RAMANUJAN-TYPE PARTIAL THETA IDENTITIES AND RANK DIFFERENCES FOR SPECIAL UNIMODAL SEQUENCES

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Abstract. We consider three types of unimodal sequences related to Ramanujan-type partial theta identities. In each case we compute generating functions related to the rank and use the partial theta identity to give formulas for certain rank differences.

1. Introduction

1.1. **Background and motivation.** Let $U(n)$ denote the number of unimodal sequences of the form

$$a_1 \leq a_2 \leq \cdots \leq a_r \leq \overline{c} \geq b_1 \geq b_2 \geq \cdots \geq b_s \tag{1.1}$$

with weight $n = c + \sum_{i=1}^{r} a_i + \sum_{i=1}^{s} b_i$. For example, $U(4) = 12$, the relevant sequences being

$$(\overline{4}), (1, \overline{3}), (\overline{3}, 1), (1, \overline{2}, 1), (\overline{2}, 2), (2, \overline{2}),$$
$$(1, 1, \overline{2}), (\overline{2}, 1, 1), (\overline{1}, 1, 1, 1), (1, \overline{1}, 1, 1), (1, 1, \overline{1}, 1), (1, 1, 1, \overline{1}).$$

Define the rank of a unimodal sequence to be $s - r$, and assume that the empty sequence has rank 0. Let $U(m, n)$ be the number of unimodal sequences of weight n with rank m, and let $U(t, m, n)$ be the number of unimodal sequences of weight n with rank congruent to t modulo m. Note the symmetries $U(m, n) = U(-m, n)$ and $U(m - t, m, n) = U(t, m, n)$, which follow upon exchanging the partitions $\sum_{i=1}^{r} a_i$ and $\sum_{i=1}^{s} b_i$ in (1.1).

Define the rank difference $U_{t_1 t_2, m}(x)$ by

$$U_{t_1 t_2, m}(x) := \sum_{n \geq 0} \Big(U(t_1, m, mn + x) - U(t_2, m, mn + x) \Big) q^{mn+x}.$$

In a recent paper [18], we showed that the rank differences $U_{t_1 t_2, 5}(x)$ have surprisingly simple expressions in terms of partial theta functions and modular forms. Recall the usual q-series notation,

$$(a_1, a_2, \ldots, a_k)_n := (a_1, a_2, \ldots, a_k; q)_n := \prod_{i=1}^{k} (1 - a_i)(1 - a_i q) \cdots (1 - a_i q^{n-1}).$$

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Theorem 1.1 (See Theorem 1.1 of [18]). *We have*

$$
\begin{aligned}
U_{02,5}(0) &= \left(\sum_{n\geq 0} - \sum_{n\leq -1}\right)(-1)^n q^{5n(15n+1)/2}(1+q^{45n+15}),\\
U_{12,5}(0) &= \frac{\sum_{n\geq 0}(-1)^n q^{5n(5n+1)/2}}{(q^5,q^{20};q^{25})_\infty} - \sum_{n\geq 0}(-1)^n q^{5n(15n+1)/2}(1+q^{25n+5}),\\
U_{02,5}(1) &= \frac{\sum_{n\geq 0}(-1)^n q^{(5n+3)(5n+4)/2}}{(q^5,q^{20};q^{25})_\infty} + \frac{q\sum_{n\geq 0}(-1)^n q^{(5n+4)(5n+5)/2}}{(q^{10},q^{15};q^{25})_\infty} + q(q^{25},q^{50},q^{75};q^{75})_\infty,\\
U_{12,5}(1) &= \frac{\sum_{n\geq 0}(-1)^n q^{(5n+3)(5n+4)/2}}{(q^5,q^{20};q^{25})_\infty} - \frac{q\sum_{n\geq 0}(-1)^n q^{5n(5n+1)/2}}{(q^{10},q^{15};q^{25})_\infty} + q(q^{25},q^{50},q^{75};q^{75})_\infty,\\
U_{02,5}(2) &= \frac{q\sum_{n\geq 0}(-1)^n q^{(5n+3)(5n+4)/2}}{(q^{10},q^{15};q^{25})_\infty} + \sum_{n\geq 0}(-1)^n q^{(5n+1)(15n+4)/2}(1+q^{25n+10}),\\
U_{12,5}(2) &= \left(\sum_{n\geq 0} - \sum_{n\leq -1}\right)(-1)^n q^{(5n+1)(15n+4)/2}(1+q^{15n+5}),\\
U_{02,5}(3) &= \frac{\sum_{n\geq 0}(-1)^n q^{(5n+2)(5n+3)/2}}{(q^5,q^{20};q^{25})_\infty},\\
U_{12,5}(3) &= 0,\\
U_{02,5}(4) &= 0,\\
U_{12,5}(4) &= \frac{q\sum_{n\geq 0}(-1)^n q^{(5n+2)(5n+3)/2}}{(q^{10},q^{15};q^{25})_\infty}.
\end{aligned}
$$

These resemble rank difference identities for partitions [5, 13, 19, 25, 27, 28] and overpartitions [22, 23, 24], but with partial theta functions taking the place of mock theta functions. Indeed, Theorem 1.1 is an application of the partial theta identity on p. 37 of Ramanujan's lost notebook [4, Entry 6.3.2],

$$
\sum_{n\geq 0}\frac{q^n}{(xq)_n(q/x)_n} = (1-x)\sum_{n\geq 0}(-1)^n x^{3n}q^{n(3n+1)/2}(1-x^2q^{2n+1}) + \frac{\sum_{n\geq 0}(-1)^n x^{2n+1}q^{\binom{n+1}{2}}}{(xq)_\infty(q/x)_\infty}. \quad (1.2)
$$

(Note that the left-hand side of this equation is the generating function for $U(m,n)$.)

There are several other partial theta identities like (1.2) in the lost notebook, and further examples have been found by Warnaar [30, 31] and the second author [21]. In general these identities may be interpreted combinatorially as generating functions for special unimodal sequences, and in theory they may be used to compute the corresponding rank differences (see Section 5 for further remarks on this). The question is whether any of these rank differences are as simple and elegant as those in Theorem 1.1. In this paper we highlight three cases where this is so.

1.2. **Statement of Results.** The first type of unimodal sequence we consider corresponds to the following partial theta identity due to Warnaar [30],

$$\sum_{n\geq 0} \frac{q^{2n}}{(xq,q/x)_n} = (1-x^2) + (1+x^2)(1-x)\sum_{n\geq 1}(-1)^n x^{3n-2} q^{n(3n-1)/2}(1+xq^n) + \frac{x^2 + (1+x^2)\sum_{n\geq 1}(-1)^n x^{2n} q^{\binom{n+1}{2}}}{(xq,q/x)_\infty}. \tag{1.3}$$

Let $W(n)$ be the number of unimodal sequences with a double peak, i.e., sequences of the form

$$a_1 \leq a_2 \leq \cdots \leq a_r \leq \overline{c}\ \overline{c} \geq b_1 \geq b_2 \geq \cdots \geq b_s, \tag{1.4}$$

with weight $n = 2c + \sum_{i=1}^r a_i + \sum_{i=1}^s b_i$. For example, $W(6) = 11$, the relevant sequences being

$$(\overline{3},\overline{3}), (\overline{2},\overline{2},2), (2,\overline{2},\overline{2}), (\overline{2},\overline{2},1,1), (1,\overline{2},\overline{2},1), (1,1,\overline{2},\overline{2}),$$
$$(\overline{1},\overline{1},1,1,1,1), (1,\overline{1},\overline{1},1,1,1)(1,1,\overline{1},\overline{1},1,1), (1,1,1,\overline{1},\overline{1},1), (1,1,1,1,\overline{1},\overline{1}).$$

Define the rank of such a unimodal sequence to be $s - r$, and assume that the empty sequence has rank 0. Let $W(m,n)$ denote the number of sequences counted by $W(n)$ with rank m, and let $W(t,m,n)$ be the number of sequences counted by $W(n)$ with rank congruent to t modulo m. Note the symmetries $W(m,n) = W(-m,n)$ and $W(m-t,m,n) = W(t,m,n)$, which follow upon exchanging the partitions $\sum_{i=1}^r a_i$ and $\sum_{i=1}^s b_i$ in (1.4).

Define the rank difference $W_{t_1t_2,m}(x)$ by

$$W_{t_1t_2,m}(x) := \sum_{n\geq 0}\Big(W(t_1,m,mn+x) - W(t_2,m,mn+x)\Big)q^{mn+x}. \tag{1.5}$$

With our first result we compute all of the rank differences $W_{t_1t_2,5}(x)$.

