OVERPARTITIONS

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1. Introduction

An *overpartition* of n is a partition where the first occurrence of each integer may be overlined. For example, there are 14 overpartitions of 4,

$$(4), (\overline{4}), (3,1), (\overline{3},1), (3,\overline{1}), (\overline{3},\overline{1}), (2,2), (\overline{2},2),$$
$$(2,1,1), (\overline{2},1,1), (2,\overline{1},1), (\overline{2},\overline{1},1), (1,1,1,1), (\overline{1},1,1,1).$$

The word overpartition, a contraction of *overlined partition*, was coined by the first author in [16]. Though they have been called a number of other things, like dotted partitions [8], joint partitions [55], or superpartitions [23], the term overpartitions has become standard.

Let $\overline{p}(n)$ denote the number of overpartitions of n, with the usual convention that $\overline{p}(0) = 1$. Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, we have the generating function

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{p}(n) q^n &= \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right) \\ &= 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 24q^5 + \cdots .\end{aligned} \tag{1.1}$$

Note that this generating function is the reciprocal of the classical theta function,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} \left(\frac{1-q^n}{1+q^n} \right).$$

This fact was also noted by Hardy and Ramanujan, who anticipated the importance of the generating function for overpartitions in their celebrated paper that gave birth to the circle method [32]. They wrote that (1.1) "has no very simple arithmetical interpretation; but the series is none the less, as the direct reciprocal of simple ϑ-function, of particular interest" and proceeded to give its asymptotics among the applications of their new method.

While combinatorial objects generated by the series in (1.1) appeared occasionally in the 20th century literature (e.g. [8, 40]), it was perhaps not until a series of papers in the early 2000s that the significance and potential of overpartitions were recognized [16, 17, 18]. In these papers, identities like the q-binomial series, Lebesgue's identity, Ramanujan's ${}_1\psi_1$ summation, and the q-Gauss summation were given combinatorial proofs using overpartitions. Once it became clear that overpartitions were intimately related to q-series, the floodgates were opened. In the two subsequent decades, further q-series identities found combinatorial proofs thanks to overpartitions [19, 29, 63, 67]; Andrews' theory of q-difference equations for well-poised basic hypergeometric series, which encompasses the Rogers-Ramanujan-Gordon identities, was applied to overpartitions [6, 12, 13, 20, 21, 41, 42, 45, 46, 49]; classical partition

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identities like Schur's theorem were found to have overpartition versions [24, 25, 26, 27, 33, 43, 48, 56]; a theory of plane overpartitions was developed [7, 22]; Ramanujan-type congruences and other arithmetic properties of overpartitions were studied [14, 15, 30, 36, 37, 52, 53, 58, 60, 61, 62]; the Dyson-Atkin-Swinnerton-Dyer theory of partition ranks was extended to overpartitions [39, 44, 47, 50, 59, 66]; connections were made between overpartitions and Maass forms [9, 10, 11, 38, 54], and much, much more, including many discoveries that have no counterpart for ordinary partitions.

Any attempt to summarize all of this in the space of a few pages is bound to fall woefully short. We therefore limit ourselves to brief discussions of a few of the highlights – Ramanujan's ${}_1\psi_1$ summation, overpartition ranks and mock theta functions, overpartition identities of the Rogers-Ramanujan type, and plane overpartitions.

2. Combinatorics and q-series identities – Ramanujan's ${}_1\psi_1$ summation

To illustrate the interaction between q-series identities and overpartitions, it seems fitting to go back to one of the very first examples – Ramanujan's ${}_1\psi_1$ summation. Using the standard notation,

$$(a;q)_\infty = \prod_{k\geq 0}(1-aq^k)$$

and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

this summation may be written

$$\frac{(-aq;q)_\infty(-bq;q)_\infty}{(abq;q)_\infty(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1/a;q)_n(azq)^n}{(-bq;q)_n} = \frac{(-zq;q)_\infty(-1/z;q)_\infty}{(b/z;q)_\infty(azq;q)_\infty}, \tag{2.1}$$

where $|b| < |z| < \frac{1}{|aq|}$ and $|q| < 1$.

A short calculation reveals that the identity is true if and only if the constant terms in the variable z agree on both sides. On the left hand side, the constant term is

$$\frac{(-aq;q)_\infty(-bq;q)_\infty}{(abq;q)_\infty(q;q)_\infty}, \tag{2.2}$$

the generating function for pairs of overpartitions. (The variables a and b track certain statistics, but we suppress this here.) The constant term on the right-hand side is more subtle, but it turns out that it is the generating function for certain Frobenius symbols; namely, two-rowed arrays

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix},$$

where each row is an overpartition into non-negative parts.

