A THEOREM ON SEVEN-COLORED OVERPARTITIONS AND ITS APPLICATIONS

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Abstract. A q-series identity in four parameters is established and interpreted it as a statement about 7-colored overpartitions. As corollaries some overpartition theorems of the Rogers-Ramanujan type and some weighted overpartition theorems are exhibited. Among these are overpartition analogues of classical partition theorems of Schur and Göllnitz.

1. Introduction

An overpartition is a partition in which the first occurrence of a number may be overlined. It has recently been shown [15, 16] that some of the classical techniques Andrews used to study Rogers-Ramaujan type identities for ordinary partitions are equally well-suited to the study of overpartitions. Here we turn our attention to the method of colored partitions that has been developed during the last decade [5, 6, 7] and show that overpartitions have a natural place there as well.

We begin by establishing the following q-series identity in four parameters, wherein we employ the common notation

$$T_m := m(m+1)/2 \tag{1.1}$$

and

$$(a_1, a_2, ..., a_k)_n := (a_1, a_2, ..., a_k; q)_n := \prod_{j=0}^{n-1} (1-a_1q^j)(1-a_2q^j)\cdots(1-a_kq^j), \tag{1.2}$$

the first definition being valid for all integers m and the second for non-negative integers n.

Lemma 1.1.

$$\frac{(-aq,-bq,-cq)_\infty}{(adq,bdq,cdq)_\infty} = \tag{1.3}$$

$$\sum_{\substack{\alpha,\beta,\gamma,\delta,\epsilon,\phi,\ell\geq 0\\ s=\alpha+\beta+\gamma+\delta+\epsilon+\phi+\ell}} \frac{q^{s+T_\delta+T_\epsilon+T_{\phi-1}+\ell}(1-q^\alpha(1-q^\phi))a^{\alpha+\delta+\epsilon+\ell}b^{\beta+\delta+\phi+\ell}c^{\gamma+\epsilon+\phi+\ell}d^{s+\ell}(-1/d)_s(-1)^\ell}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\epsilon(q)_\phi(q)_\ell}.$$

Then we show that the identity may be interpreted in terms of certain overpartitions whose parts occur in seven colors. We denote these seven colors by a, b, c, ab, ac, bc, and $abcd$, with the convention that parts are ordered first according to size and then according to color, where the colors are ordered by

Date: April 19, 2007.
2000 *Mathematics Subject Classification.* 11P81, 05A17.

$$abcd < ab < ac < a < bc < b < c. \tag{1.4}$$

The relevant seven-colored overpartitions are defined below. For any part λ_i, we use c_i to denote its color.

Definition 1.2. *Let $\overline{C}(\alpha, \beta, \gamma, \delta, \epsilon, \phi, \ell; k; n)$ denote the number of overpartitions of n occurring in the seven colors above with α a-parts, β b-parts, γ c-parts, δ ab-parts, ϵ ac-parts, ϕ bc-parts, ℓ abcd-parts, k non-overlined parts, and the following additional properties:*

- *(i) 1 cannot occur, overlined or not, in colors $ab, ac,$ or $abcd$.*
- *(ii) 1 and $\overline{1}$ can occur in color bc only if 1 also occurs in color a.*
- *(iii) $\lambda_i - \lambda_{i+1}$ is at least one if the smaller is overlined OR if $c_i = c_{i+1} \in \{ab, ac, bc\}$ OR if $c_i < c_{i+1}$ in the order (1.4),*
- *(iv) $\lambda_i - \lambda_{i+1}$ is at least two if the smaller is overlined AND $c_i = c_{i+1} \in \{ab, ac, bc\}$ or $c_i < c_{i+1}$ in the order (1.4).*

The principal result in this paper is the following, which asserts that a weighted generating function for the overpartitions defined above is an infinite product.

Theorem 1.3.

$$\sum_{\alpha,\beta,\gamma,\delta,\epsilon,\phi,\ell,k,n\geq 0} \overline{C}(\alpha, \beta, \gamma, \delta, \epsilon, \phi, \ell; k; n) a^{\alpha+\delta+\epsilon+\ell} b^{\beta+\delta+\phi+\ell} c^{\gamma+\epsilon+\phi+\ell} d^{k+\ell} (-1)^{\ell} = \frac{(-aq, -bq, -cq)_\infty}{(adq, bdq, cdq)_\infty}.$$

Our first two applications of Theorem 1.3 are a generalization of a partition theorem of Schur [17] and a result that is very much like a partition theorem of Göllnitz [6, Theorem G].

