PARTITIONS WITH ROUNDED OCCURRENCES AND ATTACHED PARTS

JEREMY LOVEJOY

To George Andrews, with admiration, on the occasion of his 70th birthday

Abstract. We introduce the number of (k, i)-rounded occurrences of a part in a partition and use q-difference equations to interpret a certain q-series $S_{k,i}(a; x; q)$ as the generating function for partitions with bounded (k, i)-rounded occurrences and attached parts. When $a = 0$ these partitions are the same as those studied by Bressoud in his extension of the Rogers-Ramanujan-Gordon identities to even moduli. When $a = 1/q$ we obtain a new family of partition identities.

1. Introduction and statement of results

In 1968, greatly generalizing work of Rogers [25] and Selberg [26], Andrews defined a family of basic hypergeometric series $J_{\lambda,k,i}(a_1, a_2, \dots, a_\lambda; x; q)$ and established q-difference equations involving them [5]. This work became one of the foundations of modern partition theory. Andrews had already seen how to use some of these q-difference equations to prove families of partition identities [1, 2, 3, 4], including Gordon's combinatorial generalization of the Rogers-Ramanujan identities, and over the next decade many further partition identities [6, 8, 9, 14, 16] would be deduced from the $J_{\lambda,k,i}(a_1, a_2, \dots, a_\lambda; x; q)$ and their q-difference equations.

With the focus on analytic identities, motivated in large part by the burgeoning applications in statistical mechanics [10] and the advent of the powerful Bailey pair method [11], the 80's and 90's saw the study of q-difference equations fall out of favor. Over the last decade or so, however, a string of papers have shown that there is still much to be discovered in Andrews' $J_{\lambda,k,i}(a_1, a_2, \dots, a_\lambda; x; q)$ [12, 13, 18, 20, 21, 22, 23, 24]. The present work is yet another contribution to this list.

We study the series $S_{k,i}(a; x; q)$, defined for $k \geq 2$ and $1 \leq i \leq k$ using the usual q-series notation [19] by

$$S_{k,i}(a;x;q) := \frac{1}{(xq)_\infty} \sum_{n\geq 0} \frac{a^n x^{(k-1)n} q^{(k-1)n^2+(k-i+1)n} (x^2q^2, 1/a; q^2)_n}{(q^2, ax^2q^2; q^2)_n} \times \left(1 + \frac{ax^i q^{(2n+1)i-2n}(1-q^{2n}/a)}{(1-ax^2q^{2n+2})}\right). \tag{1.1}$$

In terms of Andrews' series, we have

$$S_{k,i}(a;x;q) := \frac{(-xq)_\infty}{(ax^2q^2;q^2)_\infty} J_{1,\frac{k-1}{2},\frac{i}{2}}(1/a; x^2; q^2). \tag{1.2}$$

Date: July 4, 2009.

We will describe the coefficient of $a^t x^m q^n$ of $S_{k,i}(a;x;q)$ in terms of partition pairs, using the number of (k,i)-*rounded occurrences* of a part j in a partition λ.

Definition 1.1. *Denote by $f_j(\lambda)$ the number of occurrences of j in λ. The number of (k,i)-rounded occurrences of a part j in a partition λ, denoted $f_j^{(k,i)}(\lambda)$, is defined using the usual charactersitic function by*

$$f_{2j}^{(k,i)}(\lambda) \ := \ f_{2j}(\lambda) + \chi(f_{2j} \not\equiv (k-i) \pmod 2), \tag{1.3}$$

$$f_{2j+1}^{(k,i)}(\lambda) \ := \ f_{2j+1}(\lambda) + \chi(f_{2j+1} \not\equiv (i-1) \pmod 2). \tag{1.4}$$

Notice that with this definition we require nothing about the parity of the number of occurrences of a part, only that the number of occurrences be *regarded* as having a certain parity by rounding up, if necessary. To illustrate the definition, consider the partition $\lambda = (6,6,6,4,4,3,3,3,3,1)$. Then we have, for example, $f_1^{(4,3)} = 2$, $f_2^{(4,2)} = 0$, $f_3^{(5,2)} = 5$, $f_4^{(4,4)} = 2$, $f_5^{(4,4)} = 1$, and $f_6^{(5,3)} = 4$.

We now define the partitions pairs of interest.

