OVERPARTITIONS INTO DISTINCT PARTS WITHOUT SHORT SEQUENCES

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Abstract. In the first part of this paper we introduce overpartitions into distinct parts without k-sequences. When $k = 1$ these are the partitions into parts differing by at least two which occur in the Rogers-Ramanujan identities. For general k we compute a three-variable double sum q-hypergeometric generating function and give asymptotic estimates for the number of such overpartitions. When $k = 2$ we obtain several more double sum generating functions as well as a combinatorial identity. In the second part of the paper, we establish arithmetic and combinatorial properties of some related q-hypergeometric double sums.

1. Introduction

For $k \geq 2$, a k-sequence in a partition λ is a string of k consecutive integers occurring in λ. For example, the partition $\lambda = (13, 12, 10, 10, 9, 9, 9, 7, 6, 5, 5, 3, 1)$ contains the 3-sequence $(7, 6, 5)$ and four 2-sequences, $(13, 12)$, $(10, 9)$, $(7, 6)$, and $(6, 5)$. Let $p_k(n)$ denote the number of partitions of n without k-sequences. The case $k = 2$ was first studied by MacMahon [22, Ch. IV], who proved (among other things) that

$$\sum_{n\geq 0} p_2(n)q^n = 1 + \sum_{n\geq 1} \frac{(q^6;q^6)_{n-1}q^n}{(q^3;q^3)_{n-1}(q^2;q^2)_{n-1}(1-q^n)}. \tag{1.1}$$

Here and throughout we use the standard q-series notation,

$$(a)_n := (a;q)_n := \prod_{k=1}^{n}(1 - aq^{k-1}). \tag{1.2}$$

For higher k, partitions without k-sequences first arose in work of Holroyd, Liggett, and Romik [16] in relation to a certain probabilistic model, which led them to asymptotic estimates for $\log p_k(n)$. Andrews then made an extensive study of $p_k(n)$ in his inaugural article for the National Academy of Sciences [5]. He established double sum q-hypergeometric generating functions and made a conjecture on the asymptotics of $p_k(n)$, which was eventually proven in full by Kane and Rhoades [18]. In proving his own conjecture in the case $k = 2$, Andrews made the important observation that the generating function for $p_2(n)$ is not modular, but the product of a modular form and a mock theta function. Namely, we have

$$\sum_{n\geq 0} p_2(n)q^n = \frac{(-q^3;q^3)_\infty}{(q^2;q^2)_\infty}\chi(q), \tag{1.3}$$

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where $\chi(q)$ is Ramanujan's third order mock theta function

$$\chi(q) := \sum_{n\geq 0} \frac{q^{n^2}(-q)_n}{(-q^3;q^3)_n}. \tag{1.4}$$

This motivated Bringmann and Mahlburg to extend the Hardy-Ramanujan circle method to handle functions of this type [10]. For some of the many other works inspired by Holroyd, Liggett, Romik, and Andrews, we refer to [7, 8, 9, 11, 12, 13].

In the first part of this paper we consider overpartitions into distinct parts without k-sequences. These generalize partitions into distinct parts without k-sequences, which were studied in a recent paper of Bringmann, Mahlburg, and Nataraj [13]. Recall that an overpartition is a partition in which the first occurrence of an integer may be overlined, and let $\overline{Q}(n)$ denote the number of overpartitions of n into distinct parts where successive parts differ by at least 2 if the smaller is overlined. For $k \geq 1$, we define a k-sequence in an overpartition λ counted by $\overline{Q}(n)$ to be a string of k consecutive parts of λ, if the largest part of the string is not overlined, and a string of $k+1$ consecutive parts if the largest part is overlined. As an illustration, the overpartition into distinct parts $(\overline{23}, 21, 20, 17, \overline{15}, 14, 13, \overline{11}, \overline{9}, 8, 6, 4, 3, 2)$ contains the 3-sequence $(4, 3, 2)$ and the 2-sequences $(21, 20)$, $(\overline{15}, 14, 13)$, $(14, 13)$, $(4, 3)$ and $(3, 2)$. Each non-overlined part forms a 1-sequence, as do the strings $(\overline{15}, 14)$ and $(\overline{9}, 8)$.

Let $\overline{Q}_k(n)$ denote the number of overpartitions counted by $\overline{Q}(n)$ without k-sequences, and let $\overline{Q}_k(m,n)$ (resp. $\overline{Q}_k(\ell,m,n)$) denote the number of such overpartitions with m parts (resp. m parts, ℓ of which are overlined). Our first result is a double sum generating function for

$$O_k(a,z,q) := \sum_{\ell,m,n\geq 0} \overline{Q}_k(\ell,m,n) a^\ell z^m q^n.$$

Theorem 1.1. *For a positive integer k,*

$$O_k(a,z,q) = \sum_{j,r\geq 0} \frac{(-1)^j(-a)_r z^{kj+r} q^{\binom{kj+r+1}{2}+k\binom{j}{2}}}{(q^k;q^k)_j (q)_r}.$$

When $a = 0$, this is Theorem 1.1 of [13]. When $k = 1$, overpartitions into distinct parts without 1-sequences are nothing but partitions where parts differ by at least two, which are the partitions occurring in the Rogers-Ramanujan identities. (See [1] for background on these identities.) Thus we have

$$O_1(a,z,q) = \sum_{n\geq 0} \frac{a^n z^n q^{n^2}}{(q)_n}, \tag{1.5}$$

and the Rogers-Ramanujan identities imply that

$$O_1(1,1,q) = \sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q,q^4;q^5)_\infty} \tag{1.6}$$

and

$$O_1(1,q,q) = \sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{1.7}$$

Here we have extended the notation in (1.2) to

$$(a_1, a_2, \ldots, a_k; q)_n := (a_1;q)_n (a_2;q)_n \cdots (a_k;q)_n. \tag{1.8}$$

When $k = 2$, we shall establish the following generating function.

