RAMANUJAN-TYPE PARTIAL THETA IDENTITIES AND CONJUGATE BAILEY PAIRS, II. MULTISUMS

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Abstract. In the first paper of this series we described how to find conjugate Bailey pairs from residual identities of Ramanujan-type partial theta identities. Here we carry this out for four multisum residual identities of Warnaar and two more due to the authors. Applying known Bailey pairs gives expressions in the algebra of modular forms and indefinite theta functions.

1. Introduction and statement of results

A *Bailey pair* relative to a is a pair of sequences $(\alpha_n, \beta_n)_{n\geq 0}$ satisfying

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}}. \tag{1.1}$$

Here we have used the usual q-hypergeometric notation,

$$(a)_n = (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j). \tag{1.2}$$

A *conjugate Bailey pair* relative to a is a pair of sequences $(\delta_n, \gamma_n)_{n\geq 0}$ satisfying

$$\gamma_n = \sum_{k=n}^{\infty} \frac{\delta_k}{(q)_{k-n}(aq)_{k+n}}. \tag{1.3}$$

The *Bailey transform* says that under suitable convergence conditions Bailey pairs and conjugate Bailey pairs combine to give the identity

$$\sum_{n\geq 0} \alpha_n \gamma_n = \sum_{n\geq 0} \beta_n \delta_n. \tag{1.4}$$

The classical conjugate Bailey pair is

$$\delta_n = \frac{(b)_n (c)_n (aq/bc)_{N-n}}{(aq/b)_N (aq/c)_N (q)_{N-n}} \left(\frac{aq}{bc}\right)^n \tag{1.5}$$

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and

$$\gamma_n = \frac{(b)_n(c)_n}{(aq/b)_n(aq/c)_n(aq)_{N+n}(q)_{N-n}} \left(\frac{aq}{bc}\right)^n. \tag{1.6}$$

Its proof was first described by Bailey [9], though its full power was only realized much later by Andrews [5, 6]. Inserted in (1.4) the classical conjugate Bailey pair shows that a given Bailey pair (α_n, β_n) gives rise to new non-trivial Bailey pairs

$$\alpha'_n = \frac{(b)_n(c)_n}{(aq/b)_n(aq/c)_n} \left(\frac{aq}{bc}\right)^n \alpha_n \tag{1.7}$$

and

$$\beta'_n = \sum_{j=0}^{n} \frac{(b)_j(c)_j(aq/bc)_{n-j}}{(aq/b)_n(aq/c)_n(q)_{n-j}} \left(\frac{aq}{bc}\right)^j \beta_j. \tag{1.8}$$

Iterating this gives the *Bailey chain*, which lies behind many of the applications of Bailey pairs in number theory, combinatorics, and physics.

While the classical conjugate Bailey pair has received the most attention, there are other conjugate Bailey pairs which have proven interesting and useful [10, 21, 23, 24, 25]. Most recently, the second author discussed how Ramanujan-type partial theta identities lead naturally to conjugate Bailey pairs via their *residual identities* [21]. For example, Ramanujan gave the partial theta identity [8, Entry (6.6.1)]

$$\sum_{n\geq 0} \frac{(q^{n+1})_n q^n}{(-a)_{n+1}(-q/a)_n} = \sum_{n\geq 0} (-a)^n q^{n^2+n} - \frac{a}{(-a)_\infty(-q/a)_\infty} \sum_{n\geq 0} (-1)^n a^{3n} q^{3n^2+2n}(1+aq^{2n+1}), \tag{1.9}$$

which can be used to deduce the residual identity [4]

$$\sum_{n\geq 0} \frac{(a^2q^{n+1})_n q^n}{(aq)_n(q)_n} = \frac{1}{(aq)_\infty(q)_\infty} \sum_{n\geq 0} a^{3n} q^{3n^2+2n}(1-q^{2n+1}), \tag{1.10}$$

which can in turn be used to show that if (α_n, β_n) is a Bailey pair relative to a^2, then

$$\sum_{n\geq 0} \frac{(a^2q)_{2n} q^n}{(aq)_n} \beta_n = \frac{1}{(aq)_\infty(q)_\infty} \sum_{n,r\geq 0} a^{3n} q^{3n^2+3rn+2n+r}(1-aq^{2n+r+1})\alpha_r. \tag{1.11}$$

For more details on precisely how this works, along with many examples, see [21].

The double sums like the one on the right-hand side of (1.11) lead naturally to instances of the indefinite theta function

$$f_{a,b,c}(x,y,q) = \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) (-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}. \tag{1.12}$$

These are intimately related to mock theta functions [14, 29], and so it is not suprising that the conjugate Bailey pairs arising from residual identites have been useful in a number of studies [17, 18, 19, 22].

Recently a generalization of (the case $a = 1$ of) (1.11) was proved by Hikami and the second author using a multisum residual identity of Warnaar [28]. Namely, if (α_n, β_n) is a

Bailey pair relative to 1, then we have [16, Lemma 2.6]

$$\sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0}(q^{n_k+1})_{n_k}q^{n_k}\beta_{n_k}q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}\begin{bmatrix}n_k\\ n_{k-1}\end{bmatrix}\begin{bmatrix}n_{k-1}\\ n_{k-2}\end{bmatrix}\cdots\begin{bmatrix}n_2\\ n_1\end{bmatrix}$$
$$=\sum_{i=1}^{2k}\frac{(-1)^{i-1}q^{\binom{i}{2}}(q^i,q^{2k+1-i},q^{2k+1};q^{2k+1})_\infty}{(q)_\infty^3}\sum_{r,n\geq 0}q^{kn((2k+1)n+2i)+(2k+1)rn+ri}\alpha_r. \tag{1.13}$$

Here we use the q-binomial coefficient (or Gaussian polynomial), defined by

$$\begin{bmatrix}n\\ k\end{bmatrix}=\frac{(q)_n}{(q)_k(q)_{n-k}}. \tag{1.14}$$

This was key to obtaining Hecke-type expansions for certain unified Witten-Reshetikhin-Turaev invariants [16].