Theorem 1.2. *We have*

$$W_{02,5}(0) = 2 + \frac{\sum_{n\geq 1}(-1)^n q^{\binom{5n+1}{2}}}{(q^5,q^{20};q^{25})_\infty} - \sum_{n\geq 0}(-1)^n q^{5n(15n+1)/2}(1+q^{25n+5}),$$

$$W_{12,5}(0) = -1 - \frac{\sum_{n\geq 0}(-1)^n q^{\binom{5n+5}{2}}}{(q^5,q^{20};q^{25})_\infty} + \left(\sum_{n\geq 0} - 2\sum_{n\leq -1}\right)(-1)^n q^{5n(15n+1)/2}(1+q^{25n+5}),$$

$$W_{02,5}(1) = \frac{\sum_{n\geq 0}(-1)^n q^{\binom{5n+2}{2}}}{(q^5,q^{20};q^{25})_\infty} - \frac{q\sum_{n\geq 0}(-1)^n q^{\binom{5n+5}{2}}}{(q^{10},q^{15};q^{25})_\infty} - q(q^{25};q^{25})_\infty,$$

$$W_{12,5}(1) = \frac{-q\sum_{n\geq 1}(-1)^n q^{\binom{5n}{2}}(1-q^{5n})}{(q^{10},q^{15};q^{25})_\infty},$$

$$W_{02,5}(2) = \frac{q^2(q^5,q^{20},q^{25};q^{25})_\infty}{(q^{10},q^{15};q^{25})_\infty} + 2\sum_{n\geq 0}(-1)^n q^{(5n+2)(15n+7)/2}(1+q^{25n+15}),$$

$$W_{12,5}(2) = \frac{q\sum_{n\geq 0}(-1)^n q^{\binom{5n+2}{2}}}{(q^{10},q^{15};q^{25})_\infty} - \sum_{n\geq 0}(-1)^n q^{(5n+1)(15n+4)/2}(1+q^{25n+10}),$$

$$W_{02,5}(3) = 0,$$

$$W_{12,5}(3) = \frac{\sum_{n\geq 0}(-1)^n q^{\binom{5n+3}{2}}}{(q^5,q^{20};q^{25})_\infty},$$

$$W_{02,5}(4) = \frac{q\sum_{n\geq 0}(-1)^n q^{\binom{5n+3}{2}}}{(q^{10},q^{15};q^{25})_\infty}, \tag{1.6}$$

$$W_{12,5}(4) = -\frac{q\sum_{n\geq 0}(-1)^n q^{\binom{5n+3}{2}}}{(q^{10},q^{15};q^{25})_\infty}. \tag{1.7}$$

The following is an immediate consequence of equations (1.6) and (1.7).

Corollary 1.3. *For all $n \geq 0$ we have*

$$W(0,5,5n+4) + W(1,5,5n+4) = 2W(2,5,5n+4).$$

The second type of unimodal sequence we consider corresponds to the partial theta identity on p. 12 of Ramanujan's lost notebook [4, Entry 6.6.1],

$$\sum_{n\geq 0}\frac{(q^{n+1})_n q^n}{(xq,q/x)_n} = (1-x)\sum_{n\geq 0}x^n q^{n^2+n} + \frac{x}{(xq,q/x)_\infty}\sum_{n\geq 0}x^{3n}q^{3n^2+2n}(1-xq^{2n+1}). \tag{1.8}$$

Let $V(n)$ denote the number of unimodal sequences of the form (1.1), where $\sum b_i$ is a partition into parts at most $c-k$ and k is the size of the Durfee square of the partition $\sum a_i$. For example, $V(4) = 10$, the relevant sequences being

$$(\overline{4}), (1,\overline{3}), (\overline{3},1), (1,\overline{2},1), (\overline{2},2), (2,\overline{2}), (1,1,\overline{2}), (\overline{2},1,1),$$
$$(\overline{1},1,1,1), (1,1,1,\overline{1}).$$

As usual define the rank of a sequence counted by $V(n)$ to be $s-r$, and assume that the empty sequence has rank 0. Let $V(m,n)$ denote the number of sequences counted by $V(n)$ with rank m and let $V(t,m,n)$ be the number of sequences counted by $V(n)$ with rank congruent to t modulo m. Although they are not obvious from the definition, the symmetries $V(m,n) = V(-m,n)$ and $V(m-t,m,n) = V(t,m,n)$ follow from the generating function (3.1).

Define the rank difference $V_{t_1t_2,m}(x)$ by

$$V_{t_1t_2,m}(x) := \sum_{n\geq 0}\Big(V(t_1,m,mn+x) - V(t_2,m,mn+x)\Big)q^{mn+x}.$$

Our next result gives formulas for all of the rank differences $V_{t_1t_2,5}(x)$.

Theorem 1.4. *We have*

$$V_{02,5}(0) = \sum_{n\geq 0} q^{25n^2+5n}(1-q^{40n+20}),$$

$$V_{12,5}(0) = \frac{\sum_{n\geq 0} q^{75n^2+10n}(1-q^{100n+40})}{(q^5,q^{20};q^{25})_\infty} - \sum_{n\geq 0} q^{25n^2+5n},$$

$$V_{02,5}(1) = \frac{\sum_{n\geq 0} q^{75n^2+40n+6}(1-q^{100n+60})}{(q^{10},q^{15};q^{25})_\infty} + \frac{\sum_{n\geq 0} q^{75n^2+20n+1}(1-q^{50n+15})}{(q^5,q^{20};q^{25})_\infty} - q^6\frac{(q^{100};q^{100})_\infty}{(q^{50};q^{100})_\infty},$$

$$V_{12,5}(1) = \frac{\sum_{n\geq 0} q^{75n^2+20n+1}(1-q^{50n+15})}{(q^5,q^{20};q^{25})_\infty} - \frac{\sum_{n\geq 0} q^{75n^2+10n+1}(1-q^{100n+40})}{(q^{10},q^{15};q^{25})_\infty} - q^6\frac{(q^{100};q^{100})_\infty}{(q^{50};q^{100})_\infty},$$

$$V_{02,5}(2) = \frac{\sum_{n\geq 0} q^{75n^2+80n+22}(1-q^{50n+35})}{(q^{10},q^{15};q^{25})_\infty} + \sum_{n\geq 0} q^{25n^2+15n+2},$$

$$V_{12,5}(2) = \sum_{n\geq 0} q^{25n^2+15n+2}(1-q^{20n+10}),$$

$$V_{02,5}(3) = -\frac{\sum_{n\geq 0} q^{75n^2+50n+8}(1-q^{50n+25})}{(q^5,q^{20};q^{25})_\infty},$$

$$V_{12,5}(3) = 0,$$

$$V_{02,5}(4) = 0,$$

$$V_{12,5}(4) = -\frac{\sum_{n\geq 0} q^{75n^2+50n+9}(1-q^{50n+25})}{(q^{10},q^{15};q^{25})_\infty}.$$

The last type of unimodal sequence we consider corresponds to the partial theta identity on p. 2 of Ramanujan's lost notebook [4, Entry 6.3.7],

$$\sum_{n\geq 0}\frac{(-q)_{2n}q^{2n+1}}{(xq,q/x;q^2)_{n+1}} = \frac{-x}{1+x}\sum_{n\geq 0}(-x)^n q^{n^2+n} + \frac{x(-q)_\infty}{(1+x)(xq,q/x;q^2)_\infty}\sum_{n\geq 0}(-x)^n q^{\binom{n+1}{2}}. \quad (1.9)$$

Let $\mathcal{V}(n)$ denote the number of unimodal sequences of the form (1.1) where c is odd, $\sum a_i$ is a partition without repeated even parts, and $\sum b_i$ is an overpartition into odd parts whose largest part is not $\overline{c}$. (Recall that an overpartition is a partition in which the first occurrence of a part may be overlined.) For example, $\mathcal{V}(5) = 12$, the relevant sequences being

$$(\overline{5}), (1, \overline{3}, 1), (1, 1, \overline{3}), (\overline{3}, 1, 1), (\overline{3}, \overline{1}, 1), (1, \overline{3}, \overline{1}), (2, \overline{3}),$$
$$(1, 1, 1, 1, \overline{1}), (1, 1, 1, \overline{1}, 1), (1, 1, \overline{1}, 1, 1), (1, \overline{1}, 1, 1, 1), (\overline{1}, 1, 1, 1, 1).$$

Define the rank of a sequence counted by $\mathcal{V}(n)$ to be the number of odd non-overlined parts in $\sum b_i$ minus the number of odd parts in $\sum a_i$, and assume that the empty sequence has rank 0. Let $\mathcal{V}(m, n)$ denote the number of sequences counted by $\mathcal{V}(n)$ with rank m and let $\mathcal{V}(t, m, n)$ be the number of sequences counted by $\mathcal{V}(n)$ with rank congruent to t modulo m. Note the symmetries $\mathcal{V}(m, n) = \mathcal{V}(-m, n)$ and $\mathcal{V}(m - t, m, n) = \mathcal{V}(t, m, n)$, which follow from exchanging the odd parts of $\sum a_i$ with the odd non-overlined parts of $\sum b_i$.

Define the rank difference $\mathcal{V}_{t_1t_2,m}(x)$ by

$$\mathcal{V}_{t_1t_2,m}(x) := \sum_{n\geq 0} \Big(\mathcal{V}(t_1, m, mn + x) - \mathcal{V}(t_2, m, mn + x)\Big) q^{mn+x}.$$

Our final theorem gives formulas for all of the rank differences $\mathcal{V}_{t_1t_2,3}(x)$.

Theorem 1.5. *We have*

$$\begin{aligned}
\mathcal{V}_{01,3}(0) &= \frac{\sum_{n\geq 0}(-1)^n q^{(3n+1)(3n)/2}}{(q^3; q^6)_\infty} - \sum_{n\geq 0}(-1)^n q^{9n^2+3n},\\
\mathcal{V}_{01,3}(1) &= \frac{\sum_{n\geq 0}(-1)^n q^{(3n+2)(3n+1)/2}}{(q^3; q^6)_\infty},\\
\mathcal{V}_{01,3}(2) &= -\sum_{n\geq 0}(-1)^n q^{9n^2+9n+2}.
\end{aligned} \tag{1.10}$$

Since $\mathcal{V}(1, 3, n) = \mathcal{V}(2, 3, n)$ and $\mathcal{V}(n) = \sum_{i=1}^{3} \mathcal{V}(i, 3, n)$, equation (1.10) implies the following congruence.