Definition 1.2. *Let $s_{k,i}(n)$ denote the number of partition pairs (λ,μ) of n such that:*

(i) $f_1^{(k,i)}(\lambda) \leq i-1$,

(ii) if $f_1^{(k,i)}(\lambda) = i-1$ then 1 may occur an even number of times in μ,

(iii) if $i = 1$ then 1 may occur unrestricted in μ,

(iv) for each $j \geq 1$ we have $f_j^{(k,i)}(\lambda) + f_{j+1}^{(k,i)}(\lambda) \leq k-1$,

(v) for each $j \geq 1$, if $f_j^{(k,i)}(\lambda) + f_{j+1}^{(k,i)}(\lambda) = k-1$, then $j+1$ may occur an even number of times in μ,

(vi) for each $j \geq 1$, if $f_j^{(k,i)}(\lambda) = k-1$, then $j+1$ may occur unrestricted in μ.

We are now ready to state the main theorem.

Theorem 1.3. *Let $s_{k,i}(t,m,n)$ denote the number of partition pairs counted by $s_{k,i}(n)$ such that $m = \sum_j (f_j(\lambda) + f_j(\mu))$ and $t = \sum_j \lceil \frac{f_j(\mu)}{2} \rceil$. Then*

$$\sum_{t,m,n \geq 0} s_{k,i}(t,m,n) a^t x^m q^n = S_{k,i}(a;x;q). \tag{1.5}$$

Theorem 1.3 shows that special cases of the functions $S_{k,i}(a;x;q)$ are generating functions for some well-known partitions. For example, a few moments' consideration (or, to bypass Theorem 1.3, an appeal to (1.1)) reveals that ordinary partitions are generated by $S_{k,i}(1;x;q)$ (for any k and i). It is also not hard to see that partitions into distinct parts are generated by $S_{3,2}(0;1;q)$. More generally, the partitions generated by $S_{k,i}(0;x;q)$ may be identitifed with those studied by Bressoud in his extension to even moduli of Gordon's generalization of the Rogers-Ramanujan identities (i.e., the partitions counted by $b_{k,i}(m,n)$ in [14]). Setting $a = 0$ and $x = 1$ in (1.1) and appealing to the triple product identity [19, p.239, Eq. (II.28)],

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = (-q/z, -zq, q^2; q^2)_\infty, \tag{1.6}$$

we then recover Bressoud's result [14, Theorem, $j = 0$] in the following form:

Corollary 1.4 (Bressoud, [14]). *For $k \geq 2$ and $1 \leq i < k$, let $B_{k,i}(n)$ denote the number of partitions λ of n such that:*

(i) $f_1^{(k,i)}(\lambda) \leq i - 1$,

(ii) for each $j \geq 1$ we have $f_j^{(k,i)}(\lambda) + f_{j+1}^{(k,i)}(\lambda) \leq k - 1$.

Let $A_{k,i}(n)$ denote the number of partitions of n into parts not congruent to 0 or $\pm i$ modulo $2k$. Then $A_{k,i}(n) = B_{k,i}(n)$.

Another interesting consequence of Theorem 1.3 arises when we set $a = 1/q$. It is convenient to state this result in terms of overpartitions, which are partitions in which the first occurrence of a part may be overlined.

Corollary 1.5. *For $k \geq 2$ and $1 < i < k$, let $\mathcal{B}_{k,i}(n)$ denote the number of overpartition pairs (λ, μ) of n such that:*

(i) λ is an ordinary partition counted by $B_{k,i}(n)$ (see Corollary 1.4),

(ii) if $f_1^{(k,i)}(\lambda) = i - 1$, then 1 may occur (non-overlined and unrestricted) in μ,

(iii) for $j \geq 1$, if $f_j^{(k,i)}(\lambda) + f_{j+1}^{(k,i)}(\lambda) = k - 1$, then $2j + 1$ may occur (non-overlined and unrestricted) in μ,

(iv) for $j \geq 1$ if $f_j^{(k,i)}(\lambda) = k - 1$ then $\overline{j}$ may appear in μ.

Let $\mathcal{A}_{k,i}(n)$ denote the number of overpartitions of n where non-overlined parts are not divisible by $2k - 2$ and overlined parts are $\pm(i - 1) \pmod{2k - 2}$. Then $\mathcal{A}_{k,i}(n) = \mathcal{B}_{k,i}(n)$.

