PARTITIONS AND OVERPARTITIONS WITH ATTACHED PARTS

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Abstract. We show how to interpret a certain q-series as a generating function for overpartitions with attached parts. A number of families of partition theorems follow as corollaries.

1. Introduction

The purpose of this paper is to unify and generalize some families of partition theorems, including Gordon's generalization of the Rogers-Ramanujan identities [9], Andrews' generalization of the Göllnitz-Gordon identities [1], the two Gordon's theorems for overpartitions [10], the Andrews-Santos identities for partitions with attached odd parts [3], and their overpartition analogue [11]. This will be accomplished through a combinatorial study of the series $G_{k,i}(a, b; x; q)$, defined by

$$G_{k,i}(a,b;x;q) = \frac{(-bxq)_\infty}{(xq)_\infty} \sum_{n=0}^{\infty} \frac{x^{kn} q^{n((2k-1)n+3)/2+(k-i)n} (xq)_n (1/a,-1/b)_n (ab)^n}{(q,axq,-bxq)_n} \times \left(1 + \frac{abx^i q^{(2n+1)i-2n}(1-q^n/a)(1+q^n/b)}{(1-axq^{n+1})(1+bxq^{n+1})}\right). \tag{1.1}$$

Here we have employed the usual q-series notation [8]. In the context of Andrews' theory of well-poised basic hypergeometric series [2], the series $G_{k,i}(a, b; x; q)$ can be expressed in terms of his $J_{k,i}(a, b; x; q)_2$ as

$$G_{k,i}(a,b;x;q) = \frac{J_{k,i}(1/a,-1/b;x;q)_2}{(axq)_\infty}. \tag{1.2}$$

Our main result is an interpretation of the coefficient of $a^r b^s x^t q^n$ in $G_{k,i}(a, b; x; q)$ using the framework of overpartitions and overpartition pairs [5, 12]. Recall that by an overpartition we mean a partition in which the first occurrence of a number may be overlined. We employ the notation $f_j(\lambda)$ to denote the number of occurrences of j in λ.

Theorem 1.1. *For $k \geq 2$ and $1 \leq i \leq k$, let $g_{k,i}(r, s, t, n)$ denote the number of overpartition pairs (λ, μ) of n with $k-i$ non-overlined zeros in λ and t positive parts, r of which are in μ, and s of which are overlined, where for each $j \geq 0$ we have (i) $f_j(\lambda) + f_{\overline{j+1}}(\lambda) + f_{j+1}(\lambda) \leq k-1$, (ii) if $f_j(\lambda) + f_{\overline{j+1}}(\lambda) + f_{j+1}(\lambda) = k-1$, then $j+1$ may occur in μ, and (iii) if $f_j(\lambda) = k-1$, then $\overline{j+1}$ may occur in μ. Then*

$$G_{k,i}(a,b;x;q) = \sum_{r,s,t,n \geq 0} g_{k,i}(r,s,t,n) a^r b^s x^t q^n. \tag{1.3}$$

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Following [3], we say that the parts of the overpartition μ are *attached* to the overpartition λ.

We might make a few remarks here. First, the $k-i$ "phantom" zeros in the statement of Theorem 1.1 are just an artifice to streamline the definition of the $g_{k,i}(r,s,t,n)$. Second, one may compare this theorem, where achieving a boundary condition affords more freedom to the combinatorial objects under consideration, to results like those in [4, 6] or [12], where achieving a boundary condition induces further restrictions on the objects. Finally, we note that the case $a=0$ of Theorem 1.1 is Corollary 1.1 of [7].