Corollary 1.6. *For $3n + 2 \geq 0$ not of the form $9k^2 + 9k + 2$ for any $k \geq 0$, we have*

$$\mathcal{V}(3n + 2) \equiv 0 \pmod 3.$$

1.3. **Outline.** In the next three sections we treat the three types of unimodal sequences introduced above. For $X = W, V$, and $\mathcal{V}$, we first establish useful generating functions for $X(m, n)$ and $X(t, m, n)$, and then we prove the rank difference identities in Theorems 1.2, 1.4, and 1.5. In Section 5 we make some remarks on rank differences $X_{rs,m}(x)$ for arbitrary m and we close in Section 6 with some suggestions for future research.

2. Unimodal sequences for the partial theta identity (1.3)

2.1. **Generating functions.** We begin by establishing three generating functions for $W(m, n)$. Define $W(x, q)$ by

$$W(x, q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} W(m, n)x^m q^n.$$

Proposition 2.1. *We have*

$$W(x,q) = \sum_{n\geq 0} \frac{q^{2n}}{(xq,q/x)_n} \tag{2.1}$$

$$= (1-x^2) + (1+x^2)(1-x)\sum_{n\geq 1}(-1)^n x^{3n-2} q^{n(3n-1)/2}(1+xq^n) + \frac{x^2 + (1+x^2)\sum_{n\geq 1}(-1)^n x^{2n} q^{\binom{n+1}{2}}}{(xq,q/x)_\infty} \tag{2.2}$$

$$= \frac{(1-x)}{(q)_\infty^2}\left(\sum_{n,r\geq 0} - \sum_{r,n<0}\right)\frac{(-1)^{n+r}q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r})}{1-xq^r} - \frac{1}{(xq,q/x)_\infty}. \tag{2.3}$$

Proof. Equation (2.1) is straightforward from the definition and equation (2.2) is just (1.3). For (2.3) we use Bailey pairs. It is not necessary to go into detail on these (the interested reader may consult [2] or [30]), only to note that if (α_n, β_n) is a Bailey pair relative to 1, then [21, Eq. (1.15)]

$$\sum_{n\geq 0} q^{2n}\beta_n = \frac{1}{(q)_\infty^2}\left(\sum_{r\geq 0} q^{2r}\alpha_r + \sum_{\substack{n\geq 1\\ r\geq 0}}(-1)^n q^{\binom{n+1}{2}+2nr}(1+q^{2r})\alpha_r\right), \tag{2.4}$$

and that the sequences

$$\alpha_n = \begin{cases} 1, & \text{if } n=0, \\ \frac{q^{\binom{n+1}{2}}(-1)^n(1+q^n)(1-x)(1-1/x)}{(1-xq^n)(1-q^n/x)}, & \text{otherwise,} \end{cases} \tag{2.5}$$

and

$$\beta_n = \frac{1}{(xq)_n(q/x)_n} \tag{2.6}$$

form a Bailey pair relative to 1 (see [29, Eq. (4.1)] with $(a,c,d) = (1,x,1/x)$). Substituting this Bailey pair into (2.4) and using the fact that for $r \geq 1$

$$\frac{(1+q^r)(1-x)(1-1/x)}{(1-xq^r)(1-q^r/x)} = \frac{1-x}{1-xq^r} + \frac{1-1/x}{1-q^r/x}, \tag{2.7}$$

we have

$$\begin{aligned}
W(x,q) &= \frac{1}{(q)_\infty^2}\Bigg(\sum_{r\geq 0}(-1)^r q^{\binom{r+1}{2}+2r}\frac{1-x}{1-xq^r} + \sum_{r\geq 1}(-1)^r q^{\binom{r+1}{2}+2r}\frac{1-1/x}{1-q^r/x} + 2\sum_{n\geq 1}(-1)^n q^{\binom{n+1}{2}} \\
&\qquad + \sum_{\substack{n\geq 1\\ r\geq 1}}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r})\left(\frac{1-x}{1-xq^r}+\frac{1-1/x}{1-q^r/x}\right)\Bigg) \\
&= \frac{1}{(q)_\infty^2}\Bigg(\sum_{r\geq 0}(-1)^r q^{\binom{r+1}{2}+2r}\frac{1-x}{1-xq^r} + \sum_{r\geq 1}(-1)^r q^{\binom{r+1}{2}+2r}\frac{1-1/x}{1-q^r/x} \\
&\qquad + \sum_{\substack{n\geq 1\\ r\geq 0}}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r})\frac{1-x}{1-xq^r} \\
&\qquad\qquad + \sum_{\substack{n\geq 1\\ r\geq 1}}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r})\frac{1-1/x}{1-q^r/x}\Bigg) \\
&= \frac{1}{(q)_\infty^2}\Bigg(\sum_{\substack{n\geq 0\\ r\geq 0}}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r})\frac{1-x}{1-xq^r} - \sum_{r\geq 0}(-1)^r q^{\binom{r+1}{2}}\frac{1-x}{1-xq^r} \\
&+ \sum_{\substack{n\geq 0\\ r\geq 1}}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r})\frac{1-1/x}{1-q^r/x} - \sum_{r\geq 1}(-1)^r q^{\binom{r+1}{2}}\frac{1-1/x}{1-q^r/x}\Bigg) \qquad (2.8) \\
&= \frac{(1-x)}{(q)_\infty^2}\Bigg(\Bigg(\sum_{n,r\geq 0} - \sum_{r,n<0}\Bigg)\frac{(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r})}{1-xq^r} - \sum_{r\in\mathbb{Z}}\frac{(-1)^r q^{\binom{r+1}{2}}}{1-xq^r}\Bigg),
\end{aligned}$$

and (2.3) now follows upon applying the classical identity,

$$\sum_{r\in\mathbb{Z}}\frac{(-1)^r q^{\binom{r+1}{2}}}{1-xq^r} = \frac{(q)_\infty^2}{(x,q/x)_\infty}. \qquad (2.9)$$

□

Next we establish generating functions for $W(m,n)$ and $W(t,m,n)$ for fixed m and t. Recall the characteristic function $\chi(P)$, which is equal to 1 when P is true and 0 if P is false.

Proposition 2.2.

(1) *For $m \in \mathbb{Z}$ we have*

$$\sum_{n\geq 0} W(m,n)q^n = \chi(m=0) + \frac{-1}{(q)_\infty^2}\Bigg(\sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^{n+r}q^{\binom{n+1}{2}+2nr+\binom{r}{2}+|m|r}(1+q^{2r})(1-q^r) - \sum_{n\geq 1}(-1)^n q^{\binom{n}{2}+|m|n}(1-q^n)\Bigg). \tag{2.10}$$

(2) *For $m \geq 1$ and $0 \leq t \leq m-1$ we have*

$$\sum_{n\geq 0} W(t,m,n)q^n = \chi(t=0) + \frac{-1}{(q)_\infty^2}\Bigg(\sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^{n+r}q^{\binom{n+1}{2}+2nr+\binom{r}{2}}(1+q^{2r})(1-q^r)\frac{(q^{rt}+q^{r(m-t)})}{1-q^{rm}} - \sum_{n\geq 1}(-1)^n q^{\binom{n}{2}}(1-q^n)\frac{(q^{nt}+q^{n(m-t)})}{1-q^{nm}}\Bigg). \tag{2.11}$$

Proof. For $m \geq 1$ equation (2.10) follows from (2.3) and (2.9) after expanding

$$\frac{1-x}{1-xq^r} = (1-x)\sum_{m\geq 0} x^m q^{mr}$$

and picking off the coefficient of x^m. The case $m < 0$ follows from the symmetry $W(m,n) = W(-m,n)$. The case $m = 0$ is trickier. For this we need the identity

$$\left(\sum_{r,n\geq 0} - \sum_{r,n<0}\right)(-1)^{n+r}q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r}) = (q)_\infty^2, \tag{2.12}$$

which follows from (2.4) and the unit Bailey pair relative to 1, [2, Theorem 1],

$$\alpha_n = \begin{cases} 1, & \text{if } n=0,\\ (-1)^n q^{\binom{n}{2}}(1+q^n), & \text{otherwise}\end{cases} \tag{2.13}$$

and

$$\beta_n = \chi(n=0). \tag{2.14}$$

Specifically, using (2.13) and (2.14) in (2.4) we obtain

$$
\begin{aligned}
(q)_\infty^2 &= \sum_{r\geq 0}(-1)^r q^{\binom{r}{2}+2r} + \sum_{r\geq 1}(-1)^r q^{\binom{r+1}{2}+2r} + 2\sum_{n\geq 1}(-1)^n q^{\binom{n+1}{2}} \\
&\qquad + \sum_{\substack{r\geq 1\\ n\geq 1}}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r}{2}}(1+q^{2r})(1+q^r) \\
&= \sum_{r\geq 0}(-1)^r q^{\binom{r}{2}+2r} + \sum_{r\geq 1}(-1)^r q^{\binom{r+1}{2}+2r} + 2\sum_{n\geq 1}(-1)^n q^{\binom{n+1}{2}} \\
&\qquad + \sum_{\substack{r\geq 1\\ n\geq 1}}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r}{2}}(1+q^{2r}) + \sum_{\substack{r\geq 1\\ n\geq 1}}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r}) \\
&= \sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r}{2}}(1+q^{2r}) + \sum_{r,n\geq 0}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r}{2}}(1+q^{2r}).
\end{aligned}
$$

Replacing (r, n) by $(-r, -n-1)$ in the first sum gives (2.12).