Despite the requirement that $1 < i < k$ above, there is still an identity when $i = 1$ or k. Indeed, the proof of Corollary 1.5 presented in Section 2 applies equally well when $i = 1$ or k. The definition of $\mathcal{B}_{k,i}(n)$ is still valid (with a suitable modification for $i = 1$ arising from condition *(iii)* in Definition 1.2), and the generating functions for $\mathcal{A}_{k,1}(n)$ and $\mathcal{A}_{k,k}(n)$ are

$$\mathcal{A}_{k,1}(n) = \frac{2(-q^{2k-2}; q^{2k-2})_\infty^2 (q^{2k-2}; q^{2k-2})_\infty}{(q)_\infty}$$

and

$$\mathcal{A}_{k,k}(n) = \frac{(-q^{k-1}; q^{2k-2})_\infty^2 (q^{2k-2}; q^{2k-2})_\infty}{(q)_\infty}.$$

We let the reader interpret these products as he pleases.

2. Proofs of Theorem 1.3 and Corollary 1.5

Using (1.2) and [5, Theorem 1] one may compute that

$$S_{k,1}(a;x;q) = \frac{(1+axq)}{(1-ax^2q^2)} S_{k,k}(a;xq;q), \tag{2.1}$$

$$S_{k,2}(a;x;q) = \frac{(1+xq)}{(1-ax^2q^2)} S_{k,k-1}(a;xq;q), \tag{2.2}$$

and for $3 \leq i \leq k$,

$$\begin{aligned} S_{k,i}(a;x;q) - S_{k,i-2}(a;x;q) &= \frac{(xq)^{i-2}(1+xq)}{(1-ax^2q^2)} S_{k,k-i+1}(a;xq;q) \\ &\quad - \frac{a(xq)^{i-2}(1+xq)}{(1-ax^2q^2)} S_{k,k-i+3}(a;xq;q). \end{aligned} \tag{2.3}$$

The final q-difference equation is not terribly useful combinatorially. However, there is another q-difference equation which may be easily deduced from (2.1), (2.2), and (2.3) using induction (equation (2.3) providing the induction step). This method of eliminating the minus sign is inspired by [13].

Lemma 2.1. *If $i \geq 2$ is even then*

$$\begin{aligned} S_{k,i}(a;x;q) = &\frac{(xq)^{i-2}(1+xq)}{(1-ax^2q^2)} S_{k,k-i+1}(a;xq;q) \\ &+ \sum_{v=1}^{(i-2)/2} (xq)^{2v-2}(1+xq) S_{k,k-2v+1}(a;xq;q), \end{aligned} \tag{2.4}$$

and if $i \geq 3$ is odd then

$$\begin{aligned} S_{k,i}(a;x;q) = S_{k,k}(a;xq;q) &+ \frac{(xq)^{i-2}(1+xq)}{(1-ax^2q^2)} S_{k,k-i+1}(a;xq;q) \\ &+ \sum_{v=1}^{(i-3)/2} (xq)^{2v-1}(1+xq) S_{k,k-2v}(a;xq;q). \end{aligned} \tag{2.5}$$

Proof of Theorem 1.3. Notice that together with the initial condition $S_{k,i}(a;0;q) = 1$, the q-difference equations (2.1), (2.4), and (2.5) uniquely define the functions $S_{k,i}(a;x;q)$. To prove Theorem 1.3 then, we define

$$\widehat{S}_{k,i}(a;x;q) := \sum_{t,m,n \geq 0} s_{k,i}(t,m,n) a^t x^m q^n$$

and show that the $\widehat{S}_{k,i}(a;x;q)$ satisfy the same defining conditions. That $\widehat{S}_{k,i}(a;0;q) = 1$ follows from the fact that the only partition without any parts whatsoever is the empty partition of 0.

We now turn to (2.1). Let (λ, μ) be a partition pair counted by $\widehat{S}_{k,1}(a;x;q)$. By definition, we have $f_1(\lambda) = 0$, $f_2^{(k,1)}(\lambda) \leq k-1$, and $f_1(\mu)$ is unrestricted. Removing the 1's and subtracting one from each part ≥ 2, we see that

$$\widehat{S}_{k,1}(a;x;q) = \frac{(1+axq)}{(1-ax^2q^2)} \widehat{S}_{k,k}(a;xq;q).$$

(Notice that for $(k,i) = (k,1)$ and (k,k), the residue classes modulo 2 of $(k-i)$ and $(i-1)$ are interchanged, so that subtracting one from each part is consistent with the definition of the number of (k,i)-rounded occurrences in Definition 1.1 and the conditions on the $s_{k,i}(t,m,n)$ in Theorem 1.3. This will be the case throughout the proof, though we shall not mention it again.)

