THE NUMBER OF PARTITIONS INTO DISTINCT PARTS MODULO POWERS OF 5

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Abstract. We establish a relationship between the factorization of $24n+1$ and the 5-divisibility of $Q(n)$, where $Q(n)$ is the number of partitions of n into distinct parts. As an application we explicitly exhibit an abundance of infinite families of congruences for $Q(n)$ modulo powers of 5.

1. Introduction

A partition of a natural number n into distinct parts is a decreasing sequence of natural numbers whose sum is n. The number of such partitions is denoted $Q(n)$, and we adopt the usual conventions that $Q(0) = 1$ and $Q(\alpha) = 0$ whenever α is not a natural number. The types of questions we consider about partition functions like $Q(n)$ have been greatly influenced by the work of Ramanujan, who observed that for certain small primes p, the p-divisibility of $24n - 1$ is related to the p-divisibility of $p(n)$, where $p(n)$ denotes the number of ordinary partitions of n. In particular, he conjectured the existence of three infinite families of congruences in arithmetic progressions for $p(n)$. If $\beta_\ell(j)$ denotes the inverse of 24 modulo ℓ^j, then for all non-negative integers n we have

$$p(5^j n + \beta_5(j)) \equiv 0 \pmod{5^j}, \tag{1.1}$$

$$p(7^j n + \beta_7(j)) \equiv 0 \pmod{7^{[j/2]+1}}, \tag{1.2}$$

and

$$p(11^j n + \beta_{11}(j)) \equiv 0 \pmod{11^j}. \tag{1.3}$$

Ramanujan himself had a proof of (1.1) [4], while (1.2) was proven by Watson [16]. The third family was not proven until 1967 by Atkin [2], who developed an idea of Lehner [10] into a technique which could theoretically be utilized to obtain a proof of any simple family of congruences in arithmetic progressions for the Fourier coefficients of a modular function modulo powers of a prime.

Applying many of these ideas, Rødseth [14] uncovered an infinite family of congruences for the number of partitions into distinct parts. He proved that for all non-negative integers n we have

$$Q(5^{2j+1} n + \gamma_{2j+1}) \equiv 0 \pmod{5^j} \tag{1.4}$$

and

$$Q(5^{10j} n + \gamma_{10j}) \equiv 0 \pmod{5^{5j}}, \tag{1.5}$$

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where γ_j is the inverse of -24 modulo 5^j,

$$\gamma_j = \frac{25^{[(j+1)/2]} - 1}{24}.$$

The congruences (1.4) were subsequently rediscovered by Gordon and Hughes [6], but to our knowledge no further families have been identified. Recent works [1, 7, 11] p-adically relating certain values of $Q(n)$ to Fourier coefficients of holomorphic modular forms guarantee that for any modulus M coprime to 3, there are indeed infinitely many independent congruences of the form $Q(an+b) \equiv 0 \pmod{M}$. However, it was believed that to explicitly identify examples in this theory would require a case-by-case computation of the action of the Hecke operators $T(p)$ (see Proposition 5 for the definition) on the relevant forms. This is true in general, although for the modulus 5 it turns out that there is a significant amount of regularity. Our main theorems demonstrate that the 5-divisibility of $24n+1$ is not the only factor which determines the 5-divisibility of $Q(n)$.

Theorem 1. *Let $j \geq 1$ and let p be a prime for which*

$$p \not\equiv \{1, 11, 13, 23, 37, 47, 49, 59\} \pmod{120}. \tag{1.6}$$

If $ord_p(n)$ is odd, then

$$Q\left(\frac{5^{2j}n - 1}{24}\right) \equiv 0 \pmod{5^j}. \tag{1.7}$$

Corollary 2. *Let p be any prime satisfying (1.6) and let m be any odd natural number. If b is an integer with $b < 24p$, $(b,p) = 1$, and $b \equiv p \pmod{24}$, then for all non-negative integers n we have*

$$Q(5^{2j}p^{m+1}n + (5^{2j}p^m b - 1)/24) \equiv 0 \pmod{5^j} \tag{1.8}$$

To illustrate Corollary 2, if we take $p = 7$, $m = 1$, and $b = 31$, then for all non-negative integers n

$$\begin{aligned} Q(1225n + 226) &\equiv 0 \pmod{5}, \\ Q(30625n + 5651) &\equiv 0 \pmod{25}, \\ &\vdots \\ Q(5^{2j} \cdot 49n + k_j) &\equiv 0 \pmod{5^j}, \end{aligned}$$

where $k_j = (5^{2j} \cdot 217 - 1)/24$.