Theorem 1.2. *We have*

$$O_2(a,z,q) = \sum_{r,s\geq 0} \frac{a^r z^{r+s} q^{r^2+rs+s^2}}{(q)_r (q)_s}.$$

This is also related to Rogers-Ramanujan type partitions. Namely, if we define $G(z)$ by

$$G(z) := \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q)_n}, \tag{1.9}$$

then

$$O_2(a,z,q) = \sum_{r\geq 0} \frac{a^r z^r q^{r^2}}{(q)_r} G(zq^r). \tag{1.10}$$

As a consequence, we obtain a second characterization of the overpartitions counted by $\overline{Q}_2(\ell,m,n)$.

Corollary 1.3. *Let $\overline{Q}_2'(\ell,m,n)$ denote the number of overpartitions of n into m parts, ℓ of which are overlined, satisfying*

1. *Each natural number $< \ell$ appears as a non-overlined part exactly once,*
2. *There is no non-overlined part of size ℓ,*
3. *Non-overlined parts $> \ell$ have gap ≥ 2.*

If $m' = m + \ell - \chi(\ell > 0)$, then

$$\overline{Q}_2(\ell,m,n) = \overline{Q}_2'(\ell,m',n), \tag{1.11}$$

where $\chi(P)$ is the characteristic function, which is equal to 1 when P is true and 0 if P is false.

For example, $\overline{Q}_2(2,4,14) = 5$ from the partitions $(\overline{7},4,\overline{2},1)$, $(7,\overline{4},\overline{2},1)$, $(\overline{6},\overline{4},3,1)$, $(6,\overline{4},3,\overline{1})$, and $(\overline{6},5,\overline{2},1)$. We see that $\overline{Q}_2'(2,5,14)$ also equals 5 from the partitions $(7,3,\overline{2},1,\overline{1})$, $(6,4,\overline{2},1,\overline{1})$, $(6,3,\overline{3},1,\overline{1})$, $(5,\overline{4},3,1,\overline{1})$, and $(5,3,\overline{3},\overline{2},1)$.

While the generating function for $\overline{Q}_1(n)$ is modular by (1.6), we do not have any information on the modular transformations of the generating function for $\overline{Q}_2(n)$. Nevertheless we are able to use expressions for $G(z)$ in Ramanujan's lost notebook to obtain the following elegant expressions involving positive definite binary quadratic forms and Gaussian polynomials, the latter defined by

$$\begin{bmatrix} m \\ n \end{bmatrix} := \begin{cases} \frac{(q)_m}{(q)_n (q)_{m-n}}, & \text{if } 0 \leq n \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1.4. *The following identities are true.*

$$O_2(1,1,q) = \frac{1}{(q)_\infty} \sum_{k,n\geq 0} (-1)^n \begin{bmatrix} k \\ n \end{bmatrix} q^{k^2+n(3n-1)/2} (1-q^{3n+3k+3}) \tag{1.12}$$

$$= \frac{1}{(q)_\infty} \sum_{k,n\geq 0} (-1)^n \begin{bmatrix} k+n \\ n \end{bmatrix} q^{k^2+n(5n+1)/2+2kn} (1-q^{4n+2k+2}). \tag{1.13}$$

Other identities in the literature involving $G(z)$ could be used to give further expressions for $O_2(1,1,q)$. We record one more straightforward example later in Prop. 3.1, by directly substituting results of Andrews [2] and Garrett, Ismail and Stanton [14] into (1.10).

For $k \geq 2$ we study the asymptotics of $\overline{Q}_k(n)$, following the methods of [8] and [13]. We first define two functions

$$g_k(u) := -2\pi^2 u^2 + \mathrm{Li}_2(e^{2\pi i u}) - \frac{1}{k}\mathrm{Li}_2(e^{2\pi i k u}) - \mathrm{Li}_2(-e^{2\pi i u})$$

and

$$h_k(x) := x^{k+2} + x^{k+1} - x^2 - 2x + 1,$$

where $\mathrm{Li}_2(x)$ is the dilogarithm function defined by

$$\mathrm{Li}_2(x) = \sum_{n\geq 0} \frac{x^n}{n^2}.$$

By adopting the same argument in [13, Prop 2.1], we can easily see that

(1) $h_k(x)$ has the unique root $w_k \in (0,1)$ for $k \geq 2$.
(2) On the positive real axis, $g_k(u)$ has the unique critical point at v_k, where $w_k = e^{2\pi i v_k}$. Moreover, $g_k''(v_k) < 0$.