To prove Ramanujan's ${}_1\psi_1$ summation, then, one needs to prove that (2.2) is the generating function for such Frobenius symbols. This was accomplished in [17] via a combinatorial mapping between the Frobenius symbols and pairs of overpartitions. As a bonus, the very same mapping furnishes a combinatorial proof of the q-Gauss summation,

$$\sum_{n\geq 0}\frac{(a;q)_n(b;q)_n(c/ab)^n}{(c;q)_n(q;q)_n} = \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}.$$

When $a, b \to 0$ in (2.1), one recovers Jacobi's triple product identity,

$$\frac{1}{(q;q)_\infty} \sum_{n \in \mathbb{Z}} z^n q^{\binom{n+1}{2}} = (-1/z;q)_\infty (-zq;q)_\infty,$$

and in this case the mapping reduces to the classical correspondence between a partition and its Frobenius symbol.

3. Rank differences and mock theta functions

Freeman Dyson defined the rank of a partition to be the largest part minus the number of parts, and famously conjectured that the rank could be used to refine two of Ramanujan's partition congruences [28],

$$p(5n+4) \equiv 0 \pmod{5}, \tag{3.1}$$

$$p(7n+5) \equiv 0 \pmod{7}. \tag{3.2}$$

Specifically, if $N(a,m,n)$ denotes the number of partitions of n whose rank is congruent to a modulo m, then Dyson conjectured that for all $0 \leq a, a' \leq 4$ and $0 \leq b, b' \leq 6$ one has

$$N(a,5,5n+4) - N(a',5,5n+4) = 0$$

and

$$N(b,7,7n+5) - N(b',7,7n+5) = 0.$$

Dyson's conjecture was proven by Atkin and Swinnerton-Dyer [5], who not only showed that the above rank differences are 0 but also computed formulas for the generating functions for the rank differences in every arithmetic progression $5n+d$ and $7n+d$. One of the interesting aspects of these generating functions is that a number of them are related to Ramanujan's 5th or 7th order mock theta functions. To give just one example [35],

$$\sum_{n \geq 0} \left(N(0,7,7n) - N(2,7,7n)\right) q^n = \mathcal{F}_0(q),$$

where

$$\mathcal{F}_0(q) = \sum_{n \geq 0} \frac{q^{n^2}}{(q^{n+1};q)_n}$$

is one of Ramanujan's 7th order mock theta functions.

In the case of overpartitions, the methods of Atkin and Swinnerton-Dyer can be adapted to prove formulas for the generating functions for rank differences [39, 50, 59]. While there are no simple congruences like (3.1) or (3.2) for $\overline{p}(n)$, the rank differences still have nice generating functions and connections to mock theta functions. For example, if $\overline{N}(a,m,n)$ denotes the number of overpartitions of n with rank congruent to a modulo m, we have [51, 59]

$$\sum_{n \geq 0} \left(\overline{N}(0,5,5n+1) - \overline{N}(2,5,5n+1)\right) q^n = 2\phi(q)$$

and

$$\sum_{n \geq 0} \left(\overline{N}(1,6,3n+2) - \overline{N}(3,6,3n+2)\right) q^n = 2\omega(q),$$

where

$$\phi(q) = \sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}}$$

is one of Ramanujan's 10th order mock theta functions and

$$\omega(q) = \sum_{n\geq 0} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}$$

is one of Ramanujan's 3rd order mock theta functions.

The most striking results arise, however, when considering the overpartition rank differences modulo 2. Here the rank differences turn out to be related to the Hurwitz class numbers $H(n)$ [10]. We recall that $H(n)$ is the number of equivalence classes of positive definite binary quadratic forms of discriminant $-n$, with classes containing a multiple of x^2+y^2 or x^2+xy+y^2 counted with weight 1/2 and 1/3, respectively. The generating function for $H(n)$,

$$\mathcal{H}(q) = -\frac{1}{12} + \sum_{n\geq 1} H(n)q^n,$$

is known as the Zagier-Eisenstein series. This was perhaps the first example of a mock modular form that explicitly appeared in the literature [65], long before the notion was formalized. Specifically, it is a weight 3/2 mock modular form with shadow proportional to the classical theta function

$$\Theta(q) = \sum_{n\in\mathbb{Z}} q^{n^2}.$$

It turns out that the same is true for the generating function for $(-1)^n(\overline{N}(0,2,n)-\overline{N}(1,2,n))$, and this gives rise to the identity

$$\sum_{n\geq 0} \left(\overline{N}(0,2,n) - \overline{N}(1,2,n)\right)(-1)^n q^n = -16\mathcal{H}(q) - \frac{1}{3}\Theta^3(q).$$