In the next section we use q-difference equations to establish Theorem 1.1. In Section 3 we mention how the various results listed in the opening paragraph are corollaries of Theorem 1.1 and offer another example of a family of overpartition identities that can be deduced from this theorem:

Theorem 1.2. *For all $k \geq 2$ and $2 \leq i \leq k$, let $H_{k,i}(n)$ denote the number of overpartitions ν of n having $k-i$ phantom zeros such that (i) for each $j \geq 0$, $f_{2j}(\nu)+f_{\overline{2j+1}}(\nu)+f_{2j+2}(\nu) \leq k-1$, (ii) if $f_{2j}(\nu)+f_{\overline{2j+1}}(\nu)+f_{2j+2}(\nu) = k-1$, then $2j+1$ may occur (without restriction), and (iii) if $f_{2j}(\nu) = k-1$, then $\overline{2j}$ may occur. Let $H'_{k,i}(n)$ denote the number of overpartitions of n where the non-overlined parts are even and not divisible by $4k-2$ and the overlined parts are odd or congruent to $\pm(2i-2)$ modulo $4k-2$. Then $H_{k,i}(n) = H'_{k,i}(n)$.*

2. Proof of Theorem 1.1

Henceforth we assume that $k \geq 2$ and that $1 \leq i \leq k$. Using equation (1.2) and the q-difference equations for the $J_{k,i}(a,b;x;q)_2$ (and related functions) recorded in [2, Eq. 2.1, 2.2, 2.3], we may deduce the following q-difference equations for the $G_{k,i}(a,b;x;q)$:

Lemma 2.1.

$$G_{k,1}(a,b;x;q) = \frac{(1+abxq)}{(1-axq)} G_{k,k}(a,b;xq;q), \tag{2.1}$$

$$\begin{aligned} G_{k,2}(a,b;x;q) &= G_{k,k}(a,b;xq;q) \\ &+ \frac{bxq}{1-axq} G_{k,k}(a,b;xq;q) \\ &+ \frac{xq}{1-axq} G_{k,k-1}(a,b;xq;q), \end{aligned} \tag{2.2}$$

and, for $i \geq 3$,

$$\begin{aligned} G_{k,i}(a,b;x;q) - G_{k,i-1}(a,b;x;q) &= \frac{(xq)^{i-1}}{1-axq} G_{k,k-i+1}(a,b;xq;q) \quad (2.3) \\ &- \frac{a(xq)^{i-1}}{1-axq} G_{k,k-i+2}(a,b;xq;q) \\ &+ \frac{b(xq)^{i-1}}{1-axq} G_{k,k-i+2}(a,b;xq;q) \quad (2.4) \\ &- \frac{ab(xq)^{i-1}}{1-axq} G_{k,k-i+3}(a,b;xq;q). \end{aligned}$$

The presence of the minus sign in this final equation is rather undesirable from a combinatorial point of view, but fortunately there is a nicer set of equations which follow by induction from Lemma 2.1.

Lemma 2.2.

$$G_{k,1}(a,b;x;q) = \frac{(1+abxq)}{(1-axq)} G_{k,k}(a,b;xq;q), \tag{2.5}$$

and, for $i \geq 2$*,*

$$\begin{aligned} G_{k,i}(a,b;x;q) &= \frac{(xq)^{i-1}}{1-axq} G_{k,k-i+1}(a,b;xq;q) \\ &+ \frac{b(xq)^{i-1}}{1-axq} G_{k,k-i+2}(a,b;xq;q) \\ &+ \sum_{v=0}^{i-3} b(xq)^{v+1} G_{k,k-v}(a,b;xq;q) \\ &+ \sum_{v=0}^{i-2} (xq)^{v} G_{k,k-v}(a,b;xq;q). \end{aligned} \tag{2.6}$$

Since

$$G_{k,i}(a,b;0;q) = 1 \tag{2.7}$$

for all k and i, equations (2.5) and (2.6) uniquely define the functions $G_{k,i}(a,b;x;q)$. Let

$$\widehat{G}_{k,i}(a,b;x;q) = \sum_{r,s,t,n \geq 0} g_{k,i}(r,s,t,n) a^r b^s x^t q^n, \tag{2.8}$$

with the $g_{k,i}(r,s,t,n)$ defined as in Theorem 1.1. To prove the theorem we will show that the $\widehat{G}_{k,i}(a,b;x;q)$ satisfy the same q-difference equations as the $G_{k,i}(a,b;x;q)$. The boundary condition (2.7) is clear since there is only one overpartition pair having no parts, the empty one.