Using the constant term method and the saddle point method as in [8] and [13], we can prove the following asymptotic formula.

Theorem 1.5. *Let $k \geq 2$. Using the notation above, as $n \to \infty$,*

$$\overline{Q}_k(n) \sim \frac{\sqrt{\pi} g_k(v_k)^{1/4}}{\sqrt{-g_k''(v_k)} n^{3/4}} e^{2\sqrt{g_k(v_k) n}}.$$

In the second part of the paper we leave overpartitions without k-sequences and turn to double q-hypergeometric sums resembling the one in Theorem 1.2. We study instances of $D_\alpha(q)$ and $E_k(q)$, defined for integers $\alpha \geq -1$ and $k > 0$ by

$$D_\alpha(q) := \sum_{n,m\geq 0} \frac{q^{n^2+\alpha nm+m^2}}{(q)_n (q)_m}, \tag{1.14}$$

and

$$E_k(z,q) := \sum_{n,m\geq 0} \frac{z^{m-n} q^{kn^2-(2k-1)nm+km^2}}{(q)_n (q)_m}. \tag{1.15}$$

These lead to some new partition functions with interesting combinatorial and arithmetic properties. To keep the introduction from running on, we postpone a discussion of this until Sections 5 and 6.

The remainder of paper up to Section 5 is organized as follows. In Section 2, we prove the generating function for $O_k(a,z,q)$ in Theorem 1.1. In Section 3, we focus on the case $k=2$, obtaining the simpler generating function for $O_2(a,z,q)$ in Theorem 1.2 and its partition theoretic implication in Corollary 1.3, as well as the generating functions for $O_2(1,1,q)$ in Theorem 1.4. In Section 4, we prove the asymptotic formula for $\overline{Q}_k(n)$.

2. The two-variable generating function

We begin by proving Theorem 1.1. We define $\overline{\mathcal{D}}_k$ to be the set of overpartitions counted by $\overline{Q}_k(n)$ for some n. For an overpartition $\lambda \in \overline{\mathcal{D}}_k$, we denote its weight by $|\lambda|$, the number of its overlined parts by $o(\lambda)$, and the number of its parts by $\ell(\lambda)$.

Note that

$$O_k(a,z,q) = \sum_{\lambda\in\overline{\mathcal{D}}_k} a^{o(\lambda)} z^{\ell(\lambda)} q^{|\lambda|}.$$

For a given partition $\lambda \in \overline{\mathcal{D}}_k$, we remove the "staircase" by subtracting $\ell(\lambda)-1$ from the largest part, $\ell(\lambda)-2$ from the second largest part, and so on down to 1 from the second smallest part and 0 from the smallest part. Denote the resulting partition by λ'. By the definition of $\overline{\mathcal{D}}_k$, in λ' we have that each non-overlined part can be repeated at most $k-1$ times and there could be an overlined part of the same size. Therefore, we see that

$$\sum_{\lambda \in \overline{\mathcal{D}}_k} a^{o(\lambda)} z^{\ell(\lambda')} q^{|\lambda'|} = \frac{(-azq)_\infty}{(zq)_\infty} (z^k q^k; q^k)_\infty.$$

Using the q-binomial theorem

$$\sum_{n \geq 0} \frac{(-a)_n}{(q)_n} (zq)^n = \frac{(-azq)_\infty}{(zq)_\infty}$$

and the case $(z,q) = (z^k, q^k)$ of the q-binomial series

$$(zq)_\infty = \sum_{n \geq 0} \frac{(-1)^n z^n q^{\binom{n+1}{2}}}{(q)_n}, \tag{2.1}$$

we can express this product as a double sum,

$$\sum_{\lambda \in \overline{\mathcal{D}}_k} a^{o(\lambda)} z^{\ell(\lambda')} q^{|\lambda'|} = \sum_{j,r \geq 0} \frac{(-1)^j z^{kj} q^{kj(j+1)/2}}{(q^k; q^k)_j} \frac{(-a)_r z^r q^r}{(q)_r}.$$

From the fact that $\ell(\lambda) = \ell(\lambda')$ and $|\lambda| = |\lambda'| + \ell(\lambda)(\ell(\lambda)-1)/2$, we deduce Theorem 1.1. □

3. The case $k = 2$

In this section we prove the results on $O_2(a,z,q)$ in Theorems 1.2 and 1.4 and Corollary 1.3. We begin with Theorem 1.2.