Now picking off the coefficient of x^0 in (2.3) (c.f. equation (2.8)), we have

$$
\begin{aligned}
\sum_{n\geq 0} W(0,n)q^n &= \frac{1}{(q)_\infty^2}\Big(\sum_{r,n\geq 0}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r}) - \sum_{r\geq 0}(-1)^r q^{\binom{r+1}{2}} \\
&\qquad + \sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^r q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r}) - \sum_{r\geq 1}(-1)^r q^{\binom{r+1}{2}}\Big) \\
&= \frac{1}{(q)_\infty^2}\Big((q)_\infty^2 + \sum_{r,n<0}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r}) - \sum_{r\geq 0}(-1)^r q^{\binom{r+1}{2}} \\
&\qquad + \sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r}) - \sum_{r\geq 1}(-1)^r q^{\binom{r+1}{2}}\Big) \\
&= 1 + \frac{-1}{(q)_\infty^2}\Big(\sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r}{2}}(1+q^{2r}) \\
&\qquad - \sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^{n+r} q^{\binom{n+1}{2}+2nr+\binom{r+1}{2}}(1+q^{2r}) - \sum_{r\geq 1}(-1)^r q^{\binom{r}{2}}(1-q^r)\Big),
\end{aligned}
$$

which gives (2.10) when $m = 0$.

Finally, equation (2.11) follows from (2.10) after noting that

$$W(t,m,n) = \sum_{v\geq 0} W(mv+t,n) + \sum_{v\geq 1} W(mv-t,n).$$

□

2.2. **Proof of Theorem 1.2.** We begin by recording a lemma of Garvan [14, Lemma (3.9)].

Lemma 2.3. *For $\zeta_5 = \exp(2\pi i/5)$, we have*

$$\frac{1}{(\zeta_5 q, \zeta_5^{-1} q)_\infty} = \frac{1}{(q^5, q^{20}; q^{25})_\infty} + \frac{(\zeta_5 + \zeta_5^{-1})q}{(q^{10}, q^{15}; q^{25})_\infty}.$$

Next we define the functions A_i, B_i, and C_i by

$$A_i := \sum_{n \geq \delta_{i0}} (-1)^n q^{(5n+i)(15n+3i-1)/2},$$
$$B_i := \sum_{n \geq \delta_{i0}} (-1)^n q^{(5n+i)(15n+3i+1)/2},$$
$$C_i := \sum_{n \geq \delta_{i0}} (-1)^n q^{(5n+i)(5n+i+1)/2}.$$

Now, observe that

$$\begin{aligned} W(\zeta_5, q) &= \sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} W(m,n) \zeta_5^m q^n \\ &= \sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} \sum_{i=0}^{4} W(5m+i, n) \zeta_5^{5m+i} q^n \\ &= \sum_{n \geq 0} \sum_{i=0}^{4} W(i, 5, n) \zeta_5^i q^n. \end{aligned}$$

Thus, setting $x = \zeta_5$ in (2.2) and applying Lemma 2.3, we obtain

$$\begin{aligned} &\sum_{i=0}^{4} \sum_{n \geq 0} W(i,5,n) \zeta_5^i q^n \\ &\quad = (1 - \zeta_5^2) + (1 - \zeta_5 + \zeta_5^2 - \zeta_5^3) \sum_{n \geq 1} (-1)^n \zeta_5^{3n-2} q^{n(3n-1)/2} (1 + \zeta_5 q^n) \\ &\quad + \left(\frac{1}{(q^5, q^{20}; q^{25})_\infty} + \frac{(\zeta_5 + \zeta_5^{-1})q}{(q^{10}, q^{15}; q^{25})_\infty} \right) \left(\zeta_5^2 + (1 + \zeta_5^2) \sum_{n \geq 1} (-1)^n \zeta_5^{2n} q^{n(n+1)/2} \right). \end{aligned} \tag{2.15}$$

The contribution to q^{5n} on the right-hand side of (2.15) is

$$(1 - \zeta_5^2) + (1 - \zeta_5 + \zeta_5^2 - \zeta_5^3)(\zeta_5^3 A_0 + \zeta_5^4 A_2 + \zeta_5^4 B_0 - \zeta_5^3 B_3) + \frac{\zeta_5^2 + (1 + \zeta_5^2)(C_0 + \zeta_5^3 C_4)}{(q^5, q^{20}; q^{25})_\infty}.$$

Thus, letting $W_i(x) := \sum_{n \geq 0} W(i, 5, 5n+x) q^{5n+x}$, we have

$$\begin{aligned} 0 = &\left(-W_0(0) + 1 + \frac{C_0 + C_4}{(q^5, q^{20}; q^{25})_\infty} + A_0 - A_2 - B_0 - B_3 \right) + \zeta_5 \left(-W_1(0) - A_0 + A_2 + B_0 + B_3 \right) \\ &+ \zeta_5^2 \left(-W_2(0) - 1 - A_2 - B_0 + \frac{1 + C_0}{(q^5, q^{20}; q^{25})_\infty} \right) + \zeta_5^3 \left(-W_3(0) + A_0 - B_3 + \frac{C_4}{(q^5, q^{20}; q^{25})_\infty} \right) \\ &+ \zeta_5^4 \left(-W_4(0) - A_0 + A_2 + B_0 + B_3 \right). \end{aligned}$$

Since the minimal polynomial of ζ_5 is $1+x+x^2+x^3+x^4$, the coefficients of ζ_5^i are all equal. Therefore, subtracting the coefficients of ζ_5^0 and ζ_5^1 from the coefficient of ζ_5^3 (and using the fact that $W(2,5,n)=W(3,5,n)$) we have

$$\begin{aligned} W_{02,5}(0) &= 1+\frac{C_0}{(q^5,q^{20};q^{25})_\infty} - B_0 - A_2 \\ &= 2+\frac{\sum_{n\geq 1}(-1)^n q^{\binom{5n+1}{2}}}{(q^5,q^{20};q^{25})_\infty} - \sum_{n\geq 0}(-1)^n q^{5n(15n+1)/2}(1+q^{25n+5}) \end{aligned}$$

and

$$\begin{aligned} W_{12,5}(0) &= \frac{-C_4}{(q^5,q^{20};q^{25})_\infty} - 2A_0 + A_2 + B_0 + 2B_3 \\ &= \frac{-\sum_{n\geq 0}(-1)^n q^{\binom{5n+5}{2}}}{(q^5,q^{20};q^{25})_\infty} - 1 + \left(\sum_{n\geq 0} - 2\sum_{n\leq -1}\right)(-1)^n q^{5n(15n+1)/2}(1+q^{25n+5}), \end{aligned}$$

as claimed.

Next, the contribution to q^{5n+1} on the right-hand side of (2.15) is

$$(1-\zeta_5+\zeta_5^2-\zeta_5^3)(-\zeta_5 A_1+\zeta_5 B_4)+\frac{(1+\zeta_5^2)(-\zeta_5^2 C_1-\zeta_5 C_3)}{(q^5,q^{20};q^{25})_\infty}+\frac{q(\zeta_5+\zeta_5^4)(\zeta_5^2+(1+\zeta_5^2)(C_0+\zeta_5^3 C_4))}{(q^{10},q^{15};q^{25})_\infty},$$

which gives

$$\begin{aligned} 0 = -W_0(1) &+ \zeta_5\left(-W_1(1) - A_1 + B_4 - \frac{C_3}{(q^5,q^{20};q^{25})_\infty} + \frac{q(1+2C_0+C_4)}{(q^{10},q^{15};q^{25})_\infty}\right) \\ &+ \zeta_5^2\left(-W_2(1) + A_1 - B_4 - \frac{C_1}{(q^5,q^{20};q^{25})_\infty} + \frac{qC_4}{(q^{10},q^{15};q^{25})_\infty}\right) \\ &+ \zeta_5^3\left(-W_3(1) - A_1 + B_4 - \frac{C_3}{(q^5,q^{20};q^{25})_\infty} + \frac{q(1+C_0)}{(q^{10},q^{15};q^{25})_\infty}\right) \\ &+ \zeta_5^4\left(-W_4(1) + A_1 - B_4 + \frac{C_1}{(q^5,q^{20};q^{25})_\infty} + \frac{q(C_0+2C_4)}{(q^{10},q^{15};q^{25})_\infty}\right). \end{aligned}$$

As before, the coefficients of ζ_5^i are all equal. By subtracting the coefficient of ζ_5^2 from the coefficient of ζ_5^0, we see that

$$W_{02,5}(1) = -(A_1-B_4) + \frac{C_1}{(q^5,q^{20};q^{25})} - \frac{qC_4}{(q^{10},q^{15};q^{25})}.$$

Note that

$$\begin{aligned}
A_1 - B_4 &= \sum_{n\geq 0}(-1)^n q^{(5n+1)(15n+2)/2} - \sum_{n\geq 0}(-1)^n q^{(5n+4)(15n+13)/2}\\
&= \sum_{n\geq 0}(-1)^n q^{(75n^2+25n+2)/2} + \sum_{n\leq -1}(-1)^n q^{(75n^2+25n+2)/2}\\
&= q\sum_{n\in\mathbb{Z}}(-1)^n q^{(75n^2+25n)/2}\\
&= q(q^{25};q^{25})_\infty.
\end{aligned}$$

The second line follows from replacing n by $-n-1$ in the second sum and the last line follows from the Jacobi triple product identity,

$$\sum_{n\in\mathbb{Z}} z^n q^{\binom{n+1}{2}} = (-1/z, -zq, q)_\infty. \tag{2.16}$$

This gives

$$W_{02,5}(1) = \frac{\sum_{n\geq 0}(-1)^n q^{\binom{5n+2}{2}}}{(q^5,q^{20};q^{25})_\infty} - \frac{q\sum_{n\geq 0}(-1)^n q^{\binom{5n+5}{2}}}{(q^{10},q^{15};q^{25})_\infty} - q(q^{25};q^{25})_\infty,$$

as claimed.