Motivated by this, we revisit the methods and results of [21] and [28] in the context of q-hypergeometric multisums. We use multisum residual identities stated by Warnaar [28] to prove the conjugate Bailey pairs in the following theorem. We extend the notation of (1.2) to negative integers via

$$(a;q)_n:=\frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{1.15}$$

Theorem 1.1. *Let $k\geq 2$, $N_j=n_j+n_{j+1}+\cdots+n_{k-1}$, and suppose that (α_n,β_n) is a Bailey pair relative to a^2.*

(1) *For $\kappa=2k+1$,*

$$\sum_{n\geq 0}(a^2q)_{2n}q^n\beta_n\sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(aq)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}}$$
$$=\sum_{i=1}^{\kappa-1}\frac{(-1)^{i+1}a^{i-1}q^{\binom{i}{2}}(q^i,q^{\kappa-i},q^\kappa;q^\kappa)_\infty}{(q,q,aq)_\infty}\sum_{r,n\geq 0}a^{\kappa n}q^{kn(\kappa n+2i)+\kappa rn+ri}\alpha_r. \tag{1.16}$$

(2) *For $\kappa=2k$,*

$$\sum_{n\geq 0}(a^2q)_{2n}q^n\beta_n\sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(aq)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-2}}(q^2;q^2)_{n_{k-1}}}$$
$$=\sum_{i=1}^{\kappa-1}\frac{(-1)^{i+1}a^{i-1}q^{\binom{i}{2}}(q^i,q^{\kappa-i},q^\kappa;q^\kappa)_\infty}{(q,q,aq)_\infty}\sum_{r,n\geq 0}(-1)^na^{\kappa n}q^{(\kappa-1)(kn+i)n+\kappa rn+ri}\alpha_r. \tag{1.17}$$

(3) *For* $\kappa = 2k - 1/2$,

$$\sum_{n\geq 0}(a^2q)_{2n}q^n\beta_n \sum_{n_1,n_2,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(aq)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}(-q^{1/2};q^{1/2})_{2n_{k-1}}}$$
$$= \sum_{i=1}^{2k-1} \frac{(-1)^{i+1}a^{i-1}q^{\binom{i}{2}}(q^i,q^{\kappa-i},q^{\kappa};q^{\kappa})_\infty}{(q,q,aq)_\infty}$$
$$\times \sum_{n,r\geq 0}(-1)^n a^{2\kappa n}q^{2(\kappa-1)(\kappa n+i)n+2\kappa rn+ri}\left(1+a^{2\kappa-2i}q^{2(\kappa-1)(\kappa-i)(2n+1)+(2\kappa-2i)r}\right)\alpha_r. \tag{1.18}$$

(4) *For* $\sigma \in \{0,1\}$ *and* $\kappa = 3k-\sigma-1$,

$$\sum_{n\geq 0}(a^2q)_{2n}q^n\beta_n \sum_{n_1,n_2,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}+\sigma N_{k-1}(N_{k-1}-1)}}{(aq)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-2}}(q)_{2n_{k-1}}}$$
$$= \sum_{i=1}^{\kappa-1} \frac{(-1)^{i+1}a^{i-1}q^{\binom{i}{2}}(q^i,q^{2\kappa-i},q^{2\kappa};q^{2\kappa})_\infty(q^{2\kappa-2i},q^{2\kappa+2i};q^{4\kappa})_\infty}{(q,q,aq)_\infty}$$
$$\times \sum_{r\geq 0}q^{ri}\left[1-\sum_{n\geq 1}a^{2\kappa n-2i}q^{(2\kappa-3)(\kappa n-i)n+(2\kappa n-2i)r}\left(1-a^{2i}q^{2(2\kappa-3)in+2ir}\right)\right]\alpha_r. \tag{1.19}$$

Note that while equations (1.16) - (1.19) require $k \geq 2$, we can extend (1.16) to $k=1$ using (1.11). Also note that when $a=1$, equation (1.16) becomes (1.13) after making the change of indices

$$(n_1,n_2,\dots,n_{k-1}) = (n_1-n_2,n_2-n_3,\dots,n_{k-2}-n_{k-1},n_{k-1}) \tag{1.20}$$

followed by

$$(n,n_1,n_2,\dots,n_{k-1}) = (n_k,n_{k-1},\dots,n_1). \tag{1.21}$$

Throughout the paper, when considering special cases of Theorem 1.1, we typically apply (1.20) and (1.21) without explicitly saying so.

As mentioned above, the conjugate Bailey pairs in Theorem 1.1 follow from multisum residual identities of Warnaar, which he deduced from applications of Bailey pairs and the classical Bailey chain to his general partial theta identity in [28]. As Warnaar notes, the number of possible applications of this type is "sheer endless," so we limit ourselves more or less to the multisum residual identities he explicitly stated. We add just one more pair of results, following from multisum residual identities arising from partial *indefinite* theta identities of the authors [20]. Recalling the indefinite theta series in (1.12), we define

$$H^1_{k,\ell}(i) := f_{1,4k+3,1}(q^{2+k+\ell+i},q^{1+k-\ell+i},q) + q^{2+2k+i}f_{1,4k+3,1}(q^{4+3k+\ell+i},q^{3+3k-\ell+i},q),$$
$$H^2_{k,\ell}(i) := f_{1,4k+1,1}(q^{k+1+\ell+i},q^{k+1-\ell+i},q) + q^{1+2k+i}f_{1,4k+1,1}(q^{3k+2+\ell+i},q^{3k+2-\ell+i},q).$$

Using this notation, we have the following conjugate Bailey pairs.

Theorem 1.2. *Supppse that* ℓ *and* k *are integers with* $0 \leq \ell < k$. *If* (α_n,β_n) *is a Bailey pair relative to* a^2, *we have*

(1)

$$\sum_{n\geq 0}(a^2q)_{2n}q^n\beta_n \sum_{n_{2k-1}\geq\cdots\geq n_1\geq 0} \frac{(-1)^{n_k} q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})+\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i}}{(aq)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}}$$
$$= \frac{1}{(q,q,aq)_\infty}\sum_{i=1}^{k}(-1)^i a^{i-1}q^{\binom{i+1}{2}}H^1_{k,\ell}(i)$$
$$\times \sum_{r,m\geq 0} a^{(2k+2)r}q^{(2k^2+3k+1)r^2+(2k+1)ir+(2k+2)rm+mi}(1-a^{2k+2-2i}q^{((2k+1)(2r+1)+2m)(k+1-i)})\alpha_m. \tag{1.22}$$

(2)

$$\sum_{n\geq 0}(a^2q)_{2n}q^n\beta_n \sum_{n_{2k-1}\geq\cdots\geq n_1\geq 0} \frac{q^{\sum_{i=1}^{k-1}n_{k+i}^2+n_{k+i}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i+\binom{n_1+1}{2}}(-1)^{n_k}(-q)_{n_k}}{(aq)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q^2;q^2)_{n_1}(-q)_{n_{k+1}}}$$
$$= \frac{1}{(q,q,aq)_\infty}\sum_{i=1}^{k}(-1)^i a^{i-1}q^{\binom{i+1}{2}}H^2_{k,\ell}(i)$$
$$\times \sum_{r,m\geq 0}(-1)^r a^{(2k+1)r}q^{(2k^2+k)r^2+2kir+(2k+1)rm+mi}(1+a^{2k+1-2i}q^{(2k+1-2i)((2r+1)k+m)})\alpha_m. \tag{1.23}$$

In the next section we prove Theorems 1.1 and 1.2 and then the remainder of the paper is devoted to applications.

2. Proofs

In this section we prove Theorems 1.1 and 1.2. The arguments are quite similar in all cases, so we only give details for (1.16) and (1.22).