Proof of Theorem 1.2. From the proof in the previous section we have that

$$O_k(a,z,q) = [y^0] \frac{(-ayz)_\infty}{(yz)_\infty} (y^k z^k; q^k)_\infty \sum_{n \in \mathbb{Z}} y^{-n} q^{\binom{n+1}{2}}. \tag{3.1}$$

Using this with $k = 2$ we have

$$\begin{aligned} O_2(a,z,q) &= [y^0](-ayz)_\infty(-yz)_\infty \sum_{n \in \mathbb{Z}} y^{-n} q^{\binom{n+1}{2}} \\ &= [y^0] \sum_{r \geq 0} \frac{a^r y^r z^r q^{\binom{r}{2}}}{(q)_r} \sum_{s \geq 0} \frac{y^s z^s q^{\binom{s}{2}}}{(q)_s} \sum_{n \in \mathbb{Z}} y^{-n} q^{\binom{n+1}{2}} \\ &= \sum_{r,s \geq 0} \frac{a^r z^{r+s} q^{r^2+rs+s^2}}{(q)_r (q)_s}. \end{aligned}$$

□

We remark that a similar argument can be used to deduce (1.5) from Theorem 1.1. Namely, using (3.1) with $k = 1$ we have

$$\begin{aligned} O_1(a,z,q) &= [y^0](-ayz)_\infty \sum_{n \in \mathbb{Z}} y^{-n} q^{\binom{n+1}{2}} \\ &= [y^0] \sum_{n \geq 0} \frac{a^n y^n z^n q^{\binom{n}{2}}}{(q)_n} \sum_{n \in \mathbb{Z}} y^{-n} q^{\binom{n+1}{2}} \\ &= \sum_{n \geq 0} \frac{a^n z^n q^{n^2}}{(q)_n}. \end{aligned}$$

Next, we establish Corollary 1.3.

Proof of Corollary 1.3. We begin by writing the generating function for $\overline{Q}_2(\ell, m, n)$ as

$$O_2(a,z,q) = \sum_{r\geq 0} \frac{a^r z^r q^{r(r+1)/2}}{(q)_r} q^{r(r-1)/2} G(zq^r). \tag{3.2}$$

Recalling (1.9), we note that $q^{r(r-1)/2}G(zq^r)$ is the generating function for partitions λ such that (i) the natural numbers $< r$ appear as parts exactly once, (ii) r does not appear as a part, and (iii) the parts $> r$ have gap ≥ 2, where the exponent of z counts the number of parts $> r$. Next, we have that $a^r z^r q^{r(r+1)/2}/(q)_r$ is the generating function for partitions μ into r distinct parts. Overlining the parts of μ and adding in the parts of λ non-overlined, we obtain the overpartitions described in the statement of the theorem. This completes the proof of Corollary 1.3. $\square$

We turn now to the identities in Theorem 1.4.

Proof of Theorem 1.4. We recall the following entry from Ramanujan's lost notebook [6, Entry 4.2.1]:

$$(aq)_\infty \sum_{n\geq 0} \frac{a^n q^{n^2}}{(q)_n} = \sum_{n\geq 0} \frac{(-1)^n (a)_n (1-aq^{2n})}{(q)_n (1-a)} a^{2n} q^{n(5n-1)/2} \tag{3.3}$$

$$= \sum_{n\geq 0} \frac{(-1)^n (aq)_n (1-a^2q^{4n+2})}{(q)_n} a^{2n} q^{n(5n+1)/2}. \tag{3.4}$$

Using the case $a = q^j$ of (3.3), we have

$$\begin{aligned} O_2(1,1,q) &= \sum_{j\geq 0} \frac{q^{j^2}}{(q)_j} \sum_{n\geq 0} \frac{q^{n(n+j)}}{(q)_n} \\ &= \sum_{j\geq 0} \frac{q^{j^2}}{(q)_j} \frac{1}{(q^{j+1})_\infty} \sum_{n\geq 0} \frac{(1-q^{2n+j})(-1)^n q^{n(5n-1)/2+2nj} (q^j)_n}{(1-q^j)(q)_n} \\ &= \frac{1}{(q)_\infty} \sum_{j,n\geq 0} \frac{(1-q^{2n+j})(-1)^n q^{j^2+n(5n-1)/2+2nj} (q)_{n+j-1}}{(q)_j (q)_n}. \end{aligned}$$

Since

$$\begin{aligned} \frac{(1-q^{2n+j})(q)_{n+j-1}}{(q)_j(q)_n} &= \frac{\{1-q^{n+j}+q^{n+j}(1-q^n)\}(q)_{n+j-1}}{(q)_j(q)_n} \\ &= \frac{(q)_{n+j}}{(q)_j(q)_n} + \frac{q^{n+j}(q)_{n+j-1}}{(q)_j(q)_{n-1}}, \end{aligned}$$

we obtain

$$\begin{aligned} &O_2(1,1,q) \\ &= \frac{1}{(q)_\infty} \left\{ \sum_{j,n\geq 0} \frac{(-1)^n q^{j^2+n(5n-1)/2+2nj} (q)_{n+j}}{(q)_j(q)_n} + \sum_{j\geq 0, n\geq 1} \frac{(-1)^n q^{j^2+n(5n+1)/2+2nj+j} (q)_{n+j-1}}{(q)_j(q)_{n-1}} \right\} \\ &= \frac{1}{(q)_\infty} \left\{ \sum_{j,n\geq 0} \frac{(-1)^n q^{j^2+n(5n-1)/2+2nj} (q)_{n+j}}{(q)_j(q)_n} - \sum_{j,n\geq 0} \frac{(-1)^n q^{j^2+n(5n+11)/2+2nj+3j+3} (q)_{n+j}}{(q)_j(q)_n} \right\}. \end{aligned}$$

