1};q^{2k-1})_\infty} \\
&= \sum_{n\geq 0} \overline{E}_k(n)q^n,
\end{aligned}$$

where the penultimate equality follows from Jacobi's triple product identity [11, p. 239, Eq. (II.28)]. □

3. Geometric counterparts

3.1. **Combinatorial preparation.** Before proceeding to the treatment of the geometric counterparts of Theorems 1.1 and 1.2, we require some combinatorial preparation. First, a generalized Frobenius partition [2] is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_k \\ b_1 & b_2 & \dots & b_k \end{pmatrix}$$

where $\sum a_i$ is a partition taken from a set A and $\sum b_i$ is a partition taken from a set B. We denote by $P_{A,B}(n)$ the number of generalized Frobenius partitions with $\sum(a_i + b_i) = n$. The number of overpartitions of n can be shown to be $P_{Q,\mathcal{O}}(n)$, where Q denotes the set of partitions into distinct parts and $\mathcal{O}$ denotes the set of overpartitions into non-negative parts [10]. This is called the Frobenius representation of an overpartition.

Next, we recall the Durfee square decomposition of a partition [7]. The Ferrers diagram of a partition λ has a largest upper-left justified square called the Durfee square. Since there is a partition below this square, we identify its Durfee square as the second Durfee square of the partition λ. Continuing in this way, we obtain a sequence of successive squares, as illustrated in Fig. 1. The generating function for partitions whose parts are at most N and which have at

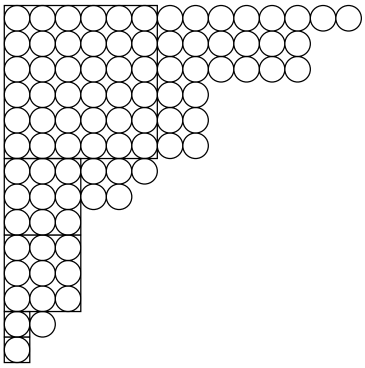

FIGURE 1. The successive Durfee squares of a partition

most $k-2$ Durfee squares is [7]

$$\sum_{n_{k-2}\geq\cdots\geq n_1} q^{n_{k-2}^2+\cdots+n_1^2}\begin{bmatrix} N \\ n_{k-2}\end{bmatrix}_q \cdots \begin{bmatrix} n_2 \\ n_1\end{bmatrix}_q. \tag{3.1}$$

Here we have employed the q-binomial coefficient

$$\begin{bmatrix} n \\ m\end{bmatrix}_q = \frac{(q)_n}{(q)_m(q)_{n-m}}, \tag{3.2}$$

which is the generating function for partitions into at most $n-m$ parts, each at most m. If we replace q by q^2 in (3.1), then we pass to partitions into even parts, and the Durfee squares become $d \times 2d$ rectangles, or $2:1$-rectangles, following [4].

Finally, a p-regular partition is a partition whose parts occur at most $p-1$ times. We will say such a partition is "good" if removing 1 from the $p-1$ smallest parts, 2 from the next $p-1$ smallest parts, and so on, results in a partition μ. We define the p-associated partition of a good p-regular partition to be the conjugate of μ.

3.2. **A review of Gordon's theorem for overpartitions.** The geometric counterpart of Gordon's theorem for overpartitions is related to the identity

$$\sum_{n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{q^{n_{k-1}(n_{k-1}+1)/2+n_{k-2}^2+\cdots+n_1^2}(-1)_{n_{k-1}}}{(q)_{n_{k-1}}}\begin{bmatrix} n_{k-1} \\ n_{k-2}\end{bmatrix}_q \cdots \begin{bmatrix} n_2 \\ n_1\end{bmatrix}_q = \prod_{\substack{n\geq 1 \\ n\nmid k}} \frac{1+q^n}{1-q^n}. \tag{3.3}$$