Proof. Warnaar [28, Cor 6.1] proved that for $k\geq 2$ and $\kappa = 2k+1$ we have the following (residual) identity:

$$\sum_{n\geq 0}\frac{(a^2q^{n+1})_nq^n}{(q)_n}\sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(aq)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}}$$
$$=\sum_{i=1}^{\kappa-1}\frac{(-1)^{i+1}a^{i-1}q^{\binom{i}{2}}(q^i,q^{\kappa-i},q^\kappa;q^\kappa)_\infty}{(q,q,aq)_\infty}\sum_{n\geq 0}a^{\kappa n}q^{kn(\kappa n+2i)}. \tag{2.1}$$

Using this, we argue as follows:

$$
\begin{aligned}
&\sum_{n\geq 0}(a^2q)_{2n}q^n\beta_n \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(aq)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} \\
&= \sum_{n\geq 0}(a^2q)_{2n}q^n \sum_{r=0}^{n} \frac{\alpha_n}{(a^2q)_{n+r}(q)_{n-r}} \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(aq)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} \\
&= \sum_{r\geq 0}\alpha_r \sum_{n\geq r} \frac{q^n(a^2q^{n+r+1})_{n-r}}{(q)_{n-r}} \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(aq)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} \\
&= \sum_{r\geq 0}\frac{q^r\alpha_r}{(aq)_r} \sum_{n\geq 0} \frac{q^n(a^2q^{n+2r+1})_{n}}{(q)_{n}} \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(aq^{r+1})_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} \\
&= \sum_{i=1}^{\kappa-1} \frac{(-1)^{i+1}a^{i-1}q^{\binom{i}{2}}(q^i,q^{\kappa-i},q^{\kappa};q^{\kappa})_\infty}{(q,q,aq)_\infty} \sum_{r,n\geq 0} a^{\kappa n}q^{kn(\kappa n+2i)+\kappa rn+ri}\alpha_r,
\end{aligned}
$$

where we employ (2.1) with $a = aq^r$ for the last identity. The remaining parts of Theorem 1.1 follow from similar arguments, applying Corollaries 6.2 – 6.4 of [28].

Turning to Theorem 1.2, we recall that for k a positive integer and $0 \leq \ell < k$, we have the residual identity [20, Theorem 6.1]

$$
\begin{aligned}
&\sum_{n\geq 0}\frac{(a^2q^{n+1})_nq^n}{(q)_n} \sum_{n_{2k-1}\geq\cdots\geq n_1\geq 0} \frac{q^{\sum_{i=1}^{k-1}n_{k+i}^2+n_{k+i}+\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i}(-1)^{n_k}}{(aq)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}} \\
&= \frac{1}{(q,q,aq)_\infty}\sum_{i=1}^{k}(-1)^ia^{i-1}q^{\binom{i+1}{2}}H_{k,\ell}^1(i) \qquad (2.2)\\
&\quad \times \sum_{r\geq 0} a^{(2k+2)r}q^{(2k^2+3k+1)r^2+(2k+1)ir}(1-a^{2k+2-2i}q^{(2k+1)(2r+1)(k+1-i)}).
\end{aligned}
$$

Using this, we argue as follows:

$$
\begin{aligned}
&\sum_{n\geq 0}(a^2q)_{2n}q^n\beta_n \sum_{n_{2k-1}\geq\cdots\geq n_1\geq 0} \frac{(-1)^{n_k}q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})+\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i}}{(aq)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}} \\
&\sum_{n\geq 0}(a^2q)_{2n}q^n \sum_{m=0}^{n}\frac{\alpha_n}{(a^2q)_{n+m}(q)_{n-m}} \sum_{n_{2k-1}\geq\cdots\geq n_1\geq 0} \frac{(-1)^{n_k}q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})+\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i}}{(aq)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}} \\
&\sum_{m\geq 0}\frac{q^m\alpha_m}{(aq)_m} \sum_{n\geq 0}\frac{q^n(a^2q^{n+2m+1})_n}{(q)_n} \sum_{n_{2k-1}\geq\cdots\geq n_1\geq 0} \frac{(-1)^{n_k}q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})+\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i}}{(aq^{m+1})_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}} \\
&= \frac{1}{(q,q,aq)_\infty}\sum_{i=1}^{k}(-1)^ia^{i-1}q^{\binom{i+1}{2}}H_{k,\ell}^1(i) \\
&\quad \times \sum_{r,m\geq 0} a^{(2k+2)r}q^{(2k^2+3k+1)r^2+(2k+1)ir+(2k+2)rm+mi}(1-a^{2k+2-2i}q^{((2k+1)(2r+1)+2m)(k+1-i)})\alpha_m,
\end{aligned}
$$

where we employ (2.2) with $a = aq^m$ for the last identity. The second part of Theorem 1.2 follows from similar arguments, applying Theorem 6.3 of [20]. $\square$

3. Applications I - Unimodal sequences

In this section we examine what happens when we insert known Bailey pairs into the first part of Theorem 1.1.

3.1. **The unit Bailey pair.** To get started we recall one of the simplest Bailey pairs, the so-called "unit" Bailey pair relative to 1,

$$\beta_n = \delta_{n0}$$

and

$$\alpha_n = \begin{cases} 1, & n = 0, \\ (-1)^n (q^{n(n-1)/2} + q^{n(n+1)/2}), & \text{otherwise.} \end{cases}$$

In this case the left-hand side of (1.16) collapses to 1 and we obtain an expression for $(q)_\infty^3$.

Corollary 3.1. *For $k \geq 1$ and $\kappa = 2k+1$,*

$$(q)_\infty^3 = \sum_{i=1}^{\kappa-1} (-1)^{i+1} q^{\binom{i}{2}} (q^i, q^{\kappa-i}, q^\kappa; q^\kappa)_\infty f_{2k\kappa,\kappa,1}(-q^{k\kappa+2ki}, q^i, q).$$

For example, when $k = 1$, we obtain

$$(q)_\infty^2 = f_{6,3,1}(-q^5, q, q) - q f_{6,3,1}(-q^7, q^2, q).$$

When $k = 2$ we derive that

$$\begin{aligned} (q)_\infty^3 = {} & (q, q^4, q^5; q^5)_\infty \left(f_{20,5,1}(-q^{14}, q, q) - q^6 f_{20,5,1}(-q^{26}, q^4, q) \right) \\ & - (q^2, q^3, q^5; q^5)_\infty \left(q f_{20,5,1}(-q^{18}, q^2, q) - q^3 f_{20,5,1}(-q^{22}, q^3, q) \right). \end{aligned}$$

Proof. By plugging the unit Bailey pair in (1.16), the right hand side of (1.16) is equal to

$$\begin{aligned} & \sum_{i=1}^{\kappa-1} \frac{(-1)^{i+1} q^{\binom{i}{2}} (q^i, q^{\kappa-i}, q^\kappa; q^\kappa)_\infty}{(q)_\infty^3} \\ & \quad \times \left(\sum_{r,n \geq 0} q^{kn(\kappa n+2i)+\kappa rn+ri} (-1)^r q^{r(r-1)/2} + \sum_{r<0,n\geq 0} q^{kn(\kappa n+2i)-\kappa rn-ri} (-1)^r q^{r(r-1)/2} \right). \end{aligned} \tag{3.1}$$