Corollary 1.4. *Let $\overline{S}(x_1, x_2; k; n)$ denote the number of overpartitions of n into parts not divisible by 3 with x_j parts congruent to j modulo 3 and k non-overlined parts. Let $\overline{T}(x_1, x_2; k; n)$ denote the number of overpartitions of n with x_j parts congruent to j or 3 modulo 3 and k non-overlined parts, where parts differ by at least 3 if the smaller is overlined OR both parts are divisible by 3, and parts differ by at least 6 if the smaller is overlined AND both parts are divisible by 3. Then $\overline{S}(x_1, x_2; k; n) = \overline{T}(x_1, x_2; k; n)$.*

Corollary 1.5. *Let $\overline{G}(n)$ be the number of overpartitions of n into parts congruent to 2, 4, or 5 modulo 6. Let $\overline{H}(n)$ be the number of overpartitions of n such that (i) there are no ones, (ii) 3 or $\overline{3}$ may occur if 2 does, (iii) the difference between consecutive parts is at least 6 if the smaller is overlined OR both parts are congruent to 0 modulo 6, both are 1 modulo 6, or both are 3 modulo 6, (iv) the difference between consecutive parts is at least 12 if the smaller is overlined AND both are 0 modulo 6, both are 1 modulo 6, or both are 3 modulo 6, and (v) any maximal set of consecutive parts, $\{\lambda_i, \lambda_{i+1}, \dots, \lambda_{i+k}\}$, where each part is congruent to 5 modulo 6 and the difference between any two consecutive parts is the minimum allowed by conditions (iii) – (iv), must have even length. Then $\overline{G}(n) = \overline{H}(n)$.*

For another application we present some weighted overpartition theorems, along the lines of [1, 2]. Here we define a *chain* in an overpartition to be a maximal set of consecutive parts where the difference between consecutive parts is 0 if the smaller is non-overlined and 1 if it is overlined. Then any overpartition may be uniquely decomposed into chains. For example, the overpartition $(\overline{5}, 5, 5, \overline{4}, 3, \overline{2}, 1)$ is comprised of the chains $(\overline{5}, 5, 5, \overline{4}), (3, \overline{2})$, and (1). In the theorems below, we will define a weight on chains, the weight of the overpartition being the product of the weights of its chains. Denote by κ the length of a chain and σ the smallest part in the chain.

Corollary 1.6. *Starting with $k = 0$, let $a(k)$ be the 6-periodic sequence that begins $1, 1, 0, -1, -1, 0, \ldots$ Then the number of overpartitions of n into parts not divisible by 3 is equal to the number of weighted overparititons of n where the weight of a chain is*

$$\begin{cases} a(\kappa), & \sigma = 1 \\ a(\kappa) + a(\kappa - 1), & \sigma \geq 2 \end{cases}$$

Corollary 1.7. *The number of overpartitions of n into parts divisible by 3 is equal to the number of overpartitions of n where the weight of a chain is*

$$\begin{cases} -1, & \kappa \equiv 1 \pmod 3 \text{ and } \sigma \geq 2 \\ 0, & \kappa \equiv 1 \pmod 3 \text{ and } \sigma = 1 \\ 1, & \text{otherwise} \end{cases}$$

As a final application, we relate a weighted count of the overpartitions of n to the number of representations of n as the sum of three squares. For an overpartition λ, define $f(\lambda)$ to be 0, if 1 and $\overline{1}$ do not occur, and the length of the chain containing 1 and/or $\overline{1}$, otherwise. We also define $r_3(n)$ to be the number of integer solutions to $x^2 + y^2 + z^2 = n$.

Corollary 1.8. *The number of overpartitions λ of n, each weighted by $(-1)^{f(\lambda)}(2f(\lambda) + 1)$, is equal to $(-1)^n r_3(n)$.*

We should point out that $r_3(n)$ is also related to class numbers of imaginary quadratic fields [14, p.389-390].