By subtracting the coefficient of ζ_5^3 from the coefficient of ζ_5, we see that

$$\begin{aligned}
W_{12,5}(1) &= \frac{q(C_0+C_4)}{(q^{10},q^{15};q^{25})_\infty}\\
&= \frac{q\sum_{n\geq 1}(-1)^n q^{\binom{5n+1}{2}} + q\sum_{n\geq 0}(-1)^n q^{\binom{5n+5}{2}}}{(q^{10},q^{15};q^{25})_\infty}\\
&= \frac{q\sum_{n\geq 1}(-1)^n q^{\binom{5n+1}{2}} - q\sum_{n\geq 1}(-1)^n q^{\binom{5n}{2}}}{(q^{10},q^{15};q^{25})_\infty}\\
&= \frac{-q\sum_{n\geq 1}(-1)^n q^{\binom{5n}{2}}(1-q^{5n})}{(q^{10},q^{15};q^{25})_\infty},
\end{aligned}$$

as claimed.

The contribution to q^{5n+2} on the right-hand side of (2.15) is

$$(1-\zeta_5+\zeta_5^2-\zeta_5^3)(-\zeta_5^2A_3+A_4-\zeta_5^2B_1+B_2) + \frac{q(\zeta_5+\zeta_5^4)(1+\zeta_5^2)(-\zeta_5^2C_1-\zeta_5C_3)}{(q^{10},q^{15};q^{25})_\infty},$$

which gives

$$
\begin{aligned}
0 =& \left(-W_0(2) + A_3 + A_4 + B_1 + B_2 - \frac{q(C_1+C_3)}{(q^{10},q^{15};q^{25})_\infty}\right)\\
&+ \zeta_5\left(-W_1(2) - A_4 - B_2 - \frac{qC_1}{(q^{10},q^{15};q^{25})_\infty}\right)\\
&+ \zeta_5^2\left(-W_2(2) - A_3 - A_4 - B_1 - B_2 - \frac{2qC_3}{(q^{10},q^{15};q^{25})_\infty}\right)\\
&+ \zeta_5^3\left(-W_3(2) + A_3 - A_4 + B_1 - B_2 - \frac{2qC_1}{(q^{10},q^{15};q^{25})_\infty}\right)\\
&+ \zeta_5^4\left(-W_4(2) + A_3 + B_1 - \frac{qC_3}{(q^{10},q^{15};q^{25})_\infty}\right).
\end{aligned}
$$

By subtracting the coefficient of ζ_5^3 from the coefficient of ζ_5^0, we see that

$$W_{02,5}(2) = 2(B_2 + A_4) + \frac{q(C_1 - C_3)}{(q^{10},q^{15};q^{25})_\infty}.$$

Note that

$$
\begin{aligned}
C_1 - C_3 &= \sum_{n\geq 0}(-1)^n q^{(5n+1)(5n+2)/2} + \sum_{n\geq 0}(-1)^n q^{(5n+3)(5n+4)/2}\\
&= \sum_{n\geq 0}(-1)^n q^{(5n+1)(5n+2)/2} + \sum_{n\leq -1}(-1)^n q^{(5n+1)(5n+2)/2}\\
&= q\sum_{n\in\mathbb{Z}}(-1)^n q^{(25n^2+15n)/2}\\
&= q(q^5,q^{20},q^{25};q^{25})_\infty,
\end{aligned}
$$

where the second line follows from replacing n by $-n-1$ and the last line follows from the Jacobi triple product identity. This gives

$$W_{02,5}(2) = \frac{q^2(q^5,q^{20},q^{25};q^{25})_\infty}{(q^{10},q^{15};q^{25})_\infty} + 2\sum_{n\geq 0}(-1)^n q^{(5n+2)(15n+7)/2}(1+q^{25n+15}),$$

as desired. By subtracting the coefficient of ζ_5^3 from the coefficient of ζ_5^1, we see that

$$W_{12,5}(2) = -B_1 - C_3 + \frac{qC_1}{(q^{10},q^{15};q^{25})_\infty},$$

which gives

$$W_{12,5}(2) = \frac{q\sum_{n\geq 0}(-1)^n q^{\binom{5n+2}{2}}}{(q^{10},q^{15};q^{25})_\infty} - \sum_{n\geq 0}(-1)^n q^{(5n+1)(15n+4)/2}(1+q^{25n+10}),$$

as claimed.

The contribution to q^{5n+3} on the right-hand side of (2.15) is

$$\frac{1}{(q^5,q^{20};q^{25})_\infty}(1+\zeta_5^2)\zeta_5^4 C_2.$$

Arguing as before, we find that

$$\begin{aligned}
W_{02,5}(3) &= 0,\\
W_{12,5}(3) &= \frac{C_2}{(q^5,q^{20};q^{25})_\infty} = \frac{\sum_{n\geq 0}(-1)^n q^{\binom{5n+3}{2}}}{(q^5,q^{20};q^{25})_\infty},
\end{aligned}$$

as desired.

Finally, the contribution to q^{5n+4} on the right-hand side of (2.15) is

$$\frac{(\zeta_5+\zeta_5^4)q}{(q^{10},q^{15};q^{25})_\infty}(1+\zeta_5^2)\zeta_5^4 C_2 = \frac{(2+\zeta^2+\zeta^3)qC_2}{(q^{10},q^{15};q^{25})_\infty},$$

which gives

$$\begin{aligned}
W_{02,5}(4) &= \frac{qU_2}{(q^{10},q^{15};q^{25})_\infty} = \frac{q\sum_{n\geq 0}(-1)^n q^{\binom{5n+3}{2}}}{(q^{10},q^{15};q^{25})_\infty},\\
W_{12,5}(4) &= -\frac{qU_2}{(q^{10},q^{15};q^{25})_\infty} = -\frac{q\sum_{n\geq 0}(-1)^n q^{\binom{5n+3}{2}}}{(q^{10},q^{15};q^{25})_\infty},
\end{aligned}$$

as claimed. □

3. Unimodal sequences for the partial theta identity (1.8)

3.1. **Generating functions.**

We begin by establishing three generating functions for $V(m,n)$. Define $V(x,q)$ by

$$V(x,q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} V(m,n)x^m q^n.$$

Proposition 3.1. *We have*

$$\begin{aligned}
V(x,q) &= \sum_{n\geq 0}\frac{(q^{n+1})_n q^n}{(xq,q/x)_n} &(3.1)\\
&= (1-x)\sum_{n\geq 0} x^n q^{n^2+n} + \frac{x}{(xq,q/x)_\infty}\sum_{n\geq 0} x^{3n}q^{3n^2+2n}(1-xq^{2n+1}) &(3.2)\\
&= \frac{(1-x)}{(q)_\infty^2}\left(\sum_{n,r\geq 0} - \sum_{n,r<0}\right)\frac{(-1)^r q^{3n^2+2n+(3n+1)r+\binom{r+1}{2}}(1-q^{2n+r+1})}{1-xq^r}. &(3.3)
\end{aligned}$$

Proof. For (3.1), we use the q-Chu-Vandermonde summation [15]

$$\sum_{k=0}^{n}\frac{(a,q^{-n})_k}{(c,q)_k}q^k = \frac{(c/a)_n}{(c)_n}a^n \qquad (3.4)$$

together with the fact that

$$(q^{1-n}/a)_k = \frac{(a)_n}{(a)_{n-k}}\left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk}. \qquad (3.5)$$

Specifically, setting $c = q/x$ and $a = q^{-n}/x$ in (3.4) and simplifying using (3.5), we obtain

$$\frac{(q^{n+1})_n}{(xq, q/x)_n} q^n = q^n \sum_{k=0}^{n} \frac{x^k q^{k^2}}{(xq)_k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{(q/x)_{n-k}},$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_{n-k}(q)_k}$$

is the usual q-binomial coefficient. It is straightforward to interpret the right-hand side in terms of unimodal sequences (1.1). First, the term q^n corresponds to a peak $\overline{n}$ of size n. Next, since $\begin{bmatrix} n \\ k \end{bmatrix}$ is the generating function for partitions inside a $k \times (n-k)$ rectangle, the term

$$\frac{x^k q^{k^2}}{(xq)_k} \begin{bmatrix} n \\ k \end{bmatrix}$$

is the generating function for partitions into parts at most n with Durfee square size k, where the exponent of x counts the number of parts. This corresponds to the partition $\sum a_i$ in (1.1). Fianlly, the term $\frac{1}{(q/x)_{n-k}}$ is the generating function for partitions into parts at most $n-k$, where the exponent of x^{-1} counts the number of parts. This corresponds to $\sum b_i$ in (1.1). Summing on n, we have the unimodal sequences generated by $V(x,q)$. This establishes (3.1).

