# CONSTANT TERMS, JAGGED PARTITIONS, AND PARTITIONS WITH DIFFERENCE TWO AT DISTANCE TWO

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Abstract. In this paper we establish a group of constant term identities that contain generating functions for "jagged" partitions and interpret these identities as theorems about partitions with difference two at distance two.

## 1. Introduction

Following [9], a *jagged partition* refers to an ordered collection of non-negative integers $(n_1, n_2, \ldots, n_m)$ with $n_m \geq p$ for some positive integer $p$, further subject to some weakly decreasing conditions that prevent it from being a genuine partition. The prototypical example, which first arose in the study of graded parafermions in mathematical physics [7, 8, 9], is the 01-jagged partition, corresponding to $p = 1$ and the conditions $n_i \geq n_{i+1} - 1$ and $n_i \geq n_{i+2}$. For example, the 8 01-jagged partitions of 3 are

$$(3), (2, 1), (1, 2), (2, 0, 1), (1, 1, 1), (1, 1, 0, 1), (1, 0, 1, 0, 1), (0, 1, 0, 1, 0, 1).$$

By a partition with difference two at distance two we mean a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying $\lambda_i - \lambda_{i+2} \geq 2$. These partitions are well-known as the successors to the Rogers-Ramanujan partitions in Gordon's hierarchy of partition identities [11]. They are naturally identified with 01-jagged partitions in the following way: if we remove a *staircase* $0, 1, \ldots, n-1$ from such a partition $\lambda$, that is, if we subtract 0 from the smallest part, 1 from the next smallest part, and so on, the result is a 01-jagged partition. For example, the 01-jagged partition associated to the partition $\lambda = (11, 10, 7, 7, 5, 3, 3, 1)$ is $(4, 4, 2, 3, 2, 1, 2, 1)$.

If $j(m, n)$ denotes the number of 01-jagged partitions of $n$ with $m$ parts, then we have the generating function

$$\sum_{m,n\geq 0} j(m, n) z^m q^n = \prod_{n\geq 1} \frac{(1 + zq^n)}{(1 - z^2q^n)}. \tag{1.1}$$

This generating function is immediate once we make the observation that a 01-jagged partition may be decomposed into "levels" indexed by the natural numbers, the general level $n$ being a sequence of $n$'s followed by a sequence of pairs $(n - 1, n)$,

$$(n, n, n, n, \ldots, n, n - 1, n, n - 1, n, \ldots, n - 1, n).$$

The pairs $(n - 1, n)$ combine to form repeatable odd parts $2n - 1$, pairs of $n$'s combine to form repeatable even parts $2n$, and the eventual left over $n$ contributes to a partition into distinct parts.

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In the first part of this paper we will prove, without any unreasonable difficulty, a handful of constant term identities involving the generating function for 01-jagged partitions (1.1). In stating these identities, we use the usual base $q$ hypergeometric notation

$$(a_1, a_2, \ldots, a_k; q)_\infty := \prod_{m \geq 0} (1 - a_1 q^m)(1 - a_2 q^m) \cdots (1 - a_k q^m) \tag{1.2}$$

and

$$(a_1, a_2, \ldots, a_k; q)_n = \frac{(a_1, a_2, \ldots, a_k; q)_\infty}{(a_1 q^n, a_2 q^n, \ldots, a_k q^n; q)_\infty}, \tag{1.3}$$

the latter being valid for all integers $n$. We also follow the custom of dropping the "$;q$" unless, of course, the base is something other than $q$.

**Theorem 1.1.** *Employing the notation $[z^0] \sum a(n) z^n = a(0)$, we have*

*(i)*

$$[z^0] \frac{(-zq, -zq, -azq, -zq/a, -1/z, q)_\infty}{(z^2 q)_\infty} = (-q)_\infty (-aq, -q/a; q^2)_\infty. \tag{1.4}$$

*(ii)*

$$[z^0] \frac{(-zq, -zq, -zq, -1/z, q)_\infty}{(z^2 q)_\infty} = \frac{1}{(q, q^4; q^5)_\infty^2}. \tag{1.5}$$