2. Proof of Lemma 1.1

Although the identity (1.3) may appear daunting, it is happily only a slight extension of the work in [6, Section 4]. Indeed, the case $d = 0$ of Lemma 1.1 is Eq. 1.4 of [6]. For the proof of (1.3), we shall require the ${}_6\psi_6$ summation [12, p.239, Eq. (II.33)],

$$\sum_{n \in \mathbb{Z}} \frac{(1 - aq^{2n})(b, c, d, e)_n (qa^2/bcde)^n}{(1 - a)(aq/b, aq/c, aq/d, aq/e)_n} = \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde)_\infty}, \tag{2.1}$$

and a certain ${}_8\phi_7$ transformation [12, p.243, Eq. (III.24)],

$$\sum_{n \geq 0} \frac{(1 - aq^{2n})(a, b, c, d, e, f)_n (a^2q^2/bcdef)^n}{(1 - a)(q, aq/b, aq/c, aq/d, aq/e, aq/f)_n} = \frac{(aq, b, bc\mu/a, bd\mu/a, be\mu/a, bf\mu/a)_\infty}{(aq/c, aq/d, aq/e, aq/f, \mu q, b\mu/a)_\infty}$$
$$\times \sum_{n \geq 0} \frac{(1 - \mu q^{2n})(\mu, aq/bc, aq/bd, aq/be, aq/bf, b\mu/a)_n b^n}{(1 - \mu)(q, bc\mu/a, bd\mu/a, be\mu/a, bf\mu/a, aq/b)_n}. \tag{2.2}$$

Here we require $\mu = a^3q^2/b^2def$. Also, we have extended (1.2) to all integers with the relation

$$(a)_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{a^n (q/a)_n}. \tag{2.3}$$

We begin with a constant term lemma, employing the usual notation

$$[z^0]\sum a(n)z^n = a(0). \tag{2.4}$$

Lemma 2.1.

$$[z^0]\frac{(-zq,-1/z,-abq/z,-acq/z,-bcq/z,q,adq,bdq,cdq)_\infty(1+abc/z^2)}{(a/z,b/z,c/z,dzq,-abcdq/z,-aq,-bq,-cq,-dq)_\infty} = 1. \tag{2.5}$$

Proof. We begin by using the ${}_6\psi_6$ summation as above, with $a=-1/z, b=-1/a, c=-1/b, d=-1/c$, and $e=1/dz$. Then the constant term identity becomes

$$\begin{aligned}
1 &= [z^0]\frac{(1+abc/z^2)}{(1-a/z)(1-b/z)(1-c/z)}\sum_{n\in\mathbb{Z}}\frac{(1+q^{2n}/z)(-1/a,-1/b,-1/c,1/dz)_n(-abcdq/z)^n}{(aq/z,bq/z,cq/z,-dq)_n} \\
&= [z^0]\frac{(1+abc/z^2)}{(1-a/z)(1-b/z)(1-c/z)}\sum_{n\geq 0}\frac{(1+q^{2n}/z)(-1/a,-1/b,-1/c,1/dz)_n(-abcdq/z)^n}{(aq/z,bq/z,cq/z,-dq)_n} \\
&+ [z^0]\frac{(1+abc/z^2)}{(1-a/z)(1-b/z)(1-c/z)}\sum_{n\leq -1}\frac{(1+q^{2n}/z)(-1/a,-1/b,-1/c,1/dz)_n(-abcdq/z)^n}{(aq/z,bq/z,cq/z,-dq)_n}.
\end{aligned}$$

Now the constant term in the middle line is obviously 1. Replacing n by $-n$, the last line can be written as

$$\begin{aligned}
&\frac{-(1+z^2/abc)(1+1/d)}{(1+aq)(1+bq)(1+cq)(1-zdq)}\sum_{n=1}^{\infty}\frac{(1+zq^{2n})(zq/a,zq/b,zq/c,-q/d)_{n-1}(-abcdq/z)^n}{(-aq^2,-bq^2,-cq^2,zdq^2)_{n-1}} \\
= &\frac{(z+abc/z)(dq)(1+1/d)}{(1+aq)(1+bq)(1+cq)(1-zdq)}\sum_{n=0}^{\infty}\frac{(1+zq^{2n+2})(zq/a,zq/b,zq/c,-q/d)_{n}(-abcdq/z)^n}{(-aq^2,-bq^2,-cq^2,zdq^2)_{n}}.
\end{aligned}$$

In the ${}_8\phi_7$ transformation (2.2) we make the substitutions $a=-zq^2, b=q, c=zq/a, d=zq/b, e=zq/c$, and $f=-q/d$. Then $\mu=abcdq^2$, and the last expression above becomes

$$\begin{aligned}
(dq+q)(z+abc/z) \quad &\times \quad \frac{(-zq^2,q,-bcdq^2,-acdq^2,-abdq^2,abcq^2/z)_\infty}{(-aq,-bq,-cq,dzq,abcdq^2,-abcdq/z)_\infty} \\
&\times \quad \sum_{n\geq 0}\frac{(1-abcdq^{2n+2})(abcdq^2,-aq,-bq,-cq,dzq,-abcdq/z)_n q^n}{(q,-bcdq^2,-acdq^2,-abdq^2,abcq^2/z,-zq^2)_n}.
\end{aligned}$$