Equation (3.2) is just (1.3). For (3.3) we use Bailey pairs. We recall that if (α_n, β_n) is a Bailey pair relative to 1, then [21, Eq. (1.10)]

$$\sum_{n \geq 0} q^n \beta_n = \frac{1}{(q)_\infty^2} \sum_{r,n \geq 0} q^{3n^2+2n+(3n+1)r}(1 - q^{2n+r+1})\alpha_r. \tag{3.6}$$

Substituting the Bailey pair (2.5) and (2.6) into (3.6) and using (2.7), we have

$$\begin{aligned} V(x,q) &= \frac{1}{(q)_\infty^2} \Bigg(\sum_{n \geq 0} q^{3n^2+2n}(1 - q^{2n+1}) \\ &\quad + \sum_{\substack{r \geq 1 \\ n \geq 0}} (-1)^r q^{3n^2+2n+(3n+1)r+\binom{r+1}{2}} (1 - q^{2n+r+1}) \left(\frac{1-x}{1-xq^r} \right) \qquad (3.7) \\ &\quad + \sum_{\substack{r \geq 1 \\ n \geq 0}} (-1)^r q^{3n^2+2n+(3n+1)r+\binom{r+1}{2}} (1 - q^{2n+r+1}) \left(\frac{1-1/x}{1-q^r/x} \right) \Bigg) \\ &= \frac{(1-x)}{(q)_\infty^2} \sum_{r,n \geq 0} \frac{(-1)^r q^{3n^2+2n+(3n+1)r+\binom{r+1}{2}}(1 - q^{2n+r+1})}{1 - xq^r} \\ &\quad + \frac{(1-1/x)}{(q)_\infty^2} \sum_{r,n<0} \frac{(-1)^r q^{3n^2+4n+(3n+1)r+\binom{r+1}{2}+1}(1 - q^{-2n-r-1})}{1 - q^{-r}/x} \\ &= \frac{(1-x)}{(q)_\infty^2} \left(\sum_{n,r \geq 0} - \sum_{n,r<0} \right) \frac{(-1)^r q^{3n^2+2n+(3n+1)r+\binom{r+1}{2}}(1 - q^{2n+r+1})}{1 - xq^r}, \end{aligned}$$

as desired. □

Next we establish generating functions for $V(m,n)$ and $V(t,m,n)$ for fixed m and t.

Proposition 3.2.

(1) *For $m \in \mathbb{Z}$ we have*

$$\sum_{n\geq 0} V(m,n)q^n = \chi(m=0) + \frac{-1}{(q)_\infty^2} \sum_{\substack{r\geq 1\\ n\geq 0}} (-1)^r q^{3n^2+2n+3rn+\binom{r+1}{2}+|m|r}(1-q^{2n+r+1})(1-q^r). \quad (3.8)$$

(2) *For $m \geq 1$ and $0 \leq t \leq m-1$ we have*

$$\sum_{n\geq 0} V(t,m,n)q^n = \chi(t=0) + \frac{-1}{(q)_\infty^2} \sum_{\substack{r\geq 1\\ n\geq 0}} (-1)^r q^{3n^2+2n+3rn+\binom{r+1}{2}}(1-q^{2n+r+1})(1-q^r)\frac{(q^{rt}+q^{r(m-t)})}{1-q^{rm}}. \quad (3.9)$$

Proof. For $m \geq 1$ equation (3.8) follows from (3.3) after expanding

$$\frac{1-x}{1-xq^r} = (1-x)\sum_{m\geq 0} x^m q^{mr}$$

and picking off the coefficient of x^m. The case $m < 0$ follows from the symmetry $V(m,n) = V(-m,n)$. The case $m = 0$ is trickier. For this we need the identity

$$\left(\sum_{r,n\geq 0} - \sum_{r,n<0}\right)(-1)^r q^{3n^2+2n+(3n+1)r+\binom{r+1}{2}}(1-q^{2n+r+1}) = (q)_\infty^2, \quad (3.10)$$

which follows from (3.6) and the unit Bailey pair in (2.13) and (2.14). Specifically, using this pair in (3.6) we obtain

$$\begin{aligned}(q)_\infty^2 &= \sum_{n\geq 0} q^{3n^2+2n}(1-q^{2n+1}) + \sum_{\substack{r\geq 1\\ n\geq 0}} (-1)^r q^{3n^2+2n+(3n+1)r+\binom{r}{2}}(1-q^{2n+r+1})(1+q^r)\\ &= \sum_{\substack{r\geq 1\\ n\geq 0}} (-1)^r q^{3n^2+2n+(3n+1)r+\binom{r}{2}}(1-q^{2n+r+1}) + \sum_{r,n\geq 0} (-1)^r q^{3n^2+2n+(3n+1)r+\binom{r+1}{2}}(1-q^{2n+r+1}).\end{aligned}$$

Replacing (r,n) by $(-r,-n-1)$ in the first sum gives (3.10).

Now picking off the coefficient of x^0 in (3.3) (c.f. equation (3.7)), we have

$$\begin{aligned}\sum_{n\geq 0} V(0,n)q^n &= \frac{1}{(q)_\infty^2}\Big(\sum_{r,n\geq 0}(-1)^r q^{3n^2+2n+(3n+1)r+\binom{r+1}{2}}(1-q^{2n+r+1}) \\ &\qquad + \sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^r q^{3n^2+2n+(3n+1)r+\binom{r+1}{2}}(1-q^{2n+r+1})\Big) \\ &= \frac{1}{(q)_\infty^2}\Big((q)_\infty^2 + \sum_{r,n<0}(-1)^r q^{3n^2+2n+(3n+1)r+\binom{r+1}{2}}(1-q^{2n+r+1}) \\ &\qquad + \sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^r q^{3n^2+2n+(3n+1)r+\binom{r+1}{2}}(1-q^{2n+r+1})\Big) \\ &= 1+\frac{1}{(q)_\infty^2}\Big(-\sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^r q^{3n^2+2n+3nr+\binom{r+1}{2}}(1-q^{2n+r+1}) \\ &\qquad + \sum_{\substack{r\geq 1\\ n\geq 0}}(-1)^r q^{3n^2+2n+(3n+1)r+\binom{r+1}{2}}(1-q^{2n+r+1})\Big),\end{aligned}$$

which gives (3.8) when $m=0$.

Finally, equation (3.9) follows from (3.8) after noting that

$$V(t,m,n) = \sum_{v\geq 0} V(mv+t,n) + \sum_{v\geq 1} V(mv-t,n).$$

□

3.2. **Proof of Theorem 1.4.** We begin by defining the following functions:

$$\begin{aligned}X_i &:= \sum_{n\geq 0} q^{(5n+i)(5n+i+1)}, \\ Y_i &:= \sum_{n\geq 0} q^{3(5n+i)^2+2(5n+i)},\end{aligned}$$

and

$$Z_i := \sum_{n\geq 0} q^{3(5n+i)^2+4(5n+i)+1}.$$

Next we observe that

$$\begin{aligned}V(\zeta_5,q) &= \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} V(m,n)\zeta_5^m q^n\\ &= \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}}\sum_{i=0}^{4} V(5m+i,n)\zeta_5^{5m+i}q^n\\ &= \sum_{n\geq 0}\sum_{i=0}^{4} V(i,5,n)\zeta_5^i q^n.\end{aligned}$$

Thus, setting $x=\zeta_5$ in (1.3) and employing Lemma 2.3, we obtain

$$\begin{aligned}&\sum_{n\geq 0}\sum_{i=0}^{4} V(i,5,n)\zeta_5^i q^n\\ &= (1-\zeta_5)\sum_{n\geq 0}\zeta_5^n q^{n^2+n} + \frac{\zeta_5}{(\zeta_5 q,\zeta_5^{-1}q)_\infty}\sum_{n\geq 0}\zeta_5^{3n}q^{3n^2+2n}(1-\zeta_5 q^{2n+1})\\ &= \sum_{n\geq 0}(\zeta_5^n-\zeta_5^{n+1})q^{n^2+n} + \left(\frac{\zeta_5}{(q^5,q^{20};q^{25})_\infty} + \frac{(1+\zeta_5^2)q}{(q^{10},q^{15};q^{25})_\infty}\right)\sum_{n\geq 0}\zeta_5^{3n}q^{3n^2+2n}(1-\zeta_5 q^{2n+1}).\end{aligned} \tag{3.11}$$

The rest of the proof is much like that of Theorem 1.2, so we give fewer details. First, the contribution to q^{5n} on the right-hand side of (3.11) is

$$(1-\zeta_5)X_0 + (\zeta_5^4-1)X_4 + \frac{\zeta_5}{(q^5,q^{20};q^{25})_\infty}(Y_0+\zeta_5^3Y_1-Z_3-\zeta_5^3Z_4).$$

Thus, by letting $V_i(x) := \sum_{n\geq 0} V(i,5,5n+x)q^{5n+x}$, we have

$$\begin{aligned}&V_0(0)+V_1(0)\zeta_5+V_2(0)\zeta_5^2+V_3(0)\zeta_5^3+V_4(0)\zeta_5^4\\ &= (X_0-X_4)+\zeta_5\left(\frac{Y_0-Z_3}{(q^5,q^{20};q^{25})_\infty}-X_0\right)+\zeta_5^4\left(\frac{Y_1-Z_4}{(q^5,q^{20};q^{25})_\infty}+X_4\right).\end{aligned}$$

As before, the minimal polynomial for ζ_5 is $1+x+x^2+x^3+x^4$, and so by comparing coefficients, we conclude that

$$V_{02,5}(0) = X_0-X_4 = \sum_{n\geq 0} q^{25n^2+5n}(1-q^{40n+20}),$$

$$V_{12,5}(0) = \frac{Y_0-Z_3}{(q^5,q^{20};q^{25})_\infty} - X_0 = \frac{\sum_{n\geq 0} q^{75n^2+10n}(1-q^{100n+40})}{(q^5,q^{20};q^{25})_\infty} - \sum_{n\geq 0} q^{25n^2+5n},$$

as desired.