Note that if we replace n by $-n-1$ in the second sum, the sum equals

$$\sum_{r<0,n<0} q^{kn(\kappa n+2(\kappa-i))+\kappa rn+(\kappa-i)r+k\kappa-2ki} (-1)^r q^{r(r-1)/2}.$$

Moreover, we find that

$$\binom{\kappa-i}{2} = \binom{i}{2} + k\kappa - 2ki.$$

Therefore, by pairing i-th term and $\kappa - i$-th term, we deduce that (3.1) is equal to

$$
\begin{aligned}
&\sum_{i=1}^{k} \frac{(-1)^{i+1}(q^i, q^{\kappa-i}, q^\kappa; q^\kappa)_\infty}{(q)_\infty^3} \\
&\times \left(q^{\binom{i}{2}} \left(\sum_{n,r\geq 0} - \sum_{n,r<0} \right) (-1)^r q^{kn(\kappa n+2i)+\kappa rn+ri} q^{r(r-1)/2} \right. \\
&\quad \left. - q^{\binom{\kappa-i}{2}} \left(\sum_{n,r\geq 0} - \sum_{n,r<0} \right) (-1)^r q^{kn(\kappa n+2(\kappa-i))+\kappa rn+(\kappa-i)r} q^{r(r-1)/2} \right) \\
&= \sum_{i=1}^{k} \frac{(-1)^{i+1}(q^i, q^{\kappa-i}, q^\kappa; q^\kappa)_\infty}{(q)_\infty^3} \left(q^{\binom{i}{2}} f_{2k\kappa,\kappa,1}(-q^{k\kappa+2ki}, q^i, q) - q^{\binom{\kappa-i}{2}} f_{2k\kappa,\kappa,1}(-q^{k\kappa+2k(\kappa-i)}, q^{(\kappa-i)}, q) \right) \\
&= \sum_{i=1}^{\kappa-1} \frac{(-1)^{i+1} q^{\binom{i}{2}} (q^i, q^{\kappa-i}, q^\kappa; q^\kappa)_\infty}{(q)_\infty^3} f_{2k\kappa,\kappa,1}(-q^{k\kappa+2ki}, q^i, q).
\end{aligned}
$$

□

3.2. **Some special unimodal sequences.** Next we take the Bailey pair relative to 1 [26],

$$\beta_n = \frac{1}{(q)_n} \tag{3.2}$$

and

$$\alpha_n = \begin{cases} 1, & n = 0, \\ (-1)^n (q^{n(3n-1)/2} + q^{n(3n+1)/2}), & \text{otherwise.} \end{cases} \tag{3.3}$$

With this Bailey pair, the α-side of (1.16) is equal to

$$
\begin{aligned}
&\sum_{i=1}^{\kappa-1} \frac{(-1)^{i+1} q^{\binom{i}{2}} (q^i, q^{\kappa-i}, q^\kappa; q^\kappa)_\infty}{(q)_\infty^3} \\
&\quad \times \left(\sum_{r,n\geq 0} q^{kn(\kappa n+2i)+\kappa rn+ri} (-1)^r q^{r(3r-1)/2} \right. \\
&\qquad \left. + \sum_{r<0,n<0} q^{kn(\kappa n+2(\kappa-i))+\kappa rn+(\kappa-i)r+k\kappa-2ki} (-1)^r q^{r(3r-1)/2} \right).
\end{aligned}
$$

Therefore, by pairing i-th term and $\kappa - i$-th term, we find that the above equals

$$\begin{aligned}
&\sum_{i=1}^{k} \frac{(-1)^{i+1}(q^i, q^{\kappa-i}, q^{\kappa}; q^{\kappa})_\infty}{(q)_\infty^3} \\
&\times \left(q^{\binom{i}{2}} \left(\sum_{n,r\geq 0} - \sum_{n,r<0} \right) (-1)^r q^{kn(\kappa n+2i)+\kappa rn+ri} q^{r(3r-1)/2} \right. \\
&\quad \left. - q^{\binom{\kappa-i}{2}} \left(\sum_{n,r\geq 0} - \sum_{n,r<0} \right) (-1)^r q^{kn(\kappa n+2(\kappa-i))+\kappa rn+(\kappa-i)r} q^{r(3r-1)/2} \right) \\
&= \sum_{i=1}^{\kappa-1} \frac{(-1)^{i+1} q^{\binom{i}{2}} (q^i, q^{\kappa-i}, q^{\kappa}; q^{\kappa})_\infty}{(q)_\infty^3} f_{2k\kappa,\kappa,3}(-q^{k\kappa+2ki}, q^{i+1}, q).
\end{aligned}$$

It is not hard to see that

$$\begin{aligned}
&\sum_{n\geq 0} (q^{n+1})_n q^n \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(q)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} \\
&= \sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0} q^{n_k} \begin{bmatrix} 2n_k \\ n_k \end{bmatrix} q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix} \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}.
\end{aligned}$$

Thus, we have proved the following identity.

Corollary 3.2. *For $k \geq 1$ and $\kappa = 2k+1$,*

$$\begin{aligned}
&\sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0} q^{n_k} \begin{bmatrix} 2n_k \\ n_k \end{bmatrix} q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix} \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix} \\
&\qquad = \sum_{i=1}^{\kappa-1} \frac{(-1)^{i+1} q^{\binom{i}{2}} (q^i, q^{\kappa-i}, q^{\kappa}; q^{\kappa})_\infty}{(q)_\infty^3} f_{2k\kappa,\kappa,3}(-q^{k\kappa+2ki}, q^{i+1}, q).
\end{aligned} \tag{3.4}$$

For example, when $k=1$, we find that

$$\sum_{n\geq 0} q^n \begin{bmatrix} 2n \\ n \end{bmatrix} = \frac{1}{(q)_\infty^2} (f_{6,3,3}(-q^5, q^2, q) - q f_{6,3,3}(-q^7, q^3, q)).$$

When $k=2$, we have

$$\begin{aligned}
\sum_{n\geq 0} q^n \begin{bmatrix} 2n \\ n \end{bmatrix} \sum_{n_1=0}^{n} q^{n_1^2+n_1} \begin{bmatrix} n \\ n_1 \end{bmatrix} &= \frac{(q, q^4, q^5; q^5)_\infty}{(q)_\infty^3} \left(f_{20,5,3}(-q^{14}, q^2, q) - q^6 f_{20,5,3}(-q^{26}, q^5, q) \right) \\
&\quad - \frac{(q^2, q^3, q^5; q^5)_\infty}{(q)_\infty^3} \left(q f_{20,5,3}(-q^{18}, q^3, q) - q^3 f_{20,5,3}(-q^{22}, q^4, q) \right).
\end{aligned}$$

The left-hand side of (3.4) can be interpreted in terms of unimodal sequences with a distinguished peak. Recall that a unimodal sequence is a sequence which is weakly increasing up to a point (called the peak), and then weakly decreasing thereafter. The weight of such a sequence is the sum of all of its terms. For example, the 12 unimodal sequences of weight 4 are

$$(\underline{4}), (\underline{3}, 1), (1, \underline{3}), (\underline{2}, 2), (2, \underline{2}), (\underline{2}, 1, 1), (1, 1, \underline{2}), (1, \underline{2}, 1),$$
$$(\underline{1}, 1, 1, 1), (1, \underline{1}, 1, 1), (1, 1, \underline{1}, 1), (1, 1, 1, \underline{1}),$$

where the peak is marked with an underline. Also recall that the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ generates the partitions which fit inside a $(n-k) \times k$ rectangle.

