3-REGULAR PARTITIONS AND A MODULAR K3 SURFACE

JEREMY LOVEJOY AND DAVID PENNISTON

1. Introduction

A k-regular partition of n ($k > 1$) is a non-increasing sequence of positive integers whose sum is n, with the condition that no summand is divisible by k. We denote the number of k-regular partitions of n by $b_k(n)$, and follow the convention that $b_k(0) = 1$. Elementary techniques in the theory of partitions [3] give the generating functions

$$\sum_{n=0}^{\infty} b_k(n)q^n = \prod_{n=1}^{\infty} \left(\frac{1-q^{kn}}{1-q^n} \right). \tag{1.1}$$

In classical representation theory, k-regular partitions of n label irreducible k-modular representations of the symmetric group S_n when k is prime [8]. More recently, such partitions have been studied for their arithmetic properties in connection with the theory of modular forms and Galois representations [1, 6, 10, 11, 12]. Although one may presumably use the ideas from [1, 10] to study the k-regular partitions modulo any prime, more focus has been placed on the most straightforward case, the p-adic behavior of p^j-regular partitions. For example, we have

Theorem 1 (Gordon-Ono [6])**.** *If $S(p, j, a)$ denotes the set of natural numbers n such that $b_{p^j}(n)$ is not divisible by p^a, then $S(p, j, a)$ has arithmetic density 0.*

In general there is no elementary characterization of the sets $S(p, j, a)$, but in the best cases we do have simple congruential formulas for $b_k(n)$. For example, the classical expansions

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \tag{1.2}$$

and

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} \tag{1.3}$$

reveal that $b_2(n)$ is even unless $24n + 1$ is a square and $b_4(n)$ is even unless $8n + 1$ is a square. The case of $b_2(n)$ has a famous combinatorial proof by Franklin [3], while K. Ono and the second author [11] have determined $b_2(n)$ modulo 8 in terms of the arithmetic of $\mathbb{Z}[\sqrt{-6}]$.

Here we undertake an investigation of the 3-adic behavior of $b_3(n)$. Let

$$\eta(z) := \prod_{n=1}^{\infty}(1-q^n)$$

1991 *Mathematics Subject Classification.* 11P83.

denote Dedekind's eta funcion, where $q := e^{2\pi i z}$. From (1.1) we have

$$\sum_{n=0}^{\infty} b_3(n) q^{12n+1} \equiv \eta^2(12z) \pmod{3},$$

where $\eta^2(12z)$ is a weight 1 modular form which is the Mellin transform of an Artin L-function for $\mathbb{Q}(i)$. Modulo 9, it turns out that the generating function for $b_3(n)$ is related to an eigenform which is essentially the Mellin transform of the "complicated factor" in the Hasse-Weil L-function for a certain $K3$ surface.

Theorem 2. *Let X be the K3 surface defined by*

$$X : s^2 = x(x+1)y(y+1)(x+8y). \tag{1.4}$$

If p is a prime such that $p \equiv 1 \pmod{12}$, then

$$b_3\left(\frac{p-1}{12}\right) \equiv \#X(\mathbb{F}_p) - (p+1)^2 \pmod{9}. \tag{1.5}$$

Using the fact that the relevant eigenform has complex multiplication, we can use Hecke theory and the arithmetic of the Gaussian integers to build a formula for the number of 3-regular partitions modulo 9.

Theorem 3. *Given a positive integer n, write*

$$12n + 1 = N^2 M$$

with M squarefree. For every prime divisor p of $12n+1$, set

$$k_p := ord_p(12n+1).$$

If $p \equiv 1 \pmod{12}$, let d_p and e_p be integers such that $3 \mid d_p$ and

$$p = d_p^2 + e_p^2.$$

(1) If there is a prime p such that $p \mid M$ and $p \equiv 5, 7$ or $11 \pmod{12}$, then $b_3(n) \equiv 0 \pmod{9}$.