The function $\widehat{G}_{k,1}(a,b;x;q)$ is the generating function for overpartition pairs (λ,μ) where λ has no ones and at most $k-1$ twos, and μ has ones without restriction. Subtracting one from each part ≥ 2, we see that

$$\widehat{G}_{k,1}(a,b;x;q) = \frac{(1+abxq)}{(1-axq)} \widehat{G}_{k,k}(a,b;xq;q). \tag{2.9}$$

This is (2.5).

Establishing (2.6) is a bit more involved. For $i \geq 2$, we divide those overpartition pairs (λ,μ) generated by $\widehat{G}_{k,i}(a,b;x;q)$ into four groups, depending on whether $f_1(\lambda) + f_{\overline{1}}(\lambda) = i-1$ and whether $\overline{1}$ occurs in λ. First, if $f_1(\lambda) + f_{\overline{1}}(\lambda) = i-1$ and $\overline{1}$ appears in λ, then there may be up to $k-i+1$ twos in λ and 1 (but not $\overline{1}$) can appear without restriction in μ. Hence by subtracting one from each part ≥ 2 we see that these overpartition pairs are generated by

$$\frac{bxq(xq)^{i-2}}{(1-axq)} \widehat{G}_{k,k-i+2}(a,b;xq;q). \tag{2.10}$$

Next, if $f_1(\lambda) + f_{\overline{1}}(\lambda) = i-1$ and $\overline{1}$ does not appear in λ, then there may be up to $k-i$ twos in λ and 1 (but not $\overline{1}$) can appear without restriction in μ. Hence by subtracting one from each part ≥ 2 we see that these overpartition pairs are generated by

$$\frac{(xq)^{i-1}}{(1-axq)}\widehat{G}_{k,k-i+1}(a,b;xq;q). \tag{2.11}$$

Now if $f_1(\lambda)+f_{\overline{1}}(\lambda) < i-1$ and $\overline{1}$ appears in λ, then there are no ones in μ. Supposing that 1 occurs v times in λ, $0 \leq v \leq i-3$, then there are at most $k-v-1$ twos in λ. Hence these overpartition pairs are generated by

$$\sum_{v=0}^{i-3} bxq(xq)^v \widehat{G}_{k,k-v}(a,b;xq;q). \tag{2.12}$$

Finally, if $f_1(\lambda)+f_{\overline{1}}(\lambda) < i-1$ and $\overline{1}$ does not appear in λ, then there are still no ones in μ. Supposing that 1 occurs v times in λ, now with $0 \leq v \leq i-2$, then there are at most $k-v-1$ twos in λ. Hence these overpartition pairs are generated by

$$\sum_{v=0}^{i-2} (xq)^v \widehat{G}_{k,k-v}(a,b;xq;q). \tag{2.13}$$

Putting these four cases together shows that the functions $\widehat{G}_{k,i}(a,b;x;q)$ satisfy (2.6) for $i \geq 2$, and we may now conclude that $G_{k,i}(a,b;x;q) = \widehat{G}_{k,i}(a,b;x;q)$, completing the proof of Theorem 1.1.

3. The Corollaries

From (1.1) and the fact that

$$(a)_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{a^n (q/a)_n},$$

it follows that

$$G_{k,i}(a,b;1;q) = \frac{(-bq)_\infty}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{q^{n((2k-1)n+3)/2+(k-i)n}(1/a,-1/b)_n (ab)^n}{(aq,-bq)_n}. \tag{3.1}$$

By employing Jacobi's triple product identity [8],

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = (-zq,-q/z,q^2;q^2)_\infty, \tag{3.2}$$

we find that these $G_{k,i}(a,b;1;q)$ are often infinite products with nice partition-theoretic interpretations. For example, when $q=q^2$ and $a=b=1/q$, then this series becomes the product

$$\frac{(-q^{2i-2},-q^{4k-2i},q^{4k-2};q^{4k-2})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}, \tag{3.3}$$

which is clearly the generating function for the $H'_{k,i}(n)$ of Theorem 1.2.