This identity is the specialization $a=1, b_k=-1$, and $N, b_1, c_1, ..., b_{k-1}, c_{k-1}, c_k \to \infty$ of Andrews' multiple series generalization of Watson's transformation [6]. The case $k=2$ is the following instance of Lebesgue's identity [8, Cor. 2.7], which appears as entry (12) in [17]:

$$\sum_{n\geq 0} \frac{(-1)_n q^{n(n+1)/2}}{(q)_n} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}. \tag{3.4}$$

In [10], it was shown that the left side of (3.3) generates overpartitions whose Frobenius representations have top rows with at most $k-2$ Durfee squares in their 2-associated partitions. It will be helpful to recall why this is true. First, $(-1)_{n_{k-1}}/(q)_{n_{k-1}}$ is the generating function for overpartitions into exactly n_{k-1} non-negative parts [10]. This corresponds to the bottom row of the Frobenius representation. The rest of the summand, which is the top row, is $q^{n_{k-1}(n_{k-1}+1)/2}$ multiplied by an instance of (3.1) (with $N = n_{k-1}$), the latter being the 2-associated partition.

3.3. **Proof of Theorem 1.3.** We are now prepared to discuss the geometric counterparts of Theorems 1.1 and 1.2. In the first case, the relevant multiple q-series identity is

$$\sum_{n_k \geq \cdots \geq n_1 \geq 0} \frac{(-1;q^2)_{n_k} q^{n_k^2+n_k+2n_{k-1}^2+\cdots+2n_2^2+2n_1^2-n_1}}{(q^2;q^2)_{n_k}(q;q^2)_{n_1}} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix}_{q^2} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_{q^2}$$

$$= \frac{(-q^2;q^2)_\infty (q^{4k-2};q^{4k-2})_\infty (-q^{2k-1};q^{4k-2})_\infty}{(q^2;q^2)_\infty (-q^{4k-2};q^{4k-2})_\infty (q^{2k-1};q^{4k-2})_\infty}. \tag{3.5}$$

This identity is the specialization $q = q^2, a = 1, c_{k+1} = -1, c_1 = q$ and $N, b_1, ..., b_{k+1}, c_2, ..., c_k \to \infty$ of the k-fold version of Andrews' multiple series generalization of Watson's transformation [6] (see also [15, Eq. (17)]). The case $k = 2$, by applying the q-Chu Vandermonde summation [11, p.236, (2.7)] to the sum over n_1, is identity (48) in [17],

$$\sum_{n \geq 0} \frac{(-1;q^2)_n q^{n^2+n}}{(q)_{2n}} = \frac{(-q^2;q^2)_\infty (-q^3,q^6;q^6)_\infty}{(q^2;q^2)_\infty (q^3,-q^6;q^6)_\infty}. \tag{3.6}$$

Observe that in Theorem 1.1, the overpartitions counted by $\overline{D}_k$ are constructed by taking "twice" an overpartition counted by $\overline{B}_k$ and attaching odd parts if some maximality condition is satisfied. The same thing happens in the geometric counterpart. The sum over the variables n_2 through n_k on the left hand side of (3.5) is an instance of the left side of (3.3), with q replaced by q^2. Twice an overpartition λ counted by $P_{Q,\mathcal{O}}$, denoted 2λ, will be obtained by doubling the number of occurrences of each part in the top and bottom rows. In the bottom row, then, we just have an overpartition into non-negative parts that occur an even number of times.

In the top row, we have a partition where parts occur at most twice. Its 3-associated partition is a partition into even parts that is twice the 2-associated partition of the top row of λ and has at most $k-2$ $2:1$-rectangles. The rest of the summand on the left side of (3.5),

$$\frac{q^{2n_1^2-n_1}}{(q;q^2)_{n_1}} \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_{q^2},$$

is the generating function for partitions into odd parts that are smaller than $2n_2$. These can be placed below the 3-associated partition, only if there are exactly $k-2$ $2:1$-rectangles. This establishes Theorem 1.3.