Letting $n+j \to k$ and $n \to \ell$, we get

$$
\begin{aligned}
&O_2(1,1,q)\\
&= \frac{1}{(q)_\infty}\left\{\sum_{k,\ell\geq 0}\frac{(-1)^\ell q^{k^2+\ell(3\ell-1)/2}(q)_k}{(q)_{k-\ell}(q)_\ell} - \sum_{k,\ell\geq 0}\frac{(-1)^\ell q^{k^2+3k+3+\ell(3\ell+5)/2}(q)_k}{(q)_{k-\ell}(q)_\ell}\right\}\\
&= \frac{1}{(q)_\infty}\left\{\sum_{k\geq 0} q^{k^2}\sum_{\ell\geq 0}(-1)^\ell q^{\ell(3\ell-\ell)/2}\begin{bmatrix} k \\ \ell \end{bmatrix} - \sum_{k\geq 0} q^{k^2+3k+3}\sum_{\ell\geq 0}(-1)^\ell q^{\ell(3\ell+5)/2}\begin{bmatrix} k \\ \ell \end{bmatrix}\right\},
\end{aligned}
$$

which establishes (1.12).

For (1.13), we set $a = q^j$ in (3.4). This gives

$$
\begin{aligned}
O_2(1,1,q) &= \sum_{j=0}^{\infty}\frac{q^{j^2}}{(q)_j}\sum_{n=0}^{\infty}\frac{q^{n(n+j)}}{(q)_n}\\
&= \sum_{j\geq 0}\frac{q^{j^2}}{(q)_j}\frac{1}{(q^{j+1})_\infty}\sum_{n\geq 0}\frac{(-1)^n (q^{j+1})_n(1-q^{4n+2j+2})}{(q)_n} q^{n(5n+1)/2+2jn}\\
&= \frac{1}{(q)_\infty}\sum_{j\geq 0} q^{j^2}\left(\sum_{n\geq 0}(-1)^n\begin{bmatrix} j+n \\ n\end{bmatrix} q^{n(5n+1)/2+2jn}(1-q^{4n+2j+2})\right),
\end{aligned}
$$

which proves (1.13). □

We end this section by recording a formula to which we alluded in the introduction.

Proposition 3.1. *We have*

$$
O_2(1,1,q) = \sum_{n\geq 0}\frac{(-1)^n q^{n(n+1)/2}}{(q)_n}\left(\frac{A_n(q)}{(q,q^4;q^5)_\infty} - \frac{B_n(q)}{(q^2,q^3;q^5)_\infty}\right),
$$

where $A_0 = 1$, and $B_0 = 0$, and for $n > 0$

$$
\begin{aligned}
A_n(q) &= \sum_{k\geq 0} q^{k^2+k}\begin{bmatrix} n-k-2 \\ k\end{bmatrix} = \sum_{k\in\mathbb{Z}}(-1)^k q^{k(5k-3)/2}\begin{bmatrix} n-1 \\ \lfloor\frac{n+1-5k}{2}\rfloor\end{bmatrix},\\
B_n(q) &= \sum_{k\geq 0} q^{k^2}\begin{bmatrix} n-k-1 \\ k\end{bmatrix} = \sum_{k\in\mathbb{Z}}(-1)^k q^{k(5k+1)/2}\begin{bmatrix} n-1 \\ \lfloor\frac{n-1-5k}{2}\rfloor\end{bmatrix}.
\end{aligned}
$$

Proof. This follows immediately from the fact that

$$
G(q^n) = (-1)^n q^{-n(n-1)/2}\left(\frac{A_n(q)}{(q,q^4;q^5)_\infty} - \frac{B_n(q)}{(q^2,q^3;q^5)_\infty}\right), \tag{3.5}
$$

which combines a result of Garrett, Ismail, and Stanton [14] and a result of Andrews [2]. □

4. Asymptotic Formulas

Here we prove asymptotic formula for $\overline{Q}_k(n)$ for $k \geq 3$. As our proof is similar with that of [13, Thm 1.2], we do not give all of the details. First we define the quantum dilogarithm

$$
\mathrm{Li}_2(x;q) := -\log(x;q)_\infty = \sum_{n\geq 1}\frac{x^n}{n(1-q^n)}.
$$

Recall that uniformly in x as $\epsilon \to 0^+$,

$$\mathrm{Li}_2(x; e^{-\epsilon}) = \frac{1}{\epsilon}\mathrm{Li}_2(x) - \frac{1}{2}\log(1-x) + O(\epsilon).$$