*(iii)*

$$[z^0] \frac{(-zq, -zq, -1/z, q)_\infty}{(z^2 q; q^2)_\infty} = \frac{(q^3; q^3)_\infty}{(q)_\infty (-q^3; q^3)_\infty}. \tag{1.6}$$

*(iv)*

$$[z^0] \frac{(-azq, -zq, -1/z, q)_\infty}{(-az^2 q; q^2)_\infty} = (-aq; q^2)_\infty. \tag{1.7}$$

*(v)*

$$[z^0] \frac{(-zq, -1/z, q)_\infty}{(z^2 q; q^2)_\infty} \sum_{s \geq 0} \frac{q^{\binom{s+1}{2}} (-z)^s}{(q^2; q^2)_s} = \frac{(q; q^2)_\infty (-q^3, -q^5, q^8; q^8)_\infty}{(q^2; q^2)_\infty}. \tag{1.8}$$

After proving these identities, we show that they have some rather attractive interpretations as theorems about partitions having the difference two at distance two condition. The fact that such identities lead to nice partition theorems should not come as a surprise, given the importance of constant terms in the modern theory of partitions [2, 3, 12].

Before stating these interpretations, let us make a few definitions and observations about partitions with difference two at distance two. First, we'll say that a part $i$ in such a partition is *isolated* if it is not repeated and neither $i-1$ nor $i+1$ occurs. Next, by a *sequence* we'll mean a set of distinct consecutive parts $i+k, i+k-1, \ldots, i$, for some $k \geq 1$, where neither $i-1$ nor $i+k+1$ occurs. Finally, a *pair* will mean the obvious thing, a pair of equal parts $(i, i)$.

Now any partition $\lambda$ with difference two at distance two may be uniquely decomposed into pairs, sequences, and isolated parts. For example, the partition $\lambda = (11, 10, 7, 7, 5, 3, 3, 1)$ is comprised of the pairs $(7, 7)$ and $(3, 3)$, the sequence $(11, 10)$, and the isolated parts 5 and 1. We shall use the notations $p(\lambda)$, $s(\lambda)$, and $i(\lambda)$ to denote the number of pairs, sequences, and isolated parts, respectively, of such a partition $\lambda$.

**Theorem 1.2.** *Let $A_1(i; n)$ denote the number of partitions of $n$ where parts may be labelled $a$, $1/a$, and $u$ (for uncolored), where no part may repeat, where colored parts are odd, and where $i$ is the number of parts colored $a$ minus the number of parts colored $1/a$. Let $B_1(i; n)$ denote the number of partitions of $n$ where parts occur in the same three colors, where again $i$ is the number of parts colored $a$ minus the number of parts colored $1/a$, where $\lambda_i - \lambda_{i+2} \geq 2$ and where the minimal difference between successive parts $\lambda_i$ and $\lambda_{i+1}$ is given by the matrix*

| | $a$ | $1/a$ | $u$ |
|---|---|---|---|
| $a$ | 2 | 1 | 1 |
| $1/a$ | 2 | 2 | 1 |
| $u$ | 2 | 2 | 0. |

*Here the $(x, y)$ entry designates the minimal difference between a part of color $x$ and a part of color $y$. Then $A_1(i; n) = B_1(i; n)$.*

This theorem has a number of interesting corollaries, some of which we record here. The first may be compared to the Capparelli theorem [1, 5]. The last two are weighted identities, where partitions $\lambda$ are counted with a certain weight $w(\lambda)$.

**Corollary 1.3.** *Let $C_1(n)$ denote the number of partitions of $\lambda$ of $n$ with no ones where $\lambda_i - \lambda_{i+2} \geq 5$, and where $\lambda_i - \lambda_{i+1} \geq 4$, unless $(\lambda_i, \lambda_{i+1}) \equiv (2, 3)$, $(3, 3)$, or $(3, 2)$ modulo 3, the latter case requiring $\lambda_i - \lambda_{i+1} \geq 7$. Let $D_1(n)$ denote the number of partitions into parts congruent to $\pm 2$ or $\pm 3$ modulo 12. Then $C_1(n) = D_1(n)$.*

**Corollary 1.4.** *Let $C_2(n)$ denote the number of partitions $\lambda$ of $n$ with difference two at distance two where $i(\lambda) = 0$. Let $D_2(n)$ denote the number of partitions of $n$ into distinct parts where odd parts are congruent to 3 modulo 6. Then $C_2(n) = D_2(n)$.*