Now we can decompose the left-hand side of (3.4) as follows. First, the term q^{n_k} corresponds to a distinguished peak of height n_k. Second, the term $\begin{bmatrix} 2n_k \\ n_k \end{bmatrix}$ is the generating function for partitions fitting inside a $n_k \times n_k$ square, and we insert the parts of such a partition to the left of the peak. Finally, the term

$$q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix} \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}$$

corresponds to a partition with a certain *Durfee dissection*. Specifically, we have a partition into parts $\leq n_k$ with $k-1$ successive Durfee rectangles of (non-negative) sizes $(n_{k-1}+1) \times n_{k-1}$, $(n_{k-2}+1) \times n_{k-2}$, ..., and $(n_1+1) \times n_1$, such that the bottom row of each rectangle is a part of the partition and there is nothing below the final rectangle. (See [3] for more details on Durfee dissections.) Inserting the parts of this partition to the right of the peak completes the unimodal sequence.

3.3. **A generalization.** Next we generalize (3.4) by iterating the Bailey pair in (3.2) and (3.3) along the Bailey chain. Repeatedly applying (1.7) and (1.8) with $b, c \to \infty$ at each step, we find that

$$\beta_n^{(k)} = \sum_{n \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_{k-1}^2+n_{k-2}^2+\cdots+n_1^2}}{(q)_{n-n_{k-1}} \cdots (q)_{n_2-n_1}(q)_{n_1}}$$

and

$$\alpha_n^{(k)} = \begin{cases} 1, & n = 0, \\ (-1)^n (q^{n((2k+1)n-1)/2} + q^{n((2k+1)n+1)/2}), & \text{otherwise}. \end{cases}$$

form a Bailey pair relative to 1 for all positive integers k. Then, by arguing as usual, we can derive the following result.

Corollary 3.3. *For $\kappa = 2k+1$ and $k, \ell, \geq 1$,*

$$\sum_{\substack{n_k \geq 0 \\ n_k \geq m_{\ell-1} \geq \cdots \geq m_1 \geq 0 \\ n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0}} q^{n_k} \begin{bmatrix} 2n_k \\ n_k \end{bmatrix} q^{m_{\ell-1}^2+m_{\ell-2}^2+\cdots+m_1^2} \begin{bmatrix} n_k \\ m_{\ell-1} \end{bmatrix} \begin{bmatrix} m_{\ell-1} \\ m_{\ell-2} \end{bmatrix} \cdots \begin{bmatrix} m_2 \\ m_1 \end{bmatrix}$$
$$\times q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix} \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix} \tag{3.5}$$
$$= \sum_{i=1}^{\kappa-1} \frac{(-1)^{i+1} q^{\binom{i}{2}} (q^i, q^{\kappa-i}, q^\kappa; q^\kappa)_\infty}{(q)_\infty^3} f_{2k\kappa,\kappa,2\ell+1}(-q^{k\kappa+2ki}, q^{i+\ell}, q).$$

For example, when $k = \ell = 2$, we obtain

$$\sum_{n\geq 0} q^n \begin{bmatrix} 2n \\ n \end{bmatrix} \sum_{m=0}^{n} q^{m^2} \begin{bmatrix} n \\ m \end{bmatrix} \sum_{k=0}^{n} q^{k^2+k} \begin{bmatrix} n \\ k \end{bmatrix}$$
$$= \frac{(q, q^4, q^5; q^5)_\infty}{(q)_\infty^3} \left(f_{20,5,5}(-q^{14}, q^3, q) - q^6 f_{20,5,5}(-q^{26}, q^6, q) \right)$$
$$- \frac{(q^2, q^3, q^5; q^5)_\infty}{(q)_\infty^3} \left(q f_{20,5,5}(-q^{18}, q^4, q) - q^3 f_{20,5,5}(-q^{22}, q^5, q) \right).$$

The left-hand side of (3.5) can be interpreted in terms of what we call *unimodal T-sequences* with a distinguished peak. Here we have a peak and then we allow a partition to the right, to the left, *and* below the peak, where the parts of the partitions are all less than or equal to the size of the peak. Then the left-hand side of (3.5) is the generating function for the number of unimodal T-sequences with a peak of size n_k, with the partitions to the left and right of the peak exactly as in the previous subsection, and with the partition below the peak having at most $\ell - 1$ successive Durfee squares of (non-negative) sizes $m_{\ell-1} \times m_{\ell-1}$, $m_{\ell-2} \times m_{\ell-2}$, $\ldots$, and $m_1 \times m_1$.

3.4. **Some other unimodal sequences.** We finish this section with a few more applications of (1.16) involving unimodal-type sequences. Both the computational and combinatorial details are similar to the previous subsections, so we shall be brief.

First, we have the following Bailey pairs [26, 27] relative to $a = 1$:

$$\beta_n = \frac{1}{(q)_{2n}}, \quad \alpha_{3k\pm 1} = -q^{(2k\pm 1)(3k\pm 1)}, \quad \alpha_{3k} = q^{k(6k-1)} + q^{k(6k+1)},$$

and

$$\beta_n = \frac{q^n}{(q)_{2n}}, \quad \alpha_{3k\pm 1} = -q^{2k(3k\pm 1)}, \quad \alpha_{3k} = q^{2k(3k-1)} + q^{2k(3k+1)},$$

where $\alpha_0 = 1$. Using these in (1.16) we find the following.

Corollary 3.4. *For all $k \geq 1$, $\kappa = 2k+1$, and $j \in \{0, 1\}$,*

$$\sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{(j+1)n_k}}{(q)_{n_k}} q^{n_{k-1}^2 + n_{k-1} + \cdots + n_1^2 + n_1} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix} \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}$$
$$= \sum_{i=1}^{\kappa-1} \frac{(-1)^{i+1} q^{\binom{i}{2}} (q^i, q^{\kappa-i}, q^\kappa; q^\kappa)_\infty}{(q)_\infty^3}$$
$$\times \left(f_{2k\kappa,3\kappa,12}(-q^{k\kappa+2ki}, -q^{3i+5-j}, q) - q^{i+1-j} f_{2k\kappa,3\kappa,12}(-q^{(k+1)\kappa+2ki}, -q^{3i+11-3j}, q) \right). \tag{3.6}$$

Note that when $k = 1$ and $j = 0$, the left-hand side is

$$\sum_{n\geq 0} \frac{q^n}{(q)_n} = \frac{1}{(q)_\infty},$$

by the case $z=q$ and $a=0$ of the q-binomial identity [2, Theorem 2.1],

$$\sum_{n\geq 0}\frac{(a)_n}{(q)_n}z^n=\frac{(az)_\infty}{(z)_\infty}. \tag{3.7}$$

Thus we have

$$\begin{aligned}(q)_\infty = f_{6,9,12}(-q^5,-q^8,q) - q f_{6,9,12}(-q^7,-q^{11},q)\\ - q^2 f_{6,9,12}(-q^8,-q^{14},q) + q^4 f_{6,9,12}(-q^{10},-q^{17},q).\end{aligned}$$

Combinatorially, the left-hand side of (3.6) is the generating function for unimodal sequences with distinguished peak of size $(j+1)n_k$ such that before the peak there is a partition with parts $\leq n_k$ and after the peak there is a partition into parts $\leq n_k$ with the same Durfee rectangle dissection discussed previously.