(2) If every prime divisor p of M satisfies $p \equiv 1 \pmod{12}$, then

$$b_3(n) \equiv (3n+1) \cdot \prod_{\substack{p \mid (12n+1) \\ p \equiv 1 \pmod{12}}} (-1)^{k_p d_p}(k_p+1) \cdot \prod_{\substack{p \mid (12n+1) \\ p \equiv 5 \pmod{12}}} (-1)^{\frac{k_p}{2}} \pmod{9}. \tag{1.6}$$

For comparison with (1.2) and (1.3) we cite the following, which is a direct consequence of Theorem 3.

Corollary 4. *$b_3(n)$ is divisible by 3 unless both of the following hold:*

- (i) *All prime divisors $p \equiv 5, 7, 11 \pmod{12}$ of $12n+1$ divide $12n+1$ with even order.*
- (ii) *All prime divisors $p \equiv 1 \pmod{12}$ of $12n+1$ divide $12n+1$ with order not congruent to 2 modulo 3.*

EXAMPLE. If $n = 5$, then $12n+1 = 61 = 6^2 + 5^2$, so $b_3(5)$ is not divisible by 3. More specifically, $b_3(5) \equiv 16 \cdot (-1)^{1 \cdot 6} \cdot 2 \equiv 5 \pmod{9}$. Indeed, the 3-regular partitions of 5 are $5, 4+1, 2+2+1, 2+1+1+1$, and $1+1+1+1+1$.

2. Proof of Theorem 2

Let

$$\eta^6(4z) := \sum_{n=1}^{\infty} a(n)q^n, \tag{2.1}$$

a weight 3 cusp form for the congruence subgroup $\Gamma_0(16)$ with character $\chi_{-1}(d) := \left(\frac{-1}{d}\right)$. We denote the space of such forms by $S_3(\Gamma_0(16), \chi_{-1})$ (see [9] for definitions related to modular forms). It is well-known [5] that $\eta^6(4z)$ has complex multiplication by $K = \mathbb{Q}(i)$. Specifically, let O_K denote the ring of integers of K, and let χ be the character on $(O_K/(2))^*$ defined by $\chi(i) = -1$. Extending χ to the set of all elements of K^* prime to (2), we find that for $d + ei \in O_K$ with $d + e$ odd, $\chi(d + ei) = (-1)^e$. Denote by c the Hecke character on K with conductor (2) and exponent 2 given by

$$c((d + ei)) = \chi(d + ei)(d + ei)^2. \tag{2.2}$$

Then

$$\eta^6(4z) = \sum c(I)q^{N(I)}, \tag{2.3}$$

where the sum is over ideals I of O_K prime to (2).

This form is the fundamental object in our work, as it relates 3-regular partitions, the $K3$ surface (1.4), and the arithmetic of the Gaussian integers.

Proof of Theorem 2. Let

$$F(z) := \frac{\eta^8(12z)}{\eta^2(36z)},$$

which is easily seen to be a modular form in $S_3(\Gamma_0(1296), \chi_{-1})$ (see [10], for example). From (1.1) and the fact that

$$\frac{\eta^9(z)}{\eta^3(3z)} \equiv 1 \pmod 9,$$

we have

$$\sum_{n=0}^{\infty} b_3(n)q^{12n+1} \equiv F(z) \pmod 9.$$

By definition, $a(n) = 0$ unless $n \equiv 1 \pmod 4$, and therefore

$$\frac{1}{2}\sum_{n=1}^{\infty}\left(\left(\frac{n}{3}\right)a(n) + \left(\frac{n}{3}\right)\left(\frac{n}{3}\right)a(n)\right)q^n = \sum_{n\equiv 1 \pmod{12}} a(n)q^n. \tag{2.4}$$