Proof of Theorem 1.5. We first investigate the asymptotic behavior of $\mathcal{O}_k(1,1,q) =: \mathcal{O}_k(q)$ with $q = e^{-\epsilon}$ as $\epsilon \to 0^+$. From (3.1), we see that

$$\mathcal{O}_k(q) = [y^0]\theta(y^{-1}q^{1/2}; q^{1/2}) \exp\left(-\mathrm{Li}_2(y^k; q^k) - \mathrm{Li}_2(-y;q) + \mathrm{Li}_2(y;q)\right),$$

where $\theta(y;q) = \sum_{n\in\mathbb{Z}} y^n q^{n^2}$. Thus, by Cauchy's formula, we derive that

$$\mathcal{O}_k(q) = \int_{[0,1]+ic} \theta(y^{-1}q^{1/2}; q^{1/2}) \exp\left(-\mathrm{Li}_2(y^k; q^k) - \mathrm{Li}_2(-y;q) + \mathrm{Li}_2(y;q)\right) du,$$

where $c = \frac{1}{2\pi}\log w_k^{-1}$ is a positive constant and $y = \exp(2\pi i u)$. By employing the transformation formula for the theta function (for example, see [13, eqn. 2.5]), we see that

$$\mathcal{O}_k(q) = \sqrt{\frac{2\pi}{\epsilon}} \int_{\mathbb{R}+ic} \exp\left(-\frac{2\pi^2}{\epsilon}\left(u - \frac{i\epsilon}{4\pi}\right)^2 - \mathrm{Li}_2(y^k; q^k) - \mathrm{Li}_2(-y;q) + \mathrm{Li}_2(y;q)\right) du.$$

Note that the integrand can be rewritten as

$$\exp\left(\frac{g_k(u)}{\epsilon} + \pi i u + \frac{1}{2}\log(1-y^k) + \frac{1}{2}\log(1+y) - \frac{1}{2}\log(1-y) + O(\epsilon)\right),$$

therefore by expanding around $u = v_k$ and replacing $u = v_k + \sqrt{\epsilon}z$, we obtain that the integrand equals

$$\sqrt{\frac{w_k(1-w_k^k)(1+w_k)}{1-w_k}} \exp\left(\frac{g_k(v_k)}{\epsilon} + \frac{g_k''(v_k)}{2}z^2 + O(\epsilon^{1/2})\right).$$

From the definition of w_k, we see that $\frac{w_k(1-w_k^k)(1+w_k)}{1-w_k} = 1$, which implies that

$$\mathcal{O}_k(q) = \frac{2\pi}{\sqrt{-g_k''(v_k)}}\left(1 + O(\epsilon^{1/2})\right)\exp\left(\frac{g_k(v_k)}{\epsilon}\right)$$

by the Gaussian integral evaluation. Since $\overline{Q}_k(n+1) \geq \overline{Q}_k(n)$ (this is because for a given overpartition counted by $\overline{Q}_k(n)$ we obtain a partition counted by $\overline{Q}_k(n+1)$ by increasing the largest part by 1), we can apply Ingham's Tauberian theorem [13, Theorem 4.2] (c.f. [17]), which proves the theorem. $\square$

5. The functions $D_\alpha(q)$

Recall the definition of $D_\alpha(q)$ in (1.14). In his proof of the Lusztig-Macdonald-Wall conjectures, Andrews [3] established the modularity of $D_{-1}(q)$ by proving the identity

$$D_{-1}(q) = \frac{(-q;q^2)_\infty^2}{(q;q^2)_\infty}. \tag{5.1}$$

This identity subsequently played a role in several works [15, 21, 26]. When $\alpha = 0$ we again have a modular function (simply the square of (1.6)), and when $\alpha = 1$ we have the generating function for overpartitions without 2-sequences studied in the first part of this paper. In this section we make some remarks on the next case, $\alpha = 2$. We define $a(n)$ by

$$\sum_{n\geq 0} a(n)q^n = D_2(q) = 1 + 2q + 2q^2 + 2q^3 + 5q^4 + 6q^5 + 9q^6 + 10q^7 + 13q^8 + \cdots. \tag{5.2}$$

Moreover, let $\mathcal{P}_n$ denote the set of all partitions of n, and for a partition λ, let $d(\lambda)$ denote the size of the Durfee square, λ_r the partition to the right of the Durfee square, and $h(\lambda)$ the first hook-length, i.e., the largest part plus the number of parts minus 1 (and -1 for the empty partition). Then we have the following:

Theorem 5.1. *We have*

(i)
$$a(2n+1) \equiv 0 \pmod 2,$$

(ii)
$$a(n) = \sum_{\substack{\lambda \vdash \mathcal{P}_n \\ h(\lambda_r) < d(\lambda)}} (d(\lambda) - h(\lambda_r)).$$

For example, among 7 partitions of 5, there are 3 partitions with $d(\lambda) > h(\lambda_r)$. The partition $1+1+1+1+1$ has $d(\lambda) = 1$ and $h(\lambda_r) = -1$, the partition $3+2$ has $d(\lambda) = 2$ and $h(\lambda_r) = 1$, and the partition $2+2+1$ has $d(\lambda) = 2$ and $h(\lambda_r) = -1$. Thus we have $a(5) = 6$, as expected.

Proof. The first part follows easily from the symmetry in the double series definition of $D_2(q)$. In (1.14), we consider the cases $m = n$ and $m \neq n$ separately, which gives

$$\begin{aligned}
\sum_{n\geq 0} a(n)q^n &= \sum_{m,n\geq 0} \frac{q^{(n+m)^2}}{(q)_n(q)_m} \\
&= \sum_{n\geq 0} \frac{q^{4n^2}}{(q)_n^2} + 2\sum_{n>m\geq 0} \frac{q^{(n+m)^2}}{(q)_n(q)_m} \\
&\equiv \sum_{n\geq 0} \frac{q^{4n^2}}{(q^2;q^2)_n} \pmod 2.
\end{aligned}$$

We note in passing that q-expansion is the generating function for the number of partitions with gap ≥ 4 and the smallest part > 1, after replacing q^2 by q.