Next we take the Bailey pair relative to 1 [26],

$$\beta_n=\frac{1}{(q)_n(q;q^2)_n}$$

and

$$\alpha_n=\begin{cases}1, & n=0,\\ (-1)^k(q^{k(3k-1)}+q^{k(3k+1)}), & \text{if } n=2k \text{ and } k\geq 1\\ 0, & \text{otherwise.}\end{cases}$$

Using this Bailey pair, we derive that

Corollary 3.5. *For $k\geq 1$ and $\kappa=2k+1$,*

$$\begin{aligned}\sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0}\frac{(-q)_{n_k}q^{n_k}}{(q)_{n_k}}q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}\begin{bmatrix}n_k\\ n_{k-1}\end{bmatrix}\begin{bmatrix}n_{k-1}\\ n_{k-2}\end{bmatrix}\cdots\begin{bmatrix}n_2\\ n_1\end{bmatrix}\\ =\sum_{i=1}^{\kappa-1}\frac{(-1)^{i+1}q^{\binom{i}{2}}(q^i,q^{\kappa-i},q^\kappa;q^\kappa)_\infty}{(q)_\infty^3}f_{2k\kappa,2\kappa,6}(-q^{k\kappa+2ki},q^{2i+2},q).\end{aligned} \tag{3.8}$$

When $k=1$, the left-hand side is a product by (3.7),

$$\sum_{n\geq 0}\frac{(-q)_n}{(q)_n}q^n=\frac{(-q^2)_\infty}{(q)_\infty}.$$

This yields

$$(q^2;q^2)_\infty=(1+q)\left(f_{6,6,6}(-q^5,q^4,q)-qf_{6,6,6}(-q^7,q^6,q)\right).$$

Combinatorially, the left-hand side of (3.8) is a generating function for the number of unimodal sequences with the peak of size n_k such that before the peak, there is an *overpartition* with parts $\leq n_k$ and after the peak there is a partition into parts $\leq n_k$ and the usual Durfee rectangle dissection. We remind the reader that an overpartition is a partition in which the first occurrence of each integer may be overlined.

Finally, consider the Bailey pair relative to 1 [26]

$$\beta_n=\frac{(-q)_{n-1}}{(q;q^2)_n(q)_n},\quad \alpha_{2n-1}=0,\quad \alpha_{2n}=q^{T_{2n-1}}+q^{T_{2n}},$$

where $\alpha_0=\beta_0=1$ and $T_k=k(k+1)/2$. This time, we obtain

Corollary 3.6. *For $k \geq 1$ and $\kappa = 2k+1$,*

$$\sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{(-q)_{n_k} q^{n_k}}{(q)_{n_k}} q^{n_{k-1}^2 + n_{k-1} + \cdots + n_1^2 + n_1} (-q)_{n_k - 1} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix} \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}$$
$$= \sum_{i=1}^{\kappa-1} \frac{(-1)^{i+1} q^{\binom{i}{2}} (q^i, q^{\kappa-i}, q^{\kappa}; q^{\kappa})_\infty}{(q)_\infty^3} f_{2k\kappa, 2\kappa, 4}(-q^{k\kappa + 2ki}, -q^{2i+1}, q). \tag{3.9}$$

The left-hand side of (3.9) is the generating function for the number of unimodal sequences with distinguished peak of size n_k, such that before the peak there is an overpartition with parts $\leq n_k$ and after the peak there is an overpartition into parts $\leq n_k$, where overlined parts are $< n_k$ and the partition consiting of the non-overlined parts parts has the usual Durfee rectangle dissection.

4. Applications II - Rogers-Ramanujan type modular functions

The celebrated Rogers-Ramanujan identities are

$$\sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q, q^4; q^5)_\infty}$$

and

$$\sum_{n \geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}.$$

Combinatorially, they imply deep partition identities between partitions with gap conditions and partitions with congruence conditions. Analytically, they give a relation between q-hypergeometric series and modular functions. See [2] for background on these identities.

The Rogers-Ramanujan identities were generalized anaytically to odd moduli by Andrews [1] and extended to even moduli by Bressoud [12]. (For combinatorial generalizations, see work of Gordon [13] and Bressoud [11].) The following are Andrews' identites for $\delta = 1$ and Bressoud's identities for $\delta = 0$ [12]:

$$\sum_{n_1, n_2, \cdots, n_{k-1} = 0}^{\infty} \frac{q^{N_1^2 + \cdots + N_{k-1}^2 + N_r + \cdots + N_{k-1}}}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_{k-2}} (q^{2-\delta}; q^{2-\delta})_{n_{k-1}}} = \frac{(q^r, q^{2k+\delta-r}, q^{2k+\delta}; q^{2k+\delta})_\infty}{(q)_\infty}, \tag{4.1}$$

where $\delta = 0$ or 1 and r and k are integers such that $0 < r < (2k+\delta)/2$.

In this section, we use Bailey pairs in equations (1.16) and (1.17) so that the left-hand side is an instance of (4.1), resulting in expressions for certain modular functions in terms of indefinite theta series.

4.1. **Special cases of Andrews' identities.**

We consider two Bailey pairs relative to $a = 1$ [26, 27],

$$\beta_n = \frac{q^{n^2-n}}{(q)_{2n}}, \quad \alpha_{3k\pm 1} = -q^{(k\pm 1)(3k\pm 1)}, \quad \alpha_{3k} = q^{k(3k-2)} + q^{k(3k+2)},$$

and

$$\beta_n = \frac{q^{n^2}}{(q)_{2n}}, \quad \alpha_{3k\pm 1} = -q^{k(3k\pm 1)}, \quad \alpha_{3k} = q^{k(3k-1)} + q^{k(3k+1)},$$

where $\alpha_0 = 1$ for all cases. Using these Bailey pairs in (1.16) and calculating in the usual way we obtain that

$$\begin{aligned}\sum_{n\geq 0} q^{n^2+jn} &\sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(q)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} \\ &= \sum_{i=1}^{\kappa-1} \frac{(-1)^{i+1} q^{\binom{i}{2}} (q^i, q^{\kappa-i}, q^{\kappa}; q^{\kappa})_\infty}{(q)_\infty^3} \\ &\quad \times \left(f_{2k\kappa,3\kappa,6}(-q^{k\kappa+2ki}, -q^{3i+1+j}, q) - q^{i+1-j} f_{2k\kappa,3\kappa,6}(-q^{(k+1)\kappa+2ki}, -q^{3i+7-3j}, q) \right),\end{aligned} \tag{4.2}$$