**Corollary 1.5.** *Let $C_3(n)$ denote the number of partitions $\lambda$ of $n$ with difference two at distance two where $w(\lambda) = 2^{s(\lambda)}$. Let $D_3(n)$ denote the number of partitions of $n$ where even parts are distinct and congruent to 2 modulo 4. Then $C_3(n) = D_3(n)$.*

**Corollary 1.6.** *Let $C_4(n)$ denote the number of partitions $\lambda$ of $n$ with difference two at distance two, where $w(\lambda) = 2^{i(\lambda)} 3^{s(\lambda)}$. Let $D_4(n)$ denote the number of partitions $\lambda$ of $n$ where only odd parts not congruent to 3 modulo 6 may repeat, weighted by $2^{f(\lambda)}$, where $f(\lambda)$ is the number of repeatable parts. Then $C_4(n) = D_4(n)$.*

The next theorem, corresponding to (1.5), gives an interpretation of the square of the product in the first Rogers-Ramanujan identity.

**Theorem 1.7.** *Let $A_2(n)$ denote the number of partitions $\lambda$ of $n$ where parts occur in two colors, say overlined and non-overlined, where $\lambda_i - \lambda_{i+2} \geq 2$ and where*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 2, & \lambda_{i+1} \text{ overlined} \\ 1, & \lambda_{i+1} \text{ non-overlined and } \lambda_i \text{ overlined} \\ 0, & \text{otherwise.} \end{cases}$$

*Let $B_2(n)$ denote the number of partitions where parts occur in two colors and where all parts are equivalent to 1 or 4 modulo 5. Then $A_2(n) = B_2(n)$.*

The last three theorems concern partitions with no sequences.

**Theorem 1.8.** *Let $A_3(n)$ denote the number of partitions of $n$ having no sequences and where parts occur at most twice. Let $B_3^{\pm}(n)$ denote the number of partitions of $n$ with an even/odd number of parts divisible by 3. Then $A_3(n) = B_3^{+}(n) - B_3^{-}(n)$.*

**Theorem 1.9.** *Let $A_4^{\pm}(m,n)$ denote the number of partitions $\lambda$ counted by $A_3(n)$ where the number of integers that are repeated is even/odd and where $p(\lambda)+i(\lambda) = m$ . Let $B_4(m,n)$ denote the number of partitions of $n$ into $m$ distinct odd parts. Then $A_4^{+}(m,n) - A_4^{-}(m,n) = B_4(m,n)$.*

Notice that, *a priori*, there is no reason to believe that the functions $B_3^{+}(n) - B_3^{-}(n)$ or $A_4^{+}(m,n) - A_4^{-}(m,n)$ are non-negative, but the above theorems ensure that they are.

**Theorem 1.10.** *Let $A_5(n)$ denote the number of partitions of $n$ counted by $A_3(n)$ where even parts must be repeated. Let $A_5^{\pm}(n)$ denote the number of partitions of counted by $A_5(n)$ where the number of non-repeated (odd) parts is even/odd. Let $B_5(n)$ denote the number of partitions of $n$ where odd parts are distinct and not $\pm 3$ or $\pm 5$ modulo 16, and where even parts are congruent to $\pm 2$ or $\pm 4$ modulo 16. Let $B_5^{\pm}(n)$ denote the number of partitions counted by $B_5(n)$ where the number of odd parts is even (odd). Then $B_5^{+}(n) - B_5^{-}(n) = A_5^{+}(n) - A_5^{-}(n)$.*

In the next section we prove the constant term identities in Theorem 1.1, and in Section 3 the partition-theoretic interpretations and corollaries are established. In Section 4 we briefly discuss an alternative way to interpret the constant term identities - in terms of generalized Frobenius partitions. Some final remarks are offered in Section 5.