Next we treat (2.4). Suppose that (λ,μ) is a partition pair counted by $\widehat{S}_{k,i}(a;x;q)$, where $i \geq 2$ is even. We have $0 \leq f_1(\lambda) \leq i-1$. For each v with $1 \leq v \leq i/2$, if $f_1(\lambda) = 2v-1$ or $2v-2$ then $f_1^{(k,i)}(\lambda) = 2v-1$. In the case $v = i/2$, we have $f_2^{(k,i)}(\lambda) \leq k-i$ and $f_1(\mu)$ is even. Removing the 1's and subtracting one from each remaining part we see that these pairs are generated by $((xq)^{i-2} + (xq)^{i-1})/(1-ax^2q^2)\widehat{S}_{k,k-i+1}(a;xq;q)$. Now for $1 \leq v \leq (i-2)/2$, we have $f_1(\mu) = 0$ and $f_2^{(k,i)}(\lambda) \leq k-2v$. Again removing the 1's and subtracting one from each part, these pairs are generated by $(xq)^{2v-2}(1+xq)\widehat{S}_{k,k-2v+1}(a;xq;q)$. This gives (2.4).

To prove (2.5), suppose that (λ,μ) is a partition pair counted by $\widehat{S}_{k,i}(a;x;q)$, where $i \geq 3$ is odd. For each v with $1 \leq v \leq (i-1)/2$, if $f_1(\lambda) = 2v$ or $2v-1$, then $f_1^{(k,i)}(\lambda) = 2v$. The argument now proceeds as above, except that we have left out the case $f_1(\lambda) = 0$ beacuse i is odd. This accounts for the extra term $\widehat{S}_{k,k}(a;xq;q)$. This concludes the proof of Theorem 1.3. □

We now turn to Corollary 1.5. First, setting $a = 1/q$ and $x = 1$ on the right-hand side of (1.1), we have

$$\begin{aligned} S_{k,i}(1/q;1;q) &= \frac{1}{(q)_\infty}\sum_{n\geq 0} q^{(k-1)n^2+(k-i)n}(1+q^{(2n+1)(i-1)}) \\ &= \frac{1}{(q)_\infty}\sum_{n\in\mathbb{Z}} q^{(k-1)n^2+(k-i)n} \\ &= \frac{(-q^{i-1},-q^{2k-i+1},q^{2k-2};q^{2k-2})_\infty}{(q)_\infty} \\ &= \sum_{n\geq 0}\mathcal{A}_{k,i}(n)q^n, \end{aligned}$$

where the penultimate line follows from the Jacobi triple product identity (1.6). On the other hand, if we let $a = 1/q$ and $x = 1$ in Theorem 1.3 and consider the effect on partition pairs counted by $s_{k,i}(t,m,n)$, then parts $j+1$ occurring an even number of times in μ may be regarded as repeatable parts of the form $2j+1$, while the eventual leftover occurrence of $j+1$ becomes $\overline{j}$. This gives the pairs counted by $\mathcal{B}_{k,i}(n)$ and completes the proof of Corollary 1.5. □

3. Conclusion

In addition to Andrews' $J_{\lambda,k,i}(a_1,a_2,\ldots,a_\lambda;x;q)$, there are several other families of q-series whose q-difference equations are worth exploring. We indicate three of these here. First, Andrews has developed q-difference equations for some series $K_{\lambda,k,i}(a_1,a_2,\ldots,a_\lambda;x;q)$ [7, Section 3] which may be regarded as bilateral series analogues of the $J_{\lambda,k,i}(a_1,a_2,\ldots,a_\lambda;x;q)$. Second, Bressoud's $F_{\lambda,k,i}(c_1,c_2,a_1,a_2,\ldots,a_\lambda;x;q)$ [15] reduce to Andrews' $J_{\lambda,k,i}(a_1,a_2,\ldots,a_\lambda;x;q)$ when $c_1,c_2 \to \infty$ and $x = xq$. When $c_1 \to \infty$ and $c_2 = -q$, q-difference equations and their combinatorial implications have been worked out for $\lambda = 1$ in [17] and for $\lambda = 2$ in [24]. Surely many more instances of Bressoud's series satisfy meaningful q-difference equations. Finally, there are nice q-difference equations for a family of series containing both $F_{1,k,i}(-q,\infty,a_1;xq;q)$ and $F_{1,k,i}(\infty,\infty,a_1;xq;q)$ presented in [17, Section 6].