For the second part, we begin by writing

$$\sum_{m,n\geq 0} \frac{q^{(n+m)^2}}{(q)_m(q)_n} = \sum_{m\geq 0} \frac{q^{m^2}}{(q)_m} \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}.$$

The right-hand side can be regarded as a weighted count of partitions λ as follows. First recall that if $P(M,N)$ is the set of partitions of whose Ferrers diagram fits inside an $M \times N$ rectangle, then it is well-known that $\begin{bmatrix} m \\ k \end{bmatrix}$ is the generating function for the partitions in $P(m-k,k)$ (or equivalently $P(k,m-k)$). Now, the index of summation $m = d(\lambda)$ is the size of the Durfee square, the term $1/(q)_m$ corresponds to parts under the Durfee square, and the term $\sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}$ corresponds to the partition to the right of the Durfee square, λ_r. Observe that λ_r generated by $\begin{bmatrix} d(\lambda) \\ k \end{bmatrix}$ satisfies $h(\lambda_r) < d(\lambda)$, and such a λ_r is contained in the set $P(k, d(\lambda)-k)$ for precisely $d(\lambda) - h(\lambda_r)$ k's. Therefore, we have proven that

$$a(n) = \sum_{\substack{\lambda \vdash \mathcal{P}_n \\ h(\lambda_r) < d(\lambda)}} (d(\lambda) - h(\lambda_r)),$$

as claimed. $\square$

6. The functions $E_k(q)$

Recall the definition of $E_k(z,q)$ in equation (1.15). In the same paper where he established the modularity of $D_{-1}(q)$ in (5.1) (note that $D_{-1}(q) = E_1(1,q)$), Andrews [3] observed (without giving a proof) that the general $E_k(1,q)$ can be expressed as infinite product. In this section we give a short proof of Andrews' observation (for general z) and then study the combinatorics of these series, using the notion of a Rogers-Ramanujan subpartition following ideas of Kolitsch [20] and Kim and Kim [19]. We begin with the infinite product representation.

Theorem 6.1. *For $k > 0$ we have*

$$E_k(z,q) = \frac{1}{(q)_\infty} \sum_{n\in\mathbb{Z}} z^n q^{kn^2} \tag{6.1}$$

$$= \frac{(-zq^k, -q^k/z, q^{2k}; q^{2k})_\infty}{(q)_\infty}. \tag{6.2}$$

Proof. The case $k=1$ was established by the third author [21, p.304], and his simple proof readily generalizes as follows. Employing the q-binomial theorem in (2.1) and the Jacobi triple product identity,

$$\sum_{n\in\mathbb{Z}} z^n q^{\binom{n+1}{2}} = (-1/z, -zq, q)_\infty, \tag{6.3}$$

we calculate

$$\begin{aligned}
\sum_{n,m\geq 0} \frac{z^{m-n} q^{kn^2-(2k-1)nm+km^2}}{(q)_n (q)_m} &= [x^0] \sum_{\ell\in\mathbb{Z}} x^{-\ell} q^{(2k-1)\binom{\ell}{2}} \sum_{m\geq 0} \frac{\left(zq^{-k+1}\right)^m x^{-m} q^{\binom{m}{2}}}{(q)_m} \sum_{n\geq 0} \frac{\left(zq^{-k+1}\right)^{-n} x^n q^{\binom{n+1}{2}}}{(q)_n} \\
&= [x^0] \sum_{\ell\in\mathbb{Z}} x^{-\ell} q^{(2k-1)\binom{\ell}{2}} (-zq^{-k+1}/x)_\infty (-xq^k/z)_\infty \\
&= \frac{1}{(q)_\infty} [x^0] \sum_{\ell\in\mathbb{Z}} x^{-\ell} q^{(2k-1)\binom{\ell}{2}} \sum_{n\in\mathbb{Z}} (xq^{k-1}/z)^n q^{\binom{n+1}{2}} \\
&= \frac{1}{(q)_\infty} \sum_{n\in\mathbb{Z}} z^{-n} q^{kn^2} \\
&= \frac{(-zq^k, -q^k/z, q^{2k}; q^{2k})_\infty}{(q)_\infty}.
\end{aligned}$$

□

Let $a_1 \geq a_2 \geq \cdots \geq a_m$ be an ordinary partition. In a recent paper [19], the second author and E. Kim defined a subpartition of an ordinary partition as follows. Let us fix positive integers d and t. Then, for a given partition, the subpartition with gap d and tail condition t is defined as the longest sequence satisfying $a_1 > a_2 > \cdots > a_s$ and $a_s > a_{s+1}$, where $a_i - a_j \geq d$ for all $i < j \leq s$ and $a_s - a_{s+1} \geq t$. Here a_{s+1} must be understood to be zero if it comes after the final part. This is a generalization of L. Kolitsch's Rogers-Ramanujan subpartition [20], which is the case $d = 2$ and $t = 1$. We define the length of the subpartition with gap d as the number of parts in the subpartition.