## 2. Proof of Theorem 1.1

In all cases, the proofs are accomplished by using various corollaries of the ${}_1\psi_1$ summation [10],

$$\sum_{n\in\mathbb{Z}} \frac{(-1/a)_n (azq)^n}{(-bq)_n} = \frac{(-zq,-1/z,q,abq)_\infty}{(azq,b/z,-aq,-bq)_\infty}, \tag{2.1}$$

to expand the left hand side as a multiple series and then simplifying. These corollaries of (2.1) are the case $a = b = 0$,

$$\sum_{n\in\mathbb{Z}} z^{-n} q^{\binom{n}{2}} = (-zq,-1/z,q)_\infty, \tag{2.2}$$

the case $a \to 0$ and $b = -1$,

$$\sum_{n\geq 0} \frac{z^n q^{\binom{n+1}{2}}}{(q)_n} = (-zq)_\infty, \tag{2.3}$$

and the case $z = z/aq$, $b = -1$, and $a \to \infty$,

$$\sum_{n\geq 0} \frac{z^n}{(q)_n} = \frac{1}{(z)_\infty}. \tag{2.4}$$

We begin with (1.4). Using (2.2) to expand $(-zq, -1/z, q)_\infty$, (2.3) to expand the terms $(-zq)_\infty$, $(-azq)_\infty$, and $(zq/a)_\infty$, and (2.4) to expand $1/(z^2q)_\infty$, we can write the left hand side of (1.4) as

$$[z^0]\sum_{n\in\mathbb{Z}} z^{-n}q^{\binom{n}{2}} \sum_{r\geq 0} \frac{a^r q^{\binom{r+1}{2}} z^r}{(q)_r} \sum_{s\geq 0} \frac{a^{-s} q^{\binom{s+1}{2}} z^s}{(q)_s} \sum_{t\geq 0} \frac{q^{\binom{t+1}{2}} z^t}{(q)_t} \sum_{u\geq 0} \frac{q^u z^{2u}}{(q)_u}, \tag{2.5}$$

which is

$$\sum_{r,s,t\geq 0} \frac{q^{\binom{r+s+t+2u}{2}+\binom{r+1}{2}+\binom{s+1}{2}+\binom{t+1}{2}+u} a^{r-s}}{(q)_r(q)_s(q)_t(q)_u}.$$

We now make the shift of variables $r \to r-u$ and $s \to s-u$. Using the fact that

$$(q)_{n-k} = \frac{(q)_n(-1)^k q^{\binom{k}{2}-nk}}{(q^{-n})_k}, \tag{2.6}$$

we end up with

$$\sum_{\substack{r,s,t,u\geq 0\\ u\leq \min\{r,s\}}} \frac{q^{\binom{r+s+t}{2}+\binom{r+1}{2}+\binom{s+1}{2}+\binom{t+1}{2}+u} a^{r-s}(q^{-r}, q^{-s})_u}{(q)_r(q)_s(q)_t(q)_u}.$$

We can now sum over $u$ using the case $c = 0$ of the $q$-Chu-Vandermonde identity,

$$\sum_{k=0}^{n} \frac{(q^{-n}, a)_k q^k}{(q, c)_k} = \frac{a^n (c/a)_n}{(c)_n}, \tag{2.7}$$

to get

$$\sum_{r,s,t\geq 0} \frac{q^{r^2+s^2+t^2+rt+st} a^{r-s}}{(q)_r(q)_s(q)_t}.$$

Now shift the variables $r$ and $s$ again, this time letting $r \to r-t$ and $s \to s-t$. Again using (2.6) to simplify and summing over $t$ using (2.7), we have

$$\begin{aligned}
\sum_{r,s\geq 0} \frac{q^{r^2+s^2-rs} a^{r-s}}{(q)_r(q)_s} &= [z^0]\sum_{n\in\mathbb{Z}} z^{-n}q^{\binom{n}{2}} \sum_{r\geq 0} \frac{a^r z^{-r} q^{\binom{r}{2}}}{(q)_r} \sum_{s\geq 0} \frac{a^{-s} z^s q^{\binom{s+1}{2}}}{(q)_s} \\
&= [z^0](q, -zq, -1/z, -a/z, -zq/a)_\infty \\
&= \frac{1}{(q)_\infty}[z^0]\sum_{n\in\mathbb{Z}} z^{-n}q^{\binom{n}{2}} \sum_{n\in\mathbb{Z}} z^n a^{-n} q^{\binom{n+1}{2}} \\
&= \frac{1}{(q)_\infty}\sum_{n\in\mathbb{Z}} a^{-n}q^{n^2} \\
&= (-q)_\infty(-aq, -q/a; q^2)_\infty,
\end{aligned}$$

and the proof of (1.4) is complete.