From [9], p. 127, (2.4) is a modular form in $S_3(\Gamma_0(1296), \chi_{-1})$. By computation, the first 648 coefficients of $F(z)$ and (2.4) are equivalent modulo 9, and hence by a theorem of Sturm [13] we have for every n,

$$b_3(n) \equiv a(12n + 1) \pmod 9. \tag{2.5}$$

To complete the proof, we recall the modularity of the surface (1.4) [2]. For every prime $p \geq 5$, we have

$$\#X(\mathbb{F}_p) = 1 + p^2 + 20p + a(p). \tag{2.6}$$

□

REMARK. Since the L-series for X is the symmetric square of the L-series for the congruent number elliptic curve given by the equation $E : y^2 = x^3 - x$ [2], the congruence (2.5) is dictated by Galois actions on certain points on E. Specifically, let $g(n)$ denote the Fourier coefficients of the associated eigenform:

$$\eta^2(4z)\eta^2(8z) := \sum_{n=1}^{\infty} g(n)q^n.$$

Then for every prime $p \geq 5$, $a(p) = g(p)^2 - 2p$. Denote by $G_{\mathbb{Q}}$ the absolute Galois group of $\mathbb{Q}$, and by $E[n]$ the group of n-division points of E for any $n \geq 1$ (as a group, $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$). If ℓ is prime, $G_{\mathbb{Q}}$ acts on the Tate module

$$T_\ell(E) = \varprojlim_m E[\ell^m] \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell,$$

and therefore we obtain a representation

$$\rho_\ell : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_\ell).$$

If frob_p denotes a Frobenius element for p ($p \neq \ell$), then $trace(\rho_\ell(\mathrm{frob}_p)) = g(p)$. With (2.5), this shows that the behavior of $b_3(n)$ modulo 9 is determined by the Galois action on the 3-division points of E.

3. Proof of Theorem 3

Since $\eta^6(4z) = \sum_{n=1}^{\infty} a(n)q^n \in S_3(\Gamma_0(16), \chi_{-1})$ is a Hecke eigenform, we have that

$$a(mn) = a(m)a(n) \qquad \text{if } (m,n) = 1 \tag{3.1}$$

and

$$a(p^{k+1}) = a(p)a(p^k) - \chi_{-1}(p)a(p^{k-1})p^2 \qquad \text{if } p \geq 5 \text{ is prime and } k \geq 0. \tag{3.2}$$

In light of (2.5), (3.1), and (3.2), we begin by studying the $a(p)$ for p prime.

Proposition 5. *Let p be an odd prime.*

(1) If $p \equiv 3 \pmod 4$, then $a(p) = 0$.

(2) If $p \equiv 5 \pmod{12}$, then $3 \mid a(p)$.

(3) If $p \equiv 1 \pmod{12}$ and we write $p = d_p^2 + e_p^2$ with $3 \mid d_p$, then $a(p) \equiv (-1)^{d_p} \cdot 2p \pmod 9$.

Proof. For *(1)*, see (2.1), or recall (2.3) and note that since (p) is prime in O_K, there are no ideals of norm p in O_K.

Now suppose $p \equiv 1 \pmod 4$. Then there are integers d_p and e_p with $p = d_p^2 + e_p^2$, and hence the prime ideals of O_K of norm p are $(d_p \pm e_p i)$. Since $\chi(d_p \pm e_p i) = (-1)^{e_p}$, (2.2) and (2.3) give us that

$$a(p) = (-1)^{e_p}(2d_p^2 - 2e_p^2) = (-1)^{e_p}(4d_p^2 - 2p). \tag{3.3}$$

If $p \equiv 5 \pmod{12}$, then since $p \equiv 2 \pmod 3$, it follows that $3 \nmid d_p e_p$. Hence $d_p^2 \equiv e_p^2 \equiv 1 \pmod 3$, and the proof of *(2)* is complete.