Observe that this entire expression is negated under the substitution $z\to -abc/z$, which shows that its constant term is 0 and completes the proof of the lemma. □

Proof of Lemma 1.1. Now to prove Lemma 1.1, we change it to a constant term statement:

$$\begin{aligned}
&\frac{(-aq,-bq,-cq)_\infty}{(adq,bdq,cdq)_\infty} = \\
&[z^0]\sum_{\substack{s\in\mathbb{Z}\\ \alpha,\beta,\gamma,\delta,\epsilon,\phi,\ell\geq 0}}\frac{q^{s+T_\delta+T_\epsilon+T_{\phi-1}+\ell}(1-q^\alpha(1-q^\phi))a^{\alpha+\delta+\epsilon+\ell}b^{\beta+\delta+\phi+\ell}c^{\gamma+\epsilon+\phi+\ell}d^{s+\ell}(-1/d)_s(-1)^\ell}{z^{\alpha+\beta+\gamma+\delta+\epsilon+\phi+\ell-s}(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\epsilon(q)_\phi(q)_\ell}.
\end{aligned} \tag{2.6}$$

Each of the eight sums on the right can be written as an infinite product by using special cases of [12, p.239, Eq. (II.29)]; the sum over s by

$$\sum_{n\in\mathbb{Z}}(-1/a)_n(azq)^n=\frac{(-zq,-1/z,q)_\infty}{(azq,-aq)_\infty},$$

the sums over α,β,γ, and ℓ by

$$\sum_{n\geq 0}\frac{z^n}{(q)_n}=\frac{1}{(z)_\infty},$$

and the sums over δ,ϵ, and ϕ by

$$\sum_{n\geq 0}\frac{z^n q^{T_n}}{(q)_n}=(-zq)_\infty.$$

After a bit of simplification, this gives

$$[z^0]\frac{(-zq,-1/z,-abq/z,-acq/z,-bcq/z,q,adq,bdq,cdq)_\infty(1+abc/z^2)}{(a/z,b/z,c/z,dzq,-abcdq/z,-aq,-bq,-cq,-dq)_\infty}=1, \tag{2.7}$$

which is precisely Lemma 2.1. □

3. Proof of Theorem 1.3

First we recall from [6] that the series

$$\sum_{\substack{\alpha,\beta,\gamma,\delta,\epsilon,\phi\geq 0\\ s=\alpha+\beta+\gamma+\delta+\epsilon+\phi}}\frac{q^{T_s+T_\delta+T_\epsilon+T_{\phi-1}}(1-q^\alpha(1-q^\phi))a^{\alpha+\delta+\epsilon}b^{\beta+\delta+\phi}c^{\gamma+\epsilon+\phi}}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\epsilon(q)_\phi} \tag{3.1}$$

is the generating function for those partitions λ counted by $\overline{C}(\alpha,\beta,\gamma,\delta,\epsilon,\phi,0;0;n)$. For such a λ, we remove a "staircase", *i.e.*, 1 from the smallest part, 2 from the next smallest, and so on, to get a partition λ'. In terms of generating functions, this corresponds to removing the term q^{T_s} from (3.1). We then take an ordinary partition μ into positive parts having color $abcd$ and weighted by -1 if the number of parts is odd, and put its parts into λ' according to the order (1.4). This corresponds to multiplying (3.1) by $(-1)^\ell q^\ell/(q)_\ell$. Finally we turn λ' into an overpartition by adding on a "generalized" staircase corresponding to the generating function $(dq)^s(-1/d)_s$, where s is now $\alpha+\beta+\gamma+\delta+\epsilon+\phi+\ell$. This is done by augmenting each part by one and then adding, for each j occurring in the partition ν generated by $(-1/d)_s$, one to each of the first j parts and overlining the $j+1$st part. The result is an overpartition counted by $\overline{C}(\alpha,\beta,\gamma,\delta,\epsilon,\phi,\ell;k;n)$, and the generating function is the left hand side of (1.3). By the identity (1.3), the theorem is proved. □