For (1.5), the left hand side may expanded to give

$$
\begin{aligned}
[z^0] & \sum_{n\in\mathbb{Z}} q^{\binom{n}{2}} z^{-n} \sum_{r\geq 0} \frac{z^r q^{\binom{r+1}{2}}}{(q)_r} \sum_{s\geq 0} \frac{z^s q^{\binom{s+1}{2}}}{(q)_r} \sum_{t\geq 0} \frac{z^{2t}q^t}{(q)_t} \\
= & \sum_{r,s,t\geq 0} \frac{q^{\binom{r+s+2t}{2}+\binom{r+1}{2}+\binom{s+1}{2}+t}}{(q)_r(q)_s(q)_t}.
\end{aligned}
$$

Shifting $r \to r-t$ and $s \to s-t$ and simplifying using (2.6) yields

$$
\sum_{\substack{r,s,t\geq 0 \\ t\leq \min\{r,s\}}} \frac{q^{r^2+s^2+rs+t}(q^{-r},q^{-s})_t}{(q)_r(q)_s(q)_t}.
$$

Now the sum on $t$ can be carried out using (2.7), and the result is

$$
\sum_{r,s\geq 0} \frac{q^{r^2+s^2}}{(q)_r(q)_s},
$$

which, in light of the first Rogers-Ramanujan identity,

$$
\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{2.8}
$$

finishes the proof of (1.5).

Next, for (1.6), expanding the left hand side yields

$$
\begin{aligned}
[z^0] & \sum_{n\in\mathbb{Z}} z^{-n} q^{\binom{n}{2}} \sum_{r\geq 0} \frac{z^r q^{\binom{r+1}{2}}}{(q)_r} \sum_{s\geq 0} \frac{z^{2s}q^s}{(q^2;q^2)_s} \\
= & \sum_{r,s\geq 0} \frac{q^{\binom{r+2s}{2}+\binom{r+1}{2}+s}}{(q)_r(q^2;q^2)_s} \\
= & \sum_{r,s\geq 0} \frac{q^{2s^2+2rs+r^2}}{(q)_r(q^2;q^2)_s}.
\end{aligned}
$$

Now this last sum is equal to the right hand side of (1.6) by the case $k=3, \delta=0$, and $r=k$ of the discussion in [6, p.387-389].

Now, for (1.7), the left hand side equals

$$
\begin{aligned}
[z^0]\sum_{n\in\mathbb{Z}} z^{-n}q^{\binom{n}{2}}\sum_{r\geq 0}\frac{z^r q^{\binom{r+1}{2}}a^r}{(q)_r}\sum_{s\geq 0}\frac{(-1)^s z^{2s}q^s a^s}{(q^2;q^2)_s} &= \sum_{r,s\geq 0}\frac{q^{2s^2+r^2+2rs}(-1)^s a^{r+s}}{(q)_r(q^2;q^2)_s}\\
&= [z^0]\sum_{n\in\mathbb{Z}} z^{-n}q^{n^2}\sum_{r\geq 0}\frac{z^r a^r}{(q)_r}\sum_{s\geq 0}\frac{q^{s^2}z^s(-1)^s a^s}{(q^2;q^2)_s}\\
&= [z^0]\frac{(q^2,-zq,-q/z,azq;q^2)_\infty}{(az)_\infty}\\
&= [z^0]\frac{(q^2,-zq,-q/z;q^2)_\infty}{(az;q^2)_\infty}\\
&= [z^0]\sum_{n\in\mathbb{Z}} z^{-n}q^{n^2}\sum_{n\geq 0}\frac{z^n a^n}{(q^2;q^2)_n}\\
&= (-aq;q^2)_\infty.
\end{aligned}
$$