The contribution to q^{5n+1} on the right-hand side of (3.11) is

$$(\zeta_5^2-\zeta_5^3)X_2 + \frac{\zeta_5(\zeta_5Y_2+\zeta_5^2Y_4-\zeta_5Z_0-\zeta_5^2Z_2)}{(q^5,q^{20};q^{25})_\infty} - \frac{q(1+\zeta_5^2)(Y_0+\zeta_5^3Y_1-\zeta_5Z_0-\zeta_5^2Z_2)}{(q^{10},q^{15};q^{25})_\infty},$$

while the left-hand side contributes

$$V_0(1) + V_1(1)\zeta_5 + V_2(1)\zeta_5^2 + V_3(1)\zeta_5^3 + V_4(1)\zeta_5^4.$$

Therefore, we conclude that

$$\begin{aligned}
V_{02,5}(1) &= \frac{Z_0 - Y_2}{(q^5, q^{20}; q^{25})_\infty} + \frac{q(Y_1 - Z_4)}{(q^{10}, q^{15}; q^{25})_\infty} - X_2 \\
&= \frac{\sum_{n\geq 0} q^{75n^2+40n+6}(1 - q^{100n+60})}{(q^{10}, q^{15}; q^{25})_\infty} + \frac{\sum_{n\geq 0} q^{75n^2+20n+1}(1 - q^{50n+15})}{(q^5, q^{20}; q^{25})_\infty} - q^6 \frac{(q^{100}; q^{100})_\infty}{(q^{50}; q^{100})_\infty}, \\
V_{12}(1) &= \frac{Z_0 - Y_2}{(q^5, q^{20}; q^{25})_\infty} - \frac{q(Y_0 - Z_3)}{(q^{10}, q^{15}; q^{25})_\infty} - X_2 \\
&= \frac{\sum_{n\geq 0} q^{75n^2+20n+1}(1 - q^{50n+15})}{(q^5, q^{20}; q^{25})_\infty} - \frac{\sum_{n\geq 0} q^{75n^2+10n+1}(1 - q^{100n+40})}{(q^{10}, q^{15}; q^{25})_\infty} - q^6 \frac{(q^{100}; q^{100})_\infty}{(q^{50}; q^{100})_\infty},
\end{aligned}$$

where we use Gauss' identity

$$\sum_{n\geq 0} q^{\binom{n+1}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}.$$

The contribution to q^{5n+2} on the right-hand side of (3.11) is

$$(1 - \zeta_5)(\zeta_5 X_1 + \zeta_5^3 X_3) - \frac{q(1 + \zeta_5^2)}{(q^{10}, q^{15}; q^{25})_\infty}(\zeta_5 Y_2 + \zeta_5^2 Y_4 - \zeta_5 Z_0 - \zeta_5^2 Z_2),$$

which gives

$$\begin{aligned}
V_{02,5}(2) &= \frac{q(Z_2 - Y_4)}{(q^{10}, q^{15}; q^{25})_\infty} + X_1 \\
&= \frac{\sum_{n\geq 0} q^{75n^2+80n+22}(1 - q^{50n+35})}{(q^{10}, q^{15}; q^{25})_\infty} + \sum_{n\geq 0} q^{25n^2+15n+2}, \\
V_{12,5}(2) &= X_1 - X_3 = \sum_{n\geq 0} q^{25n^2+15n+2}(1 - q^{20n+10}),
\end{aligned}$$

as desired.

The contribution to q^{5n+3} on the right-hand side of (3.11) is

$$\frac{\zeta_5}{(q^5, q^{20}; q^{25})_\infty}(\zeta_5^4 Y_3 - \zeta_5^4 Z_1),$$

which gives

$$\begin{aligned}
V_{02,5}(3) &= -\frac{Y_1 - Z_3}{(q^5, q^{20}; q^{25})_\infty} = -\frac{\sum_{n\geq 0} q^{75n^2+50n+8}(1 - q^{50n+25})}{(q^5, q^{20}; q^{25})_\infty}, \\
V_{12,5}(3) &= 0,
\end{aligned}$$

as claimed.

Finally, the contribution to q^{5n+4} on the right-hand side of (3.11) is

$$\frac{q(1+\zeta_5^2)}{(q^{10},q^{15};q^{25})_\infty}(\zeta_5^4 Y_3 - \zeta_5^4 Z_1),$$

which gives

$$V_{02,5}(4) = 0,$$

$$V_{12}(4) = -\frac{q(Y_1 - Z_3)}{(q^{10},q^{15};q^{25})_\infty} = -\frac{\sum_{n\geq 0} q^{75n^2+50n+9}(1-q^{50n+25})}{(q^{10},q^{15};q^{25})_\infty}.$$

This completes the proof of Theorem 1.4. □

4. Unimodal sequences for the partial theta identity (1.9)

4.1. **Generating functions.** We begin by establishing three generating functions for $\mathcal{V}(m,n)$. Define $\mathcal{V}(x,q)$ by

$$\mathcal{V}(x,q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} \mathcal{V}(m,n)x^m q^n.$$

Proposition 4.1. *We have*

$$\mathcal{V}(x,q) = \sum_{n\geq 0} \frac{(-q)_{2n} q^{2n+1}}{(xq,q/x;q^2)_{n+1}} \tag{4.1}$$

$$= \frac{-x}{1+x}\sum_{n\geq 0}(-x)^n q^{n^2+n} + \frac{x(-q)_\infty}{(1+x)(xq,q/x;q^2)_\infty}\sum_{n\geq 0}(-x)^n q^{\binom{n+1}{2}} \tag{4.2}$$

$$= \frac{1}{(q)_\infty (q^2;q^2)_\infty}\left(\sum_{n,r\geq 0} - \sum_{n,r<0}\right)\frac{(-1)^{n+r} q^{\binom{n+1}{2}+n+2nr+r^2+3r+1}}{(1+q^{2r+1})(1-xq^{2r+1})}. \tag{4.3}$$

Proof. Equation (4.1) is immediate from the definition of $\mathcal{V}(m,n)$ and (4.2) is (1.8). For (4.3) we use the fact that if (α_n,β_n) is a Bailey pair relative to (a^2q^2,q^2), then [21, Eq. (1.7)]

$$\sum_{n\geq 0}(aq)_{2n} q^{2n}\beta_n = \frac{(aq)_\infty}{(a^2q^4,q^2;q^2)_\infty}\sum_{r,n\geq 0}\frac{q^{\binom{n+1}{2}+2nr+2r+n}a^n}{1-aq^{2r+1}}\alpha_r, \tag{4.4}$$

together with the fact that (α_n,β_n) is a Bailey pair relative to (q^2,q^2), where

$$\alpha_n = \frac{(-1)^n q^{n^2+n}(1-q^{4n+2})}{(1-q^2)(1-xq^{2n+1})(1-q^{2n+1}/x)}$$

and

$$\beta_n = \frac{1}{(xq,q/x;q^2)_{n+1}}.$$

(See [29, Eq. (4.1)] with $(a,c,d) = (q,x,1/x)$.) Substituting this Bailey pair into (4.4) with $a=-1$ and using the fact that for $r\geq 0$

$$\frac{(1-q^{4r+2})}{(1-xq^{2r+1})(1-q^{2r+1}/x)} = \frac{1}{1-xq^{2r+1}} + \frac{x^{-1}q^{2r+1}}{1-q^{2r+1}/x},$$

we have

$$\mathcal{V}(x,q) = \frac{1}{(q)_\infty (q^2;q^2)_\infty} \left(\sum_{r,n\geq 0} \frac{(-1)^{n+r} q^{\binom{n+1}{2}+n+2rn+r^2+3r+1}}{1+q^{2r+1}} \left(\frac{1}{1-xq^{2r+1}} \right) \right.$$
$$\left. + \sum_{r,n\geq 0} \frac{(-1)^{n+r} q^{\binom{n+1}{2}+n+2rn+r^2+3r+1}}{1+q^{2r+1}} \left(\frac{x^{-1}q^{2r+1}}{1-q^{2r+1}/x} \right) \right)$$
$$= \frac{1}{(q)_\infty (q^2;q^2)_\infty} \left(\sum_{n,r\geq 0} - \sum_{n,r<0} \right) \frac{(-1)^{n+r} q^{\binom{n+1}{2}+n+2rn+r^2+3r+1}}{(1+q^{2r+1})(1-xq^{2r+1})},$$

after replacing (r,n) by $(-r-1,-n-1)$ in the second sum. □

Next we establish generating functions for $\mathcal{V}(m,n)$ and $\mathcal{V}(t,m,n)$ for fixed m and t.

Proposition 4.2.

(1) *For $m \in \mathbb{Z}$ we have*

$$\sum_{n\geq 0} \mathcal{V}(m,n) q^n = \frac{1}{(q)_\infty (q^2;q^2)_\infty} \sum_{r,n\geq 0} \frac{(-1)^{n+r} q^{\binom{n+1}{2}+n+2rn+r^2+3r+1+|m|(2r+1)}}{1+q^{2r+1}}. \tag{4.5}$$

(2) *For $m \geq 1$ and $0 \leq t \leq m-1$ we have*

$$\sum_{n\geq 0} \mathcal{V}(t,m,n) q^n = \frac{1}{(q)_\infty (q^2;q^2)_\infty} \sum_{r,n\geq 0} \frac{(-1)^{n+r} q^{\binom{n+1}{2}+n+2rn+r^2+3r+1} (q^{t(2r+1)} + q^{(m-t)(2r+1)})}{(1+q^{2r+1})(1-q^{m(2r+1)})}. \tag{4.6}$$

Proof. For $m \geq 0$ equation (4.5) follows from (4.3) after expanding

$$\frac{1}{(1-xq^{2r+1})} = \sum_{m\geq 0} x^m q^{m(2r+1)}$$

and picking off the coefficient of x^m. The case $m < 0$ follows from the symmetry $\mathcal{V}(m,n) = \mathcal{V}(-m,n)$.