4. Proofs of the Corollaries

Proof of Corollary 1.4. Set $c=0, q=q^3, a=aq^{-2}$, and $b=bq^{-1}$ in Theorem 1.3. On the product side, we have

$$\frac{(-aq,bq^2;q^3)_\infty}{(adq,bdq^2;q^3)_\infty}=\sum_{x_1,x_2,k,n\geq 0}\overline{S}(x_1,x_2;k;n)a^{x_1}b^{x_2}d^kq^n.$$

On the sum side, when $c = 0$, we are left with only 3 colors, ab, a, and b. Under the remaining substitutions, n_a becomes $3n-2$, n_b becomes $3n-1$, and n_{ab} becomes $3n-3$. It is easy to check that the difference conditions (iii) and (iv) in Definition 1.2 translate into those defining $\overline{T}(x_1, x_2; k; n)$. Finally, since there are no $abcd$-parts, the exponent of d is just the number of non-overlined parts. □

Proof of Corollary 1.5. Here we make the substitutions $q = q^6, d = 1, a = q^{-4}, b = q^{-2}$, and $c = q^{-1}$ in Theorem 1.3. On the product side, we have

$$\frac{(-q^2, q^4, q^5; q^6)_\infty}{(q^2, q^4, q^5; q^6)_\infty} = \sum_{n \geq 0} \overline{G}(n) q^n.$$

On the sum side, n_a becomes $6n-4$, n_b becomes $6n-2$, n_c becomes $6n-1$, n_{ab} becomes $6n-6$, n_{ac} becomes $6n-5$, n_{bc} becomes $6n-3$, and n_{abcd} becomes $6n-7$. Conditions (i) and (ii) in the definition of $\overline{H}(n)$ correspond to conditions (i) and (ii) in Definition 1.2.

For the difference conditions, let us first focus on the parts congruent to 5 modulo 6. One can check that the required minimum difference between a part not congruent to 5 modulo 6 and one that is congruent to 5 modulo 6 is the same whether the part congruent to 5 modulo 6 came from an $abcd$ part or a c part. In the case of two parts congruent to 5 modulo 6, we have four possibilities, since each came from either a c part or an $abcd$ part. Again, it is easy to check that the required minimum differences are equal, except when the larger came from a c part and the smaller from an $abcd$ part, in which case the required minimum difference is 6 more than in the other cases. Hence, a any maximal set of consecutive parts, $\{\lambda_i, \lambda_{i+1}, \ldots, \lambda_{i+k}\}$, where each part is congruent to 5 modulo 6 and the difference between any two consecutive parts is the minimum allowed by conditions $(iii) - (iv)$ defining $\overline{H}(n)$, must consist of a string of $abcd$ parts followed by a string of c parts. Given the weight of -1 to the number of $abcd$ parts, this implies that such a maximal sequence must have even length.

Finally, it is one more time easily verified that the minimum required differences in conditions (iii) and (iv) in the definition of $\overline{H}(n)$ follow from the conditions (iii) and (iv) in Definition 1.2. □

Proof of Corollary 1.6. Set $c = 0, d = 1$, $a = -e^{2\pi i/3}$, and $b = -e^{4\pi i/3}$ in Theorem 1.3. The infinite product becomes

$$\frac{(-q)_\infty (q^3; q^3)_\infty}{(q)_\infty (-q^3; q^3)_\infty},$$

the generating function for overpartitions into parts not divisible by 3. On the sum side, only the colors a, b, and ab remain. A chain, from largest to smallest, must consist of a string of b parts followed by a string of a parts, possibly followed by an ab part. Suppose for a moment that there is no ab part in a chain of length k. Then, the weight of the chain will be

$$\begin{aligned}
\sum_{j=0}^{k} a^j b^{k-j} &= [x^k] \frac{1}{(1-ax)(1-bx)} \\
&= [x^k] \frac{1}{(1+e^{2\pi i/3}x)(1+e^{4\pi i/3}x)} \\
&= [x^k](1 + x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + x^{11} \cdots) \\
&= a(k).
\end{aligned}$$

So, taking into account the possibility of an ab part in the chain, the weight of a chain will be $a(k)+a(k-1)$, unless the smallest part of the chain is one, in which case there can be no ab part and then weight is just $a(k)$. □