It is readily verified that all congruences in Corollary 2 are independent and not implied by the family of Rødseth, with the exception of the cases $p = 5$ and $5 \mid j$, which are contained in (1.4) and (1.5). It will of course be noted that Theorem 1 is a far stronger statement than Corollary 2; however, the congruences in arithmetic progressions being of particular interest in the subject, we have chosen to emphasize the corollary.

Our methods shall also reveal the following theorem, which improves Theorem 1 in certain cases:

Theorem 3. *Let $j \geq 0$ and let p be a prime for which*

$$p \equiv \{61, 71, 73, 83, 87, 97, 109, 119\} \pmod{120}. \tag{1.9}$$

If $ord_p(n)$ is odd, then

$$Q\left(\frac{5^{10j+1}n-1}{24}\right) \equiv 0 \pmod{5^{5j+1}}. \tag{1.10}$$

Finally, we shall have occasion to mention the following companions to the original theorem of Rødseth, which is (1.4) when $r = 1$:

Theorem 4. *For $r = 1, 3, 4$, and for all non-negative integers n we have*

$$Q(5^{2j+1}n + \gamma_{2j} + r5^{2j}) \equiv 0 \pmod{5^j} \tag{1.11}$$

In the following section we indicate some required preliminaries about modular forms and Hecke operators, and in §3 we deduce the main results.

2. Modular forms and Hecke Operators

For positive integers k, N, and any Dirichlet character χ, let $M_k(\Gamma_0(N), \chi)$ denote the finite-dimensional $\mathbb{C}$-vector space of modular forms of weight k, level N, and character χ. With the exception of Proposition 5 below, we shall not require any precise details about modular forms, so we omit an elemenatary discussion and instead refer the interested reader to [13]. For any $f(z) \in M_k(\Gamma_0(N), \chi)$, we identify f with its Fourier series in the variable $q := e^{2\pi i z}$.

Proposition 5 ([13])**.** *Let p denote a prime and ψ a Dirichlet character modulo M. If $f = \sum a(n)q^n$ is the Fourier expansion of a modular form in $M_k(\Gamma_0(N), \chi)$, then*

$$f \mid T(p) := \sum \left(a(pn) + \chi(p)p^{k-1}a(n/p)\right) q^n \tag{2.1}$$

is also the Fourier expansion of a modular form in $M_k(\Gamma_0(N), \chi)$, and

$$f \otimes \psi := \sum \psi(n)a(n)q^n \tag{2.2}$$

is the Fourier expansion of a modular form in $M_k(\Gamma_0(NM^2), \chi\psi^2)$.

The linear operators $T(p)$ in (2.1) are the well-known Hecke operators for integer weight modular forms. It turns out that many modular forms of interest have the property that

$$f(z) \mid T(p) = a(p)f(z) \tag{2.3}$$

whenever $p \nmid N$. Such a function is called an eigenform.

3. Proof of the Main Results

We begin by defining some relevant η-products, where $\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, and proving some lemmas about their Fourier coefficients modulo 5.

Definition 6. *For any positive $i \leq 23$ which is coprime to 6, let*

$$f_i(z) = \frac{\eta^{16}(24z)}{\eta^8(48z)} \times \frac{\eta^i(48z)}{\eta^i(24z)} = \sum a_i(n)q^n \tag{3.1}$$

Observe that the coefficients $a_i(n)$ are supported on those natural numbers n for which $n \equiv i \pmod{24}$.

Lemma 7. *We have*

(*i*) *If i = 1 or 11 and $n \equiv 0, 2, 3 \pmod 5$, then $a_i(n) \equiv 0 \pmod 5$,* *(ii) If $i = 13$ or 23 and $n \equiv 0, 1, 4 \pmod 5$, then $a_i(n) \equiv 0 \pmod 5$.*

Proof. We give details only for the function $f_1(z)$ - the remaining cases are analogous. There is a simple criterion for deciding when an η-product is a holomorphic modular form (see, for example, [11], Th. 4). We find that $f_1(z) \in M_4(\Gamma_0(1152), \chi_2)$, where χ_2 denotes the Kronecker symbol for $\mathbb{Q}(\sqrt{2})$. If $\left(\frac{\bullet}{5}\right)$ denotes the Legendre symbol, then by Proposition 5 the function defined by

$$g_1(z) = f_1 - f_1 \otimes \left(\frac{\bullet}{5}\right) \tag{3.2}$$

is a modular form in $M_4(\Gamma_0(28800), \chi_2)$ and has Fourier expansion

$$g_1(z) = 2 \sum_{n \equiv 2,3 \pmod 5} a_1(n)q^n + \sum_{n \equiv 0 \pmod 5} a_1(n)q^n. \tag{3.3}$$

By a theorem of Sturm [15], we need only demonstrate the desired congruence for $n \leq 23040$. This is easily handled by machine computation. □

Lemma 8.