For $\ell \geq 0$ let $A_k(\ell, n)$ denote the number of partitions of n with subpartitions with gap $2k$ and tail condition k, where the length of the subpartition is at least ℓ. (When $\ell = 0$ then we are simply counting the number of partitions of n.) Then, following the combinatorial argument in [19], we have

Proposition 6.2.

$$\sum_{n\geq 0}\sum_{\ell\in\mathbb{Z}} A_k(|\ell|,n)z^\ell q^n = \frac{1}{(q)_\infty}\sum_{n\in\mathbb{Z}} z^n q^{kn^2}. \tag{6.4}$$

We consider the cases $z=1$ and -1 in (6.4). We define

$$A_k(n) := \sum_{\ell\geq 1} A_k(\ell,n) = \sum_{\ell\geq 1} \ell p(n,2k,k,\ell)$$

$$B_k(n) := \sum_{\ell\geq 1}(-1)^\ell A_k(\ell,n) = -\sum_{\substack{\ell\geq 1\\ \ell\equiv 1 \pmod 2}} p(n,2k,k,\ell)$$

where $p(n,d,t,\ell)$ be the number of partitions of n having length ℓ subpartitions with gap d and tail condition t. For example, since 4 has the subpartition 4, $3+1$ has the subpartition $3+1$ of length 2, and $2+1+1$ has the subpartition 2, we have $A_1(4)=1+2+1=4$ and $B_1(4)=-2$. Proposition 6.2 together with (6.3) imply that the generating functions for $A_1(n)$ and $B_1(n)$ are differences of infinite products:

Corollary 6.3. *We have*

$$2\sum_{n\geq 1} A_1(n)q^n = \frac{(-q;q^2)_\infty^2}{(q;q^2)_\infty} - \frac{1}{(q)_\infty},$$

$$2\sum_{n\geq 1} B_1(n)q^n = (q;q^2)_\infty - \frac{1}{(q)_\infty}.$$

We close with some simple congruences for $A_1(n)$. Note that

$$\sum_{n\geq 1} A_1(n)q^n = \frac{1}{(q)_\infty}\sum_{n\geq 1} q^{n^2}. \tag{6.5}$$

In seminal work [4], Andrews considered (among many other things) the number of 2-colored Frobenius partitions, of n, denoted $c\phi_2(n)$. Its generating function is given by

$$\sum_{n\geq 0} c\phi_2(n)q^n = \frac{1}{(q)_\infty^2}\sum_{n\in\mathbb{Z}} q^{n^2}, \tag{6.6}$$

which resembles (6.5). Motivated by congruence properties for $c\phi_2(n)$ [24], we prove the following.

Proposition 6.4. *For all nonnegative integers n,*

$$A_1(25n+9) \equiv A_1(25n+14) \equiv A_1(25n+24) \equiv 0 \pmod 5.$$

Proof. From the famous partition function congruence $p(5n+4)\equiv 0 \pmod 5$, it suffices to prove $c(25n+\ell)\equiv 0 \pmod 5$ for $\ell = 9, 14, 24$, where $c(n)$ is defined by $\sum_{n\geq 0} c(n)q^n = \frac{1}{(q)_\infty}\theta(q)$. Since $(q^5;q^5)_\infty^5 \equiv (q^{25},q^{25})_\infty \pmod 5$, this is equivalent to showing that the coefficients of $q^{25n+\ell+1}$ of

$$q\frac{(q^5;q^5)_\infty^5}{(q)_\infty}\theta(q)\theta(q^{25}) =: F(q) = \sum_{n\geq 1} a_F(n)q^n$$

are divisible by 5. As we are going to use the standard facts from the theory of modular forms [23, 25], we do not give all of the details. It is not hard to see that $F(q)$ is a holomorphic modular

form of weight 3 and level 100 for some odd character χ, and thus so is $G := \sum_{n\geq 1} a_F(5n)q^n$. Note that
$$G - G \otimes \left(\frac{\cdot}{5}\right) = \sum_{n\equiv 0 \pmod 5} a_F(5n)q^n + 2 \sum_{n\equiv 2,3 \pmod 5} a_F(5n)q^n$$
is again a modular form of weight 3 and level 100 with the same character χ, and we can prove that $G - G \otimes \left(\frac{\cdot}{5}\right) \equiv 0 \pmod 5$ by checking the first 45 coefficients [25, Corollary 9.20]. This completes the proof of the propostition. □

We remark that it appears from numerical computation that Proposition 6.4 can be extended to a family of congruences modulo powers of 5. For example, it is likely true that
$$A_1(125n + 74) \equiv A_1(125n + 124) \equiv 0 \pmod{25}$$
and
$$A_1(3125n + 1849) \equiv A_1(3125n + 3099) \equiv 0 \pmod{125}.$$
Proving such a family of congruences would amount to proving a family of congruences mod powers of 5 for $\theta(q)/(q)_\infty$, as in Paule and Radu's proof of congruences for $c\phi_2(n)$. We leave this open.

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