where $j \in \{0, 1\}$. Using (4.1) the multi-sum on the left-hand side of (4.2) is a simple infinite product. To see this, note that

$$\begin{aligned}\sum_{n\geq 0} q^{n^2+jn} &\sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(q)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} \\ &= \sum_{n_1,n_2,\ldots,n_{k-1},n_k\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_k^2+N_{2-j}+N_{3-j}+\cdots+N_k}}{(q)_{n_1}(q)_{n_2}(q)_{n_2}\cdots(q)_{n_k}} \\ &= \frac{(q^{2-j}, q^{2k+j+1}, q^{2k+3}; q^{2k+3})_\infty}{(q)_\infty},\end{aligned}$$

where the last equality follows from (4.1) with $\delta = 1$ and $r = 2 - j$. As a result, we obtain the following corollary. It is very interesting that Rogers-Ramanujan type modular functions can be expressed in terms of indefinite theta series.

Corollary 4.1. *For $\kappa = 2k + 1$ with $k \geq 1$ and $j \in \{0, 1\}$,*

$$\begin{aligned}&\frac{(q^{2-j}, q^{2k+j+1}, q^{2k+3}; q^{2k+3})_\infty}{(q)_\infty} \\ &= \sum_{i=1}^{\kappa-1} \frac{(-1)^{i+1} q^{\binom{i}{2}} (q^i, q^{\kappa-i}, q^{\kappa}; q^{\kappa})_\infty}{(q)_\infty^3} \\ &\quad \times \left(f_{2k\kappa,3\kappa,6}(-q^{k\kappa+2ki}, -q^{3i+1+j}, q) - q^{i+1-j} f_{2k\kappa,3\kappa,6}(-q^{(k+1)\kappa+2ki}, -q^{3i+7-3j}, q) \right).\end{aligned}$$

For example, when $k=1$, we find that

$$\begin{aligned}\sum_{n\geq 0}\frac{q^{n^2}}{(q)_n} &= \frac{1}{(q,q^4;q^5)_\infty}\\ &= \frac{1}{(q)_\infty^2}\Big(f_{6,9,6}(-q^5,-q^4,q)-qf_{6,9,6}(-q^7,-q^7,q)\\ &\qquad -q^2f_{6,9,6}(-q^8,-q^{10},q)+q^4f_{6,9,6}(-q^{10},-q^{13},q)\Big),\\ \sum_{n\geq 0}\frac{q^{n^2+n}}{(q)_n} &= \frac{1}{(q^2,q^3;q^5)_\infty}\\ &= \frac{1}{(q)_\infty^2}\Big(f_{6,9,6}(-q^5,-q^5,q)-qf_{6,9,6}(-q^8,-q^7,q)\\ &\qquad -qf_{6,9,6}(-q^7,-q^8,q)+q^3f_{6,9,6}(-q^{10},-q^{10},q)\Big).\end{aligned}$$

When $k=2$ and $j=1$, we derive that

$$\begin{aligned}\sum_{n\geq k\geq 0}\frac{q^{n^2+k^2+n+k}}{(q)_{n-k}(q)_k} &= \frac{(q,q^6,q^7;q^7)_\infty}{(q)_\infty}\\ &= \frac{(q,q^4,q^5;q^5)_\infty}{(q)_\infty^3}\big(f_{20,15,6}(-q^{14},-q^5,q)-qf_{20,15,6}(-q^{19},-q^7,q)\\ &\qquad -q^6f_{20,15,6}(-q^{26},-q^{14},q)+q^{10}f_{20,15,6}(-q^{31},-q^{16},q)\big)\\ &\quad -\frac{(q^2,q^3,q^5;q^5)_\infty}{(q)_\infty^3}\big(qf_{20,15,6}(-q^{18},-q^8,q)-q^3f_{20,15,6}(-q^{23},-q^{10},q)\\ &\qquad -q^3f_{20,15,6}(-q^{22},-q^{11},q)+q^6f_{20,15,6}(-q^{27},-q^{13},q)\big).\end{aligned}$$

4.2. **Special cases of Bressoud's identities.** Using the same Bailey pairs in the previous subsection with (1.17), we can derive that

$$\begin{aligned}&\sum_{n\geq 0}q^{n^2+jn}\sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty}\frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(q)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-2}}(q^2;q^2)_{n_{k-1}}}\\ &=\sum_{i=1}^{\kappa-1}\frac{(-1)^{i+1}q^{\binom{i}{2}}(q^i,q^{\kappa-i},q^\kappa;q^\kappa)_\infty}{(q)_\infty^3}\\ &\qquad\times\Big(f_{2k(\kappa-1),3\kappa,6}(q^{k(\kappa-1)+(\kappa-1)i},-q^{3i+1+j},q)\\ &\qquad\qquad -q^{i+1-j}f_{2k(\kappa-1),3\kappa,6}(q^{k(\kappa-1)+(\kappa-1)i+\kappa},-q^{3i+7-3j},q)\Big),\end{aligned}$$

where $\kappa = 2k$, $j \in \{0,1\}$, and k is an integer ≥ 2. Note that

$$\begin{aligned}
&\sum_{n\geq 0} q^{n^2+jn} \sum_{n_1,n_2,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(q)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-2}}(q^2;q^2)_{n_{k-1}}} \\
&= \sum_{n_1,n_2,\dots,n_{k-1},n_k\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_k^2+N_{2-j}+N_{3-j}+\cdots+N_k}}{(q)_{n_1}(q)_{n_2}(q)_{n_2}\cdots(q)_{n_{k-1}}(q^2;q^2)_{n_k}} \\
&= \frac{(q^{2-j},q^{2k+j},q^{2k+2};q^{2k+2})_\infty}{(q)_\infty}.
\end{aligned}$$

where the last equality follows from (4.1) with $\delta = 0$ and $r = 2-j$.

Therefore, we have proven the following result.