On the other hand, we may also interpret $G_{k,i}(1/q,1/q;1;q^2)$ using Theorem 1.1. In this case, starting with an overpartition pair (λ,μ) counted by $g_{k,i}(r,s,t,n)$, the non-overlined (positive) parts j of λ become $2j$, the overlined parts $\overline{j}$ of λ become $\overline{2j-1}$, the non-overlined parts j of μ become $2j-1$, and the overlined parts of μ become $\overline{2j-2}$. Hence we may regard the resulting overpartition pair (λ,μ) as just an overpartition ν, and the conditions $(i)-(iii)$ on (λ,μ) in

Theorem 1.1 transform into the conditions $(i) - (iii)$ on ν in Theorem 1.2. In other words, $G_{k,i}(1/q, 1/q; 1; q^2)$ is the generating function for $H_{k,i}(n)$. This establishes Theorem 1.2 □

The classes of theorems in the first paragraph of the introduction may be deduced in exactly the same way. They correspond to $G_{k,i}(0, 0; 1; q)$, $G_{k,i}(0, 1/q; 1; q^2)$, $G_{k,k}(0, 1; 1; q)$, $G_{k,1}(0, 1/q; 1; q)$, $G_{k,i}(1/q, 0; 1; q^2)$, and $G_{k,k}(1/q, 1; 1; q^2)$, respectively. The details are left to the reader.

Corollary 3.1 (Gordon, [9]). *Let $A_{k,i}(n)$ denote the number of partitions λ of n with $k - i$ zeros, such that for all $j \geq 0$ we have $f_j(\lambda) + f_{j+1}(\lambda) \leq k - 1$. Let $A'_{k,i}(n)$ denote the number of partitions of n whose parts are not congruent to 0 or $\pm i$ modulo $2k + 1$. Then $A_{k,i}(n) = A'_{k,i}(n)$.*

Corollary 3.2 (Andrews, [1]). *Let $B_{k,i}(n)$ denote the number of partitions of λ of n with $k-i$ zeros, such that for all $j \geq 0$ we have $f_{2j}(\lambda) + f_{2j+1}(\lambda) + f_{2j+2}(\lambda) \leq k - 1$. Let $B'_{k,i}(n)$ denote the number of partitions of n where parts are neither congruent to 2 modulo 4 nor congruent to 0 or $\pm(2i - 1)$ modulo $4k$. Then $B_{k,i}(n) = B'_{k,i}(n)$.*

Corollary 3.3 (Lovejoy, [10]). *Let $C_k(n)$ denote the number of overpartitions λ of n with no zeros, such that for all $j \geq 1$ we have $f_j(\lambda) + f_{j+1}(\lambda) + f_{\overline{j+1}}(\lambda) \leq k - 1$. Let $C'_k(n)$ denote the number of overpartitions of n whose parts are not divisible by k. Then $C_k(n) = C'_k(n)$.*

Corollary 3.4 (Lovejoy, [10]). *Let $D_k(n)$ denote the number of overpartitions λ of n where 1 does not occur (non-overlined) and for all $j \geq 1$ we have $f_j(\lambda) + f_{\overline{j}}(\lambda) + f_{j+1}(\lambda) \leq k - 1$. Let $D'_k(n)$ denote the number of overpartitions of n whose non-overlined parts are not congruent to 0 or ± 1 modulo $2k$. Then $D_k(n) = D'_k(n)$.*