As for equation (4.6), it follows from (4.5) after noting that

$$\mathcal{V}(t,m,n) = \sum_{v\geq 0} \mathcal{V}(mv+t,n) + \sum_{v\geq 1} \mathcal{V}(mv-t,n).$$

□

4.2. **Proof of Theorem 1.5.** We adopt the method of the previous two sections. We begin by defining some auxiliary functions:

$$\mathcal{X}_i := \sum_{n\geq 0} (-1)^n q^{(3n+i)(3n+i+1)}$$

and

$$\mathcal{Y}_i := \sum_{n\geq 0}(-1)^n q^{(3n+i)(3n+i+1)/2}.$$

Next, setting $x = \zeta_3 = \exp(2\pi i/3)$ in (4.2), we see that

$$\begin{aligned}(1+\zeta_3)\mathcal{V}(\zeta_3, q) &= (1+\zeta_3)\sum_{n\geq 0}\sum_{i=0}^{2}\mathcal{V}(i,3,n)\zeta_3^i q^n \\ &= -\zeta_3\sum_{n\geq 0}(-\zeta_3)^n q^{n^2+n} + \frac{\zeta_3(-q)_\infty}{(\zeta_3 q, q/\zeta_3; q^2)_\infty}\sum_{n\geq 0}(-\zeta_3)^n q^{\binom{n+1}{2}} \\ &= -\zeta_3\sum_{n\geq 0}(-\zeta_3)^n q^{n^2+n} + \frac{\zeta_3}{(q^3;q^6)_\infty}\sum_{n\geq 0}(-\zeta_3)^n q^{\binom{n+1}{2}}. \qquad (4.7)\end{aligned}$$

We let $\mathcal{V}_i(x) := \sum_{n\geq 0}\mathcal{V}(i,3,3n+x)q^{3n+x}$. Extracting the contribution to q^{3n} on both sides of (4.7) gives

$$(\mathcal{V}_0(0)+\mathcal{V}_2(0)) + (\mathcal{V}_1(0)+\mathcal{V}_0(0))\zeta_3 + (\mathcal{V}_1(0)+\mathcal{V}_2(0))\zeta_3^2 = (\mathcal{X}_0 + \zeta_3^2\mathcal{X}_2) + \frac{\zeta_3}{(q^3;q^6)_\infty}(\mathcal{Y}_0 + \zeta_3^2\mathcal{Y}_2).$$

Arguing as usual, we find that

$$\mathcal{V}_{01,3}(0) = \frac{\mathcal{Y}_0}{(q^3;q^6)_\infty} - \mathcal{X}_0 = \frac{\sum_{n\geq 0}(-1)^n q^{(3n+1)(3n)/2}}{(q^3;q^6)_\infty} - \sum_{n\geq 0}(-1)^n q^{9n^2+3n},$$

as claimed.

Next, considering the contribution to q^{3n+1} on both sides of (4.7) gives

$$(\mathcal{V}_0(1)+\mathcal{V}_2(1)) + (\mathcal{V}_1(1)+\mathcal{V}_0(1))\zeta_3 + (\mathcal{V}_1(1)+\mathcal{V}_2(1))\zeta_3^2 = -\zeta_3^2\mathcal{Y}_1,$$

which implies

$$\mathcal{V}_{01,3}(1) = \frac{\mathcal{Y}_1}{(q^3;q^6)_\infty} = \frac{\sum_{n\geq 0}(-1)^n q^{(3n+2)(3n+1)/2}}{(q^3;q^6)_\infty}.$$

Finally, the contributions to q^{3n+2} in (4.7) yield

$$(\mathcal{V}_0(2)+\mathcal{V}_2(2)) + (\mathcal{V}_1(2)+\mathcal{V}_0(2))\zeta_3 + (\mathcal{V}_1(2)+\mathcal{V}_2(2))\zeta_3^2 = \zeta_3^2\mathcal{X}_1.$$

From this, we deduce that

$$\mathcal{V}_{01,3}(2) = -\mathcal{X}_1 = -\sum_{n\geq 0}(-1)^n q^{9n^2+9n+2},$$

as desired. ∎

5. Discussion

Using partial theta identities like (1.2), (1.3), (1.8), and (1.9) to obtain expressions for rank differences for the corresponding unimodal sequences is theoretically possible for any given m. We discuss (1.2) with m odd, but similar comments apply for other m and other identities. Let $x = \zeta_m$ and multiply the numerator and denominator of the final term in (1.2) by

$$(\zeta_m^2 q, \zeta_m^{-2}, q)_\infty (\zeta_m^3 q, \zeta_m^{-3} q, q)_\infty \cdots (\zeta_m^{(m-1)/2}, \zeta_m^{-(m-1)/2}, q)_\infty.$$

The result is

$$\frac{\sum_{n\in\mathbb{Z}} (-1)^n \zeta_m^{2n} q^{\binom{n+1}{2}} \cdots \sum_{n\in\mathbb{Z}} (-1)^n \zeta_m^{(m-1)n/2} q^{\binom{n+1}{2}} \sum_{n\geq 0} (-1)^n \zeta_m^{2n+1} q^{\binom{n+1}{2}}}{(1-\zeta_m^{-2})\cdots(1-\zeta_m^{-(m-1)/2})(q^m;q^m)_\infty (q)_\infty^{(m-5)/2}}. \tag{5.1}$$

In other words, up to roots of unity we have $1/(q^m;q^m)_\infty$ multiplied by a product of theta functions and an "extra" modular form $1/(q)_\infty^{(m-5)/2}$. When $m = 5$ we have $(q)_\infty^{(m-5)/2} = 1$, which makes determining the 5-dissection of (5.1) straightforward. This corresponds to Theorem 1.1. When $m = 3$ then $1/(q)_\infty^{(m-5)/2}$ is a theta function and the 3-dissection is also straightforward, although the rank difference formulas end up being somewhat less elegant. (See Theorem 4.2 of [18] for a precise statement.)

For larger m one not only needs to consider a quadratic form in $(m-1)/2$ variables, but also the m-dissection of $1/(q)_\infty^{(m-5)/2}$. While the latter will always be in terms of modular forms, it is not necessarily easy to calculate and there is no guarantee that one obtains simple linear combinations of eta-quotients. (For the question of which modular spaces are generated by eta-quotients, see a recent paper of Rouse and Webb [26].) When $m = 7$, the 7-dissection of $1/(q)_\infty$ can be found in [20], but even in this relatively simple case the formulas for the rank differences become unwieldy.

The cases considered in this paper (Theorems 1.2, 1.4, and 1.5) are precisely those for which the "extra" modular form in the expression analogous to (5.1) is 1. Versions of Theorems 1.2 and 1.4 for $m = 3$ are certainly possible, as in these cases the "extra" modular form is a theta function. The formulas would resemble those in Theorem 4.2 of [18]. Something similar would happen with $x = \zeta_3$ in two other partial theta identities of Ramanujan [4, Entries (6.3.9) and (6.3.11)],

$$\sum_{n\geq 0} \frac{(-q;q^2)_n q^{2n}}{(xq^2, q^2/x; q^2)_n} = (1-x)\sum_{n\geq 0} x^n (-q)^{\binom{n+1}{2}} + \frac{x(-q;q^2)_\infty}{(xq^2, q^2/x;q^2)_\infty} \sum_{n\geq 0} x^{3n} q^{3n^2+2n}(1-xq^{2n+1}) \tag{5.2}$$

and

$$\sum_{n\geq 0} \frac{(q;q^2)_n q^n}{(xq, q/x)_n} = (1-x)\sum_{n\geq 0} x^n q^{\binom{n+1}{2}} + \frac{(1-x)}{(-q, xq, q/x)_\infty} \sum_{n\geq 0} (-1)^n x^{2n+1} q^{n(n+1)}. \tag{5.3}$$

A reasonable version of Theorem 1.5 for $m = 5$ is perhaps not out of the question. Here one would need the 5-dissection of $1/(q^2;q^2)_\infty$, which follows from the identity [6, Chap. 7]

$$\frac{1}{(q)_\infty} = \frac{(q^{25};q^{25})_\infty}{(q^5;q^5)_\infty^6} \left(x^4 + qx^3 + 2q^2x^2 + 3q^3x + 5q^4 - 3\frac{q^5}{x} + \frac{2q^6}{x^2} - \frac{q^7}{x^3} + \frac{q^8}{x^4} \right).$$

Here $x = T(q^5)$, where

$$T(q) = q^{1/5}/R(q) = \frac{(q^2, q^3; q^5)_\infty}{(q, q^4; q^5)_\infty}$$

and $R(q)$ is the Rogers-Ramanujan continued fraction. We leave the task of computing any of the above rank differences to the interested reader.

6. Suggestions for further study

We close with three ideas for further study. First, as discussed in the introduction, the symmetries $U(m, n) = U(-m, n)$, $W(m, n) = W(-m, n)$, and $\mathcal{V}(m, n) = \mathcal{V}(-m, n)$ follow readily from the definitions. However, this does not seem to be the case for the symmetry $V(m, n) = V(-m, n)$. Is there a simple and natural conjugation which implies this symmetry?

Second, numerical evidence suggests that for n fixed and large enough, the sequences $\{U(m, n)\}_{m\geq 0}$, $\{W(m, n)\}_{m\geq 0}$, $\{V(m, n)\}_{m\geq 0}$, and $\{\mathcal{V}(m, n)\}_{m\geq 0}$ are decreasing. It would be interesting to see whether methods from q-series, combinatorics, or asymptotic analysis could be applied to prove these inequalities. We refer to recent work of Chan and Mao [11] for related results in the case of ordinary partitions.

Finally, there appears to be a large number of rank difference inequalities, a typical example being $U(1, 5, 5n+4) \geq U(2, 5, 5n+4)$. There are asymptotic methods for partial theta functions (see [8, 9, 16, 17, 32, 33, 34] for example), and so given the formulas in Theorems 1.1, 1.2, 1.4, and 1.5, it is reasonable to expect that many inequalities may be established for large enough n. Is it possible to prove such inequalities for all n? Also, is it possible to obtain systematic asymptotic inequalities as in the case of partitions [7]?

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