Corollary 4.2. *For $\kappa = 2k$, $j \in \{0,1\}$, and k an integer ≥ 2,*

$$\begin{aligned}
&\frac{(q^{2-j},q^{2k+j},q^{2k+2};q^{2k+2})_\infty}{(q)_\infty} \\
&= \sum_{i=1}^{\kappa-1} \frac{(-1)^{i+1}q^{\binom{i}{2}}(q^i,q^{\kappa-i},q^\kappa;q^\kappa)_\infty}{(q)_\infty^3} \\
&\quad \times \left(f_{2k(\kappa-1),3\kappa,6}(q^{k(\kappa-1)+(\kappa-1)i},-q^{3i+1+j},q) - q^{i+1-j}f_{2k(\kappa-1),3\kappa,6}(q^{k(\kappa-1)+(\kappa-1)i+\kappa},-q^{3i+7-3j},q)\right).
\end{aligned}$$

For example, when $k=2$ and $j=0$, we obtain

$$\begin{aligned}
\sum_{n\geq k\geq 0} \frac{q^{n^2+k^2+k}}{(q)_{n-k}(q^2;q^2)_k} &= \frac{(q^2,q^4,q^6;q^6)_\infty}{(q)_\infty} = (-q)_\infty \\
&= \frac{(q,q^3,q^4;q^4)_\infty}{(q)_\infty^3}\big(f_{12,12,6}(q^9,-q^4,q) - q^2 f_{12,12,6}(q^{13},-q^{10},q) \\
&\qquad + q^3 f_{12,12,6}(q^{15},-q^{10},q) - q^7 f_{12,12,6}(q^{19},-q^{16},q)\big) \\
&\quad - \frac{(q^2,q^2,q^4;q^4)_\infty}{(q)_\infty^3}\big(q f_{12,12,6}(q^{12},-q^7,q) - q^4 f_{12,12,6}(q^{16},-q^{13},q)\big).
\end{aligned}$$

5. Applications III - Theorem 1.2

At a first glance, Theorem 1.2 may appear too complicated to derive interesting identities. However, if we adopt the unit Bailey pair, then the β-side collapses and the α-side becomes an interesting combination of products of indefinite theta series.

Corollary 5.1. *For k a positive integer, and $0 \leq \ell < k$,*

$$\begin{aligned}
(q)_\infty^3 &= \sum_{i=1}^{k}(-1)^i q^{\binom{i+1}{2}} H^1_{k,\ell}(i) \times \big(f_{4k^2+6k+2,2k+2,1}(-q^{(2k+1)(k+1+i)},q^i,q) \\
&\qquad - q^{(2k+1)(k+1-i)} f_{4k^2+6k+2,2k+2,1}(-q^{(2k+1)(3k+3-i)},q^{2k+2-i},q)\big) \\
&= \sum_{i=1}^{k}(-1)^i q^{\binom{i+1}{2}} H^2_{k,\ell}(i) \times \big(f_{4k^2+2k,2k+1,1}(q^{k(2k+1+2i)},q^i,q) \\
&\qquad + q^{(k(2k+1-2i)} f_{4k^2+2k,2k+1,1}(q^{k(6k+3-2i)},q^{2k+1-i},q)\big).
\end{aligned}$$

Note that $H^1_{k,\ell}(i)$ and $H^2_{k,\ell}(i)$ are linear combinations (up to powers of q) of indefinite theta series $f_{a,b,c}(x,y,q)$. For example, when $k=1$ we obtain that

$$\begin{aligned}(q)_\infty^3 &= -q\left(f_{1,7,1}(q^4,q^3,q)+q^5f_{1,7,1}(q^8,q^7,q)\right)\left(f_{12,4,1}(-q^9,q,q)-q^3f_{12,4,1}(-q^{15},q^3,q)\right)\\ &= -q\left(f_{1,5,1}(q^3,q^3,q)+q^4f_{1,5,1}(q^6,q^6,q)\right)\left(f_{6,3,1}(q^5,q,q)+qf_{6,3,1}(q^7,q^2,q)\right).\end{aligned}$$

As a final example, we use the Bailey pair in (3.2) and (3.3) to obtain the following identities.

Corollary 5.2. *For k a positive integer, and $0\le \ell<k$,*

$$\begin{aligned}&\sum_{n\ge0}\frac{(q)_{2n}q^n}{(q)_n}\sum_{n_{2k-1}\ge\cdots\ge n_1\ge0}\frac{(-1)^{n_k}q^{\sum_{i=1}^{k-1}(n_{k+i}^2+n_{k+i})+\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i}}{(q)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}}\\ &=\frac{1}{(q)_\infty^3}\sum_{i=1}^{k}(-1)^iq^{\binom{i+1}{2}}H^1_{k,\ell}(i)\times\Big(f_{4k^2+6k+2,2k+2,3}(-q^{(2k+1)(k+1+i)},q^{i+1},q)\\ &\qquad -q^{(2k+1)(k+1-i)}f_{4k^2+6k+2,2k+2,3}(-q^{(2k+1)(3k+3-i)},q^{2k+3-i},q)\Big)\end{aligned}$$

and

$$\begin{aligned}&\sum_{n\ge0}\frac{(q)_{2n}q^n}{(q)_n}\sum_{n_{2k-1}\ge\cdots\ge n_1\ge0}\frac{q^{\sum_{i=1}^{k-1}n_{k+i}^2+n_{k+i}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i+\binom{n_1+1}{2}}(-1)^{n_k}(-q)_{n_k}}{(q)_{n-n_{2k-1}}\cdots(q)_{n_2-n_1}(q^2;q^2)_{n_1}(-q)_{n_{k+1}}}\\ &=\frac{1}{(q)_\infty^3}\sum_{i=1}^{k}(-1)^iq^{\binom{i+1}{2}}H^2_{k,\ell}(i)\times\Big(f_{4k^2+2k,2k+1,3}(q^{k(2k+1+2i)},q^{i+1},q)\\ &\qquad +q^{(k(2k+1-2i)}f_{4k^2+2k,2k+1,3}(q^{k(6k+3-2i)},q^{2k+2-i},q)\Big).\end{aligned}$$

When $k=1$, we can use (3.7) to carry out the sum over n_1 and we obtain

$$\begin{aligned}\sum_{n\ge0}(q^{n+1})_nq^n &= -\frac{q}{(q)_\infty^3}\left(f_{1,7,1}(q^4,q^3,q)+q^5f_{1,7,1}(q^8,q^7,q)\right)\\ &\qquad\times\left(f_{12,4,3}(-q^9,q^2,q)-q^3f_{12,4,3}(-q^{15},q^4,q)\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{n\ge0}(q;q^2)_nq^n &= -\frac{q}{(q)_\infty^3}\left(f_{1,5,1}(q^3,q^3,q)+q^4f_{1,5,1}(q^6,q^6,q)\right)\\ &\qquad\times\left(f_{6,3,3}(q^5,q^2,q)+qf_{6,3,3}(q^7,q^3,q)\right).\end{aligned}$$

Note that while the expressions on the right-hand sides of the two equations above are in the algebra of modular forms and indefinite theta functions, the q-series on the left-hand sides are known to be false theta functions [15, Propositions 1 and 3] and [4, (3.25)],

$$\begin{aligned}\sum_{n\ge0}(q^{n+1})_nq^n &= \sum_{\substack{n\ge1\\ n^2\equiv49 \pmod{120}}}(-1)^{\lfloor n/30\rfloor}q^{(n^2-49)/120}\\ &=\sum_{n\ge0}(-1)^nq^{n(15n+7)/2}(1+q^{3n+1}+q^{5n+2}+q^{8n+4})\end{aligned}$$

and

$$\sum_{n\geq 0}(q;q^2)_n q^n = \sum_{n\geq 0}(-1)^n q^{3n^2+2n}(1+q^{2n+1}).$$

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