(i) If p is a prime such that $p \equiv 5, 7 \pmod{12}$, then

$$f_1(z) \mid T(p) \equiv 0 \pmod 5. \tag{3.4}$$

(ii) If p is a prime such that $p \equiv 1, 11 \pmod{24}$ and $p \equiv 2, 3 \pmod 5$ or if p is a prime such that $p \equiv 13, 23 \pmod{24}$ and $p \equiv 1, 4 \pmod 5$, then

$$f_i(z) \mid T(p) \equiv 0 \pmod 5. \tag{3.5}$$

Proof. Gordon and Sinor [8] have proven that if

$$(\alpha_1, \alpha_5, ..., \alpha_{23}) = (1, -2i\sqrt{35}, 2\sqrt{110}, 4i\sqrt{154}, -8i\sqrt{77}, 16\sqrt{55}, 16i\sqrt{70}, 32\sqrt{2}), \tag{3.6}$$

then

$$g(z) = \sum \alpha_i f_i(z) \tag{3.7}$$

is a modular form in $M_4(\Gamma_0(1152), \chi_2)$ and an eigenform for the Hecke operators $T(p)$. (We alert the reader that the function g is misprinted in [8].) Since the Fourier coefficients $a_i(n)$ are supported on those n for which $n \equiv i \pmod{24}$, the $f_i(z)$ are simply permuted, up to constant multiples, by the action of the Hecke operators. In particular, we have

$$f_i \mid T(p) = \frac{\alpha_{\overline{p}} \alpha_{\overline{pi}} a_{\overline{p}}(p)}{\alpha_i} \times f_{\overline{pi}}, \tag{3.8}$$

where $\overline{x}$ denotes the residue class of x modulo 24. The fact that $5 \mid \alpha_{\overline{p}}{}^2$ when $p \equiv 5, 7 \pmod{12}$ establishes part (i) of the Lemma, while part (ii) follows from the vanishing modulo 5 of the appropriate $a_{\overline{p}}(p)$ as indicated in Lemma 7.

□

Proof of the Main Results. In the course of establishing the family of congruences (1.4) and (1.5), Rødseth [14] proves the congruence

$$\sum_{n=0}^{\infty} Q(5^{2j}n + \gamma_{2j})q^{24n+1} \equiv c_j 5^{j-1} f_1(z) \pmod{5^j}, \tag{3.9}$$

where c_j is an integer such that $5 \mid c_j$ if and only if $5 \mid j$. Then (1.4) is easily deduced from (3.9) upon replacing n by $5n+1$ and appealing to Lemma 7 (i) in the case $n \equiv 0 \pmod 5$. By applying the full strength of Lemma 7 (i) to (3.9), we obtain Theorem 4.

By the definition of the Hecke operators and an application of Lemma 8 to (3.9), we have

$$Q\left(5^{2j}\left(\frac{pn-1}{24}\right)+\frac{5^{2j}-1}{24}\right)+\chi(p)p^3Q\left(5^{2j}\left(\frac{n/p-1}{24}\right)+\frac{5^{2j}-1}{24}\right)\equiv 0 \pmod{5^j} \quad (3.10)$$

for the relevant primes p. The case $ord_p(n) = 1$ of Theorem 1 is immediate and the full statement follows by induction. When m is odd and n is replaced by $p^m(24pn+b)$ in (1.7), we have Corollary 2. Finally, it is also shown in [14] that if $j \equiv 1 \pmod 5$, then

$$\sum_{n=0}^{\infty} Q(5^{2j-1}n+\gamma_{2j-1})q^{24n+5} \equiv b_j 5^{j-1} f_{11}(z) \pmod{5^j}, \quad (3.11)$$

where $5 \nmid b_j$. Using Lemma 8 (ii) and arguing as above gives Theorem 3. □

4. Concluding Remarks

Because of ongoing interest in the arithmetic properties of partition functions, we have elected to focus the present investigation on $Q(n)$. However, it is important to recognize that the methods used here are applicable in many of the studies of families of congruences in arithmetic progressions for the Fourier coefficients of modular functions modulo powers of primes. Atkin, Gordon, Hughes, [3, 5, 9] and others have made several such studies of the values of $p_k(n)$, where

$$\sum_{n=0}^{\infty} p_k(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^k, \quad (4.1)$$

and invariably one finds an equation analogous to (3.9) as a key ingredient in their proofs. It is a simple observation about the relevant modular form appearing in such an equation that allows one to pass to a single family of congruences, but a more detailed investigation, as exemplified here or in [12], will always yield considerably more information. In some cases there are apparently only accidents, but in various situations one finds CM forms, eigenforms (as in [12]), and other forms of integer and half-integer weight whose Fourier coefficients possess particular properties. Undoubtedly there is much more waiting to be discovered in the works cited above.

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