Since the coefficients of $\Theta^3(q)$ can also be expressed in terms of $H(n)$, this leads to results like

$$\overline{N}(0,2,8n+3) - \overline{N}(1,2,8n+3) = 24H(8n+3).$$

This is all in stark contrast to the case of partitions, where the generating function for $N(0,2,n) - N(1,2,n)$ is Ramanujan's 3rd order mock theta function $f(q)$.

4. Overpartition identities

Since Euler discovered that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts, the search for such identities has been one of the principal themes in the theory of partitions. The most famous of them all are the Rogers-Ramanujan identities, which correspond to the q-series identities of the same name,

$$\sum_{n\geq 0} \frac{q^{n^2+(2-i)n}}{(q;q)_n} = \frac{1}{(q^{3-i};q^5)_\infty (q^{2+i};q^5)_\infty}.$$

Here $i = 1$ or 2. In terms of partitions, the Rogers-Ramanujan identities state that the number of partitions of n into parts differing by at least two with at most $i-1$ occurrences of 1 is equal to the number of partitions into parts not congruent to $0, \pm i$ modulo 5.

The Rogers-Ramanujan identities were famously generalized by Gordon [31], whose result may be stated as follows: For $k \geq 2$ and $1 \leq i \leq k$, let $A_{k,i}(n)$ denote the number of partitions of n where at most $i-1$ of the parts are equal to 1 and the total number of occurrences of j and $j+1$ together is at most $k-1$ for any integer j, and let $B_{k,i}(n)$ denote the number of partitions of n into parts not congruent to $0, \pm i$ modulo $2k+1$. Then

$$A_{k,i}(n) = B_{k,i}(n).$$

Note that the case $k = 2$ recovers the Rogers-Ramanujan identities.

Andrews showed how Gordon's result could be obtained from an analysis of certain well-poised q-hypergeometric series [1],

$$H_{k,i}(a;x;q) = \sum_{n\geq 0} \frac{(-a)^n q^{kn^2+n-in} x^{kn}(1-x^i q^{2ni})(-1/a;q)_n(-axq^{n+1};q)_\infty}{(q;q)_n(xq^n;q)_\infty}$$

and

$$J_{k,i}(a;x;q) = H_{k,i}(a;xq;q) + axqH_{k,i-1}(a;xq;q).$$

It turns out that these series (i) satisfy simple q-difference equations and (ii) become infinite products under various specializations of the parameters. This not only leads to the Rogers-Ramanujan-Gordon identities, but a large number of other identities of this type [4, Chapter 7].

The series $H_{k,i}(a;x;q)$ and $J_{k,i}(a;x;q)$ and their q-difference equations apply naturally to overpartitions as well. The second author [41] and Chen-Sang-Shi [12] have used this to find an overpartition version of the Rogers-Ramanujan-Gordon theorem. For brevity we record only the case $i = k$: Let $\overline{A}_{k,k}(n)$ denote the number of overpartitions of n where parts occur at most $k-1$ times, and where the total number of occurrences of j and $j+1$ together is at most k if j occurs overlined and at most $k-1$ otherwise. Let $\overline{B}_{k,k}(n)$ denote the number of overpartitions of n into parts not divisible by k. Then

$$\overline{A}_{k,k}(n) = \overline{B}_{k,k}(n).$$

Another famous partition identity is due to Schur [57]. One version of Schur's theorem says that if $S(n)$ denotes the number of partitions of n into distinct parts not divisible by 3 and $T(n)$ denotes the number of partitions of n where consecutive parts differ by at least 3 with multiples of 3 differing by at least 6, then

$$S(n) = T(n).$$

In [43], the second author found an overpartition version of Schur's theorem. It says that if $\overline{S}(k,n)$ denotes the number of overpartitions of n into parts not divisible by 3, k of which are non-overlined, and $\overline{T}(k,n)$ denotes the number of overpartitions of n with k non-overlined parts where consecutive parts differ by at least 3 if the smaller is overlined or both parts are divisible by 3, and by at least 6 if the smaller is overlined and both parts are divisible by 3, then

$$\overline{S}(k,n) = \overline{T}(k,n).$$

Note that this is a "perfect" generalization of Schur's partition theorem, in the sense that if there are no non-overlined parts then we recover Schur's result. Schur's theorem was generalized in two different ways by Andrews, from the modulus 3 to the modulus $2^k - 1$ [2, 3], and Andrews' theorems were both generalized to overpartitions by Dousse [25, 26].