Acknowledgement

The author is indebted to Ae Ja Yee for her reading of an earlier version of this paper.

References

[1] G.E. Andrews, An analytic proof of the Rogers-Ramanujan-Gordon identities, *Amer. J. Math.* **88** (1966), 844-846.
[2] G.E. Andrews, Some new partition theorems, *J. Combin. Theory* **2** (1967), 431-436.
[3] G.E. Andrews, Partition theorems related to the Rogers-Ramanujan identitites, *J. Combin. Theory* **2** (1967), 422-430.
[4] G.E. Andrews, A generalization of the Göllnitz-Gordon partition identities, *Proc. Amer. Math. Soc.* **8** (1967), 945–952.
[5] G.E. Andrews, On q-difference equations for certain well-poised basic hypergeometric series, *Quart. J. Math.* **19** (1968), 433-447.
[6] G.E. Andrews, A generalization of the classical partition theorems, *Trans. Amer. Math. Soc.* **145** (1968), 205-221.
[7] G.E. Andrews, Applications of basic hyoergeometric series, *SIAM Review* **16** (1974), 441-484.
[8] G.E. Andrews, On the general Rogers-Ramanujan theorem, *Mem. Amer. Math. Soc.* **152** (1974), 86 pp.
[9] G.E. Andrews, On the Alder polynomials and a new generalization of the Rogers-Ramanujan identities, *Trans. Amer. Math. Soc.* **204** (1975), 40-64.
[10] G.E. Andrews, The hard-hexagon model and the Rogers-Ramanujan type identities, *Proc. Nat. Acad. Sci.* **78** (1981), 5290-5292.
[11] G.E. Andrews, Multiple series Rogers-Ramanujan type identities, *Pacific J. Math.* **114** (1984), 267-283.
[12] G.E. Andrews, Parity in partition identities, *Ramanujan J.*, to appear.
[13] G.E. Andrews and J.P.O. Santos, Rogers-Ramanujan type identities for partitions with attached odd parts, *Ramanujan J.* **1** (1997), 91-99.
[14] D.M. Bressoud, A generalization of the Rogers-Ramanujan identities for all moduli, *J. Combin. Theory* **27** (1979), 64-68.
[15] D.M. Bressoud, Analytic and combinatorial generalizations of the Rogers-Ramanujan identities, *Mem. Amer. Math. Soc.* **227** (1980), 54pp.
[16] W.G. Connor, Partition theorems related to some identities of Rogers and Watson, *Trans. Amer. Math. Soc.* **214** (1975), 95-111.
[17] S. Corteel, J. Lovejoy, and O. Mallet, An extension to overpartitions of the Rogers-Ramanujan identities for even moduli, *J. Number Theory* **128** (2008), 1602-1621.
[18] S. Corteel and O. Mallet, Overpartitions, lattice paths, and the Rogers-Ramanujan identities, *J. Combin. Theory Ser. A* **114** (2007), 14071437.
[19] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge Univ. Press, Cambridge, 1990.
[20] J. Lovejoy, Gordon's theorem for overpartitions, *J. Combin. Theory Ser. A* **103** (2003), 393-401.
[21] J. Lovejoy, Overpartition theorems of the Rogers-Ramanujan type, *J. London Math. Soc.* **69** (2004), 562-574.
[22] J. Lovejoy, Overpartition pairs, *Ann. Inst. Fourier (Grenoble)* **56** (2006), 781-794.
[23] J. Lovejoy, Partitions and overpartitions with attached parts, *Arch. Math. (Basel)* **88** (2007), 316-322.
[24] J. Lovejoy and O. Mallet, Overpartition pairs and two classes of basic hypergeometric series, *Adv. Math.* **217** (2008), 386-418.
[25] L.J. Rogers, Proof of certain identities in combinatory analysis, *Proc. Cambridge Phil. Soc.* **19** (1919), 211-214.
[26] A. Selberg, Über einige arithmetische Identitäten, *Avhl. Norske Vid.* **8** (1936), 23 pp.

CNRS, LIAFA, Université Denis Diderot - Paris 7, Case 7014, 75205 Paris Cedex 13, FRANCE
E-mail address: lovejoy@liafa.jussieu.fr