To finish the proof of *(3)*, if $p \equiv 1 \pmod{12}$, then $3 \mid d_p e_p$. We assume without loss that $3 \mid d_p$. Then by (3.3),

$$a(p) \equiv (-1)^{e_p+1} \cdot 2p = (-1)^{d_p} \cdot 2p \pmod 9.$$

□

Combining Proposition 5 with (3.2), it is straightforward induction to show

Proposition 6. *Let p be an odd prime, k a positive integer.*
(1) If $p \equiv 3 \pmod 4$, then $a(p^{2k-1}) = 0$ and $a(p^{2k}) \equiv p^{2k} \pmod 9$.
(2) If $p \equiv 5 \pmod{12}$, then $3 \mid a(p^{2k-1})$ and $a(p^{2k}) \equiv (-p^2)^k \pmod 9$.
(3) If $p \equiv 1 \pmod{12}$ and $p = d_p^2 + e_p^2$ with $3 \mid d_p$, then $a(p^k) \equiv (-1)^{kd_p}(k+1)p^k \pmod 9$.

Theorem 3 follows now from (2.5), (3.1), and Proposition 6.

We have not observed any simple congruence condition which determines the parity of d_p as a function of p, which is tantamount to distinguishing between primes of the form $x^2 + 36y^2$ and those of the form $4x^2 + 9y^2$. In this direction it is known [4] that for all but finitely many primes $p \equiv 1 \pmod 4$, p is represented by $x^2 + 36y^2$ if and only if the minimal polynomial for $j(\sqrt{-36})$ has a root modulo p.

4. Concluding remarks

Since the generating functions for partition theoretic objects are typically products and quotients of the η function, connections to objects in arithmetic geometry such as that given by Theorem 2 are not unexpected. A striking example of this is in recent work of L. Guo and K. Ono [7], where it is shown that values of the ordinary partition function reveal structure of Tate-Shafarevich groups of motives of modular forms. In our case, an examination of, for instance, the five 3-regular partitions of 5 and the 4920 $\mathbb{F}_{61}$-points on our $K3$ surface gives one little reason to expect that there is something in the combinatorics of 3-regular partitions or irreducible 3-modular representations of S_n that is related to the structure of modular surfaces or the arithmetic of $\mathbb{Q}(i)$. We must for now be content that the theory of modular forms is a meeting place for diverse mathematical objects whose connections often cannot be otherwise explained.

References

[1] S. Ahlgren and J. Lovejoy, The arithmetic of the number of partitions into distinct parts, *Mathematika*, to appear.
[2] S. Ahlgren, K. Ono, and D. Penniston, Zeta functions of an infinite family of $K3$ surfaces, *Amer. J. Math.*, to appear.
[3] G. E. Andrews, "The Theory of Partitions," Cambridge Univ. Press, Cambridge, 1998.
[4] D. A. Cox, "Primes of the form $x^2 + ny^2$", John Wiley and Sons, Inc., 1989.
[5] D. Dummit, H. Kisilevsky, and J. McKay, Multiplicative products of η-functions, *Contemp. Math.* **45** (Finite groups - coming of age), 89-98.
[6] B. Gordon and K. Ono, The divisibility of certain partition functions, *Ramanujan J.* **1** (1997), 25-34.
[7] L. Guo and K. Ono, The partition function and the arithmetic of certain modular L-functions, *Internat. Math. Res. Notices* **21** (1999), 1179-1197.
[8] G. James and A. Kerber, "The Representation of the Symmetric Group," Addison-Wesley, Reading, 1979.
[9] N. Koblitz, "Introduction to Elliptic Curves and Modular Forms," Springer-Verlag, New York, 1984.
[10] J. Lovejoy, Divisibility and distribution of partitions into distinct parts, *Adv. Math.* **158** (2001), 253-263.

[11] K. Ono and D. Penniston, The 2-adic behavior of the number of partitions into distinct parts, *J. Combin. Theory Ser. A* **92** (2000), 138-157.
[12] D. Penniston, The p^a-regular partition function modulo p^j, preprint.
[13] J. Sturm, On the congruence of modular forms, *Lect. Notes in Math.* **1240**, Springer-Verlag, Ner York, 1984, 275-280.

Department of Mathematics, University of Wisconsin, Madison, WI 53706
E-mail address: `lovejoy@math.wisc.edu`

Department of Mathematics, Furman University, Greenville, SC 29613
E-mail address: `david.penniston@furman.edu`