Corollary 3.5 (Andrews-Santos, [3]). *Let $E_{k,i}(n)$ denote the number of partitions λ of n having $k - i$ zeros, such that for all $j \geq 0$ we have (i) $f_{2j}(\lambda) + f_{2j+2}(\lambda) \leq k - 1$ and (ii) if $f_{2j}(\lambda) + f_{2j+2}(\lambda) = k - 1$, then $2j + 1$ may occur without restriction. Let $E'_{k,i}(n)$ denote the number of partitions of n into parts that are either even and not divisible by $4k$ or odd, distinct, and congruent to $\pm(2i - 1)$ modulo $4k$. Then $E_{k,i}(n) = E'_{k,i}(n)$.*

Corollary 3.6 (Lovejoy, [11]). *Let $F_k(n)$ denote the number of overpartitions λ of n with no zeros, such that for all $j \geq 0$ we have (i) $f_{2j}(\lambda) + f_{2j+2}(\lambda) + f_{\overline{2j+2}}(\lambda) \leq k - 1$, (ii) if $f_{2j}(\lambda)+f_{2j+2}(\lambda)+f_{\overline{2j+2}}(\lambda) = k-1$, then $2j+1$ may occur (non-overlined and without restriction), and (iii) if $f_{2j}(\lambda) = k-1$, then $\overline{2j + 1}$ may occur. Let $F'_k(n)$ denote the number of overpartitions of n into parts that are either even and not divisible by $4k - 2$ or odd and congruent to $2k - 1$ modulo $4k - 2$. Then $F_k(n) = F'_k(n)$.*

4. Concluding Remarks

The main point of this paper is that by taking a broad look at some of the basic hypergeometric series commonly employed in the proofs of partition identities, one is led to general theorems which unify many of these identities and easily yield further results. It should also not be overlooked that the framework of overpartitions is rather useful in this endeavor.

As previously mentioned, the case $a = 0$ of Theorem 1.1 was already treated in [7]. In fact, the authors related a number of different objects to the generating function $G_{k,i}(0, b; 1; q)$, such as lattice paths, overpartitions with bounded successive ranks, and overpartitions with a specified Durfee decomposition. One wonders if all of these concepts can be extended to the general $G_{k,i}(a, b; x; q)$ studied here.

References

[1] G.E. Andrews, A generalization of the Göllnitz-Gordon partition identities, *Proc. Amer. Math. Soc.* **8** (1967), 945–952.

[2] G.E. Andrews, On q-difference equations for certain well-poised basic hypergeometric series, *Quart. J. Math. Ser. 2* **19** (1968), 433-447.

[3] G.E. Andrews and J.P.O. Santos, Rogers-Ramanujan type identities for partitions with attached odd parts, *Ramanujan J.* **1** (1997), 91-99.

[4] D.M. Bressoud, A generalization of the Rogers–Ramanujan identities for all moduli, *J. Combin. Theory Ser. A* **27** (1979), 64–68.

[5] S. Corteel and J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.* **356** (2004), 1623-1635.

[6] S. Corteel, J. Lovejoy, and O. Mallet, An extension to overpartitions of the Rogers-Ramanujan identities for even moduli, preprint.

[7] S. Corteel and O. Mallet, Overpartitions, lattice paths, and Rogers-Ramanujan identities, preprint

[8] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge Univ. Press, Cambridge, 1990.

[9] B. Gordon, A combinatorial generalization of the Rogers-Ramanujan identities, *Amer. J. Math.* **83** (1961), 393-399.

[10] J. Lovejoy, Gordon's theorem for overpartitions, *J. Comb. Theory Ser. A.* **103** (2003), 393-401.

[11] J. Lovejoy, Overpartition theorems of the Rogers-Ramanujan type, *J. London Math. Soc.* **69** (2004), 562-574.

[12] J. Lovejoy, Overpartition pairs, *Ann. Inst. Fourier* **56** (2006), 781-794.

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