3.4. **Proof of Theorem 1.4.** We now turn to a geometric counterpart of Theorem 1.2, which corresponds to the identity:

$$\sum_{n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_{k-1}+2n_{k-2}^2+\cdots+2n_1^2}(-1)_{2n_{k-1}}}{(q^2;q^2)_{n_{k-1}}} \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix}_{q^2} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_{q^2} = \frac{(-q)_\infty (q^{2k-1};q^{2k-1})_\infty}{(q)_\infty (-q^{2k-1};q^{2k-1})_\infty}. \tag{3.7}$$

This identity is the case $q = q^2, a = 1, b_k = -1, c_k = -q$, and $N, b_1, c_1, ..., b_{k-1}, c_{k-1} \to \infty$ of Andrews multiple series generalization of Watson's transformation [6]. The case $k = 2$ is identity (24) in [17],

$$\sum_{n\geq 0}\frac{(-1)_{2n}q^n}{(q^2;q^2)_n}=\frac{(-q)_\infty(q^3;q^3)_\infty}{(q)_\infty(-q^3;q^3)_\infty}. \tag{3.8}$$

To discuss the combinatorics of (3.7) we will use a second representation for overpartitions as generalized Frobenius partitions.

Proposition 3.1. *Let A denote the set of partitions into non-negative even parts and let B denote the set of overpartitions $b_1+b_2+\cdots$ such that*

$$b_j-b_{j+1}\geq\begin{cases}1, & b_{j+1}\text{ even,}\\ 2, & b_{j+1}\text{ odd and overlined,}\\ 3, & b_{j+1}\text{ even and overlined.}\end{cases}$$

Then $P_{A,B}(n)$ is equal to the number of overpartitions of n.

Proof. The first goal is to establish that the finite product

$$\frac{q^n(-1;q)_{2n}}{(q^2;q^2)_n}=\frac{q^n(-1;q^2)_n}{(q^2;q^2)_n}\times(-q;q^2)_n \tag{3.9}$$

is the generating function for overpartitions in the set B which have exactly n parts. To start, $q^n(-1;q^2)_n/(q^2;q^2)_n$ is the generating function for overpartitions μ into exactly n odd parts. Also, $(-q;q^2)_n$ generates a partition ν into distinct odd parts less than $2n+1$. Given two such objects, write the parts of μ in non-increasing order. Remove the largest part of ν, say $2k-1$, then add 2 to the first $k-1$ parts of μ and add 1 to the kth part. Repeat this process until ν is empty. The result is easily seen to be an overpartition which satisfies the difference conditions in the proposition. Moreover, the location of the even parts indicates clearly how to reverse the process.

Since $1/(q^2;q^2)_n$ is the generating function for a partition from the set A with exactly n parts, we have

$$\sum_{n=0}^{\infty}P_{A,B}(n)q^n=\sum_{n=0}^{\infty}\frac{(-1;q)_{2n}q^n}{(q^2;q^2)_n^2}. \tag{3.10}$$

This last sum, by an application of the q-Gauss summation [11, p.236, (II.8)], is $(-q)_\infty/(q)_\infty$, the generating function for overpartitions [10]. □

Putting (3.9) together with (3.1), we have Theorem 1.4.

4. Concluding Remarks

There are several natural questions that arise from the work here and in [16]. First, is it possible to approach these overpartition theorems in another way which allows for full two-parameter analogues of the classical partition theorems? Second, is there a combinatorial explanation for the equality of any of the overpartition functions considered here? Finally, Andrews has considered more general well-poised series, $J_{k,i,d}(a_1,...,a_d;x;q)$ [2], and their applications to partition theorems [3, 5]. Namely, there is a partition theorem corresponding to each $J_{k,i,d}(-q,-q^2,...,-q^d;x;q^{d+1})$ when $k\geq d$. The case $d=0$ is Gordon's theorem [13] and the case $d=1$ is Andrews' generalization of the Gollnitz-Gordon identities [1], both of which have nice analogues for overpartitions. Are there reasonable overpartition-theoretic analogues of Andrews' general partition theorems?

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