Proof of Corollary 1.7. Set $b=d=1, a=-e^{2\pi i/6}$, and $c=-e^{-2\pi i/6}$ in Theorem 1.3. On the product side we get $(-q^3;q^3)_\infty/(q^3;q^3)_\infty$, the generating function for overpartitions whose parts are divisible by 3. On the sum side, we want to determine the weight of a chain. If a chain has no ab parts, no ac parts, and no bc parts, then from largest to smallest, it must consist of a string of c parts followed by a string of b parts followed by a string of a parts followed by a string of $abcd$ parts. The weight of such a chain is

$$\begin{aligned}[x^k]\frac{1}{(1+e^{2\pi i/6}x)(1+e^{-2\pi i/6}x)(1+x)(1-x)} &= [x^k]\frac{1}{(1-x^3)(1+x)} \\ &= [x^k](1-x+x^2+x^6-x^7+x^8+x^{12}\cdots).\end{aligned}$$

Call the coefficient of x^k in the final expression $b(k)$. Now suppose that the smallest part of the chain is at least 2 and that the chain does have one or more parts of binary color. By the ordering (1.4), the chain could end with one part in binary color in 3 ways, ab, ac, or ab; it could end in two parts with binary color in 3 ways, $(bc, ab), (ac, ab)$, or (bc, ac); or it could end with three parts of binary color in one way, (bc, ac, ab). Hence the weight of a chain whose smallest part is at least two is $b(k)+b(k-3)$.

Next suppose that the smallest part of a chain is equal to 1. Now ab, bc, ac, and $abcd$ may not be the color of the smallest part, unless 1_a occurs in which case the smallest part may be 1_{bc}. Because $abcd$ is not the smallest part in the chain, there are no $abcd$ parts in the chain. Hence if 1_{bc} is not the smallest part, the weight of the chain is the coefficient of x^k in $1/(1-x^3)=\sum c(k)q^k$. To allow for the possibility of a 1_{bc} preceding a 1_a, the weight of such a chain is $c(k-2)+c(k)$. □

We conclude with the proof of Corollary 1.8, which itself is a corollary of the following:

Corollary 4.1. *For $\ell\in\mathbb{Z}$, the coefficient of $z^\ell q^n$ in*

$$\frac{(q,-zq,-q/z)_\infty}{(-q,zq,q/z)_\infty} \tag{4.1}$$

is equal to the number of overpartitions λ of n having $f(\lambda)\geq|\ell|$, each weighted by $(-1)^{\ell+f(\lambda)}$.

Proof. Let $a=-1, b=z, c=z$, and $d=1$ in Theorem 1.3. The product is (4.1), and we interpret the sum side as in the previous two corollaries using weights of chains. If a chain of length k has no ab, ac, or bc parts, then its weight is the coefficient of x^k in

$$\frac{1}{(1+x)(1-zx)(1-x/z)(1-x)}.$$

If the smallest part of the chain is at least 2, then, allowing for the possibility of ab, ac, and bc parts, the weight is the coefficient of x^k in

$$\frac{1+x(1-z-1/z)+x^2(1-z-1/z)+x^3}{(1+x)(1-zx)(1-x/z)(1-x)}=\frac{1}{1-x},$$

which is 1. Hence the overall weight of an overpartition depends only on the chain whose smallest part is 1 (or $\overline{1}$). As before, the weight of such a chain is the coefficient of x^k in

$$\frac{(1-x)}{(1-xz)(1-x/z)},$$

which is

$$\sum_{\ell=-k}^{k} (-1)^{\ell+k} z^{\ell}.$$

□

Notice that if we set $z=1$ in the above corollary, then the product becomes the generating function for overpartitions. If we set $z=-1$, then the product becomes $(q)_\infty^3/(-q)_\infty^3$, and we can deduce Corollary 1.8:

Proof of Corollary 1.8. Let $z=-1$ in Corollary 4.1. Then each overpartition λ is weighted by

$$(-1)^{f(\lambda)} \sum_{|\ell|<f(\lambda)} 1 = (-1)^{f(\lambda)}(2f(\lambda)+1).$$

Since

$$\frac{(q)_\infty}{(-q)_\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2},$$

the corollary follows. □

5. Concluding Remarks

To finish we would like to offer some suggestions for future research. First, exhibit further overpartition (or partition) theorems that follow from Theorem 1.3. Second, try to find "finite" versions of Lemma 1.1 along the lines of [3]. Third, examine some of the other proofs of Schur's theorem and see if they can be extended to overpartitions. Fourth, find a combinatorial proof of Theorem 1.3 (or any of its corollaries). Finally, study the 11-color partition theorem in [4] and see if it has some kind of overpartition analogue.

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