5. Plane overpartitions

A plane partition is a two-dimensional array of nonnegative integers where each row and each column is a partition. An example of a plane partition Π is:

$$\begin{array}{cccc} 4 & 4 & 4 & 3 \\ 4 & 4 & 3 & 1 \\ 3 & 3 & & \\ 3 & & & \end{array}$$

Plane partitions generalize ordinary partitions. Indeed a plane partition with one row is a partition. We denote by $|\Pi|$ the sum of the entries of the plane partition. In the example above $|\Pi| = 36$. The generating function of plane partitions was computed by MacMahon :

$$\sum_{\substack{\Pi \text{ a plane} \\ \text{partition}}} q^{|\Pi|} = \prod_{i \geq 1} \left(\frac{1}{1-q^i} \right)^i .$$

A plane partition may be represented visually by the placement of a stack of unit cubes in the corner of a room or as a rhombus tiling.

A plane overpartition [22] is a filling of a Ferrers diagram where (i) each row is an overpartition where the final occurrence of an integer may be overlined, (ii) each column is an overpartition where the first occurrence of an integer can be overlined or not and all the other occurrences of this integer are overlined. An example of a plane overpartition Π is:

$$\begin{array}{cccc} 4 & 4 & \bar{4} & \bar{3} \\ \bar{4} & 3 & 3 & \bar{3} \\ \bar{4} & \bar{3} & & \\ 3 & & & \end{array}$$

We denote by $|\Pi|$ the sum of the entries of the plane overpartition and by $o(\Pi)$ the number of overlined entries. In the example above $|\Pi| = 38$ and $o(\Pi) = 6$.

The generating function of plane overpartitions is

$$\sum_{\substack{\Pi \text{ a plane} \\ \text{overpartition}}} a^{o(\Pi)} q^{|\Pi|} = \prod_{i \geq 1} \frac{(1+aq^i)^i}{(1-q^i)^{\lceil i/2 \rceil} (1-a^2q^i)^{\lfloor i/2 \rfloor}} .$$

If $a = 1$, one can think of this as the overpartition analogue of the generating function of plane partitions,

$$\sum_{\substack{\Pi \text{ a plane} \\ \text{overpartition}}} q^{|\Pi|} = \prod_{i \geq 1} \frac{(1+q^i)^i}{(1-q^i)^i}$$

However, if no entry is overlined (i.e. if $a = 0$), we recover a plane partition where all the columns are strict. These objects are naturally related to Schur functions, while plane overpartitions are related to P and Q-Schur functions and super Schur functions.

Another natural analogue of a plane partition is called a pyramid partition [64]. Pyramid partitions are infinite stacks of square bricks in the shape of a pyramid where a finite number of square bricks have been removed. These are in bijection with pairs of plane overpartitions of the same shape and arise naturally from a computation of Donaldson-Thomas invariants of orbifolds. Their generating function is :

$$\prod_{i\geq 1} \frac{(1+aq^{2i-1})^{2i-1}}{(1-q^{2i})^i(1-a^2q^{2i})^i}.$$

When $a=0$ and q is replaced by its square root, we recover the generating function of plane partitions. Both plane overpartitions and pyramid partitions are in bijection with certain domino tilings [7, 22].

6. Conclusion

Overpartitions are to be found wherever partitions play a role. It is now common practice to ask "What about the overpartition case?" when one proves a theorem about partitions. This is often much more than a simple exercise involving a routine generalization. Sometimes even discovering the right definition can be a conundrum. And as we have seen in Section 3, sometimes the overpartition case holds an unexpected surprise. Moreover, overpartitions can be crucial to understanding partitions. The best example of this is a recent proof of a very general partition identity conjectured by Bressoud, where overpartitions played a key role [34]. As the authors wrote, "It should be stressed that the overpartition analogues considered in this paper are not merely a matter of extension and specialization. In fact, they play an essential role and serve as an indispensable structure in tackling the conjecture of Bressoud formulated in terms of ordinary partitions."

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