Finally, we address (1.8). Arguing as above, we find that the left hand side of (1.8) is

$$\sum_{r,s\geq 0}\frac{(-1)^s q^{2r^2+2rs+s^2}}{(q^2;q^2)_r(q^2;q^2)_s}.$$

Now, we can sum over $s$ using (2.3), and we have

$$
\begin{aligned}
&\sum_{r\geq 0}\frac{q^{2r^2}(q^{2r+1};q^2)_\infty}{(q^2;q^2)_r}\\
&= (q;q^2)_\infty\sum_{r\geq 0}\frac{q^{2r^2}}{(q)_{2r}},
\end{aligned}
$$

and this is equal to the right hand side of (1.8) upon invoking [13, Eq. (39)]. The proof of Theorem 1.1 is now complete. □

## 3. The partition-theoretic interpretations

In each of the constant term identities (1.4) - (1.8) there appears on the left hand side the term $F(z) = (-zq, -1/z, q)_\infty$, which by the triple product identity is

$$F(z) = \sum_{n\in\mathbb{Z}} z^{-n}q^{\binom{n}{2}}.$$

This is exploited by applying the following straightforward lemma, which sets the framework for our interpretation of the constant term identities:

**Lemma 3.1.** *If $G(z)$ is a power series in $z$ whose coefficient of $z^n$ is the generating function for some kind of partitions $\lambda$ having $n$ parts, then $[z^0]F(z)G(z)$ is the generating function for those partitions obtained by adding a staircase $0, 1, \ldots, n-1$ to such partitions $\lambda$.*

For a simple illustration of Lemma 3.1, take $G(z)$ to be the power series in $z$ whose coefficient of $z^n$ is the generating function for partitions into $n$ parts, $q^n/(q)_n$. Adding a staircase $0, 1, \dots, n-1$ to such a partition gives a partition into distinct parts, and indeed, the generating function for partitions into distinct parts is

$$(-q)_\infty = \sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}}{(q)_n} = [z^0]F(z) \sum_{n\geq 0} \frac{z^n q^n}{(q)_n}.$$

Now we would like to prove Theorem 1.2, interpreting the constant term identity (1.4) in Theorem 1.1 using the framework of Lemma 3.1. To do so, consider the product

$$\frac{(-zq, -azq, -zq/a)_\infty}{(z^2q)_\infty} \tag{3.1}$$

The coefficient of $z^{n_1}$ in the term $(-zq)_\infty/(z^2q)_\infty$ is the generating function for uncolored jagged partitions $\lambda_1$ into $n_1$ parts, the coefficient of $z^{n_2}$ in $(-zq/a)_\infty$ generates partitions $\lambda_2$ into $n_2$ distinct parts colored by $1/a$, weighted by $1/a^{n_2}$, and the coefficient of $z^{n_3}$ in $(-azq)_\infty$ generates partitions $\lambda_3$ into $n_3$ distinct parts colored by $a$, weighted by $a^{n_3}$.

We assemble three such partitions $\lambda_1$, $\lambda_2$, and $\lambda_3$ into one object $\lambda$ with $n_1 + n_2 + n_3$ parts by taking the level $k$ from the uncolored jagged partition $\lambda_1$, attaching a part $k_{1/a}$, if it occurs in $\lambda_2$, on the left, and then attaching a part $k_a$, if it occurs in $\lambda_3$, again to the left. So the coefficient of $z^n$ in our product (3.1) is the generating function for partitions $\lambda$ comprised of levels indexed and ordered by the natural numbers $k$, where the level $k$, reading from right to left, consists of a sequence of uncolored pairs $(k-1, k)$, followed by a sequence of uncolored $k$'s, followed by at most one $k_{1/a}$, followed by at most one $k_a$, and where the exponent of $a$ counts the number of parts colored $a$ minus the number of parts colored $1/a$.

Now, by Lemma 3.1, taking the constant term of $F(z) \times$ (3.1) gives the generating function for those partitions obtained by adding a staircase to some partition $\lambda$ described above. It is easy to verify that these are precisely the partitions counted by $B_1(i; n)$, where $i$ is the exponent of $a$. On the other hand this constant term is also the right hand side of (1.4), which is the generating function for those partitions counted by $A_1(i; n)$. □

To prove Corollary 1.3, we let $q = q^3$ and $a = q$ in the constant term identity (1.4). On the right hand side, we end up with $(-q^2, -q^3, -q^4, -q^6; q^6)_\infty$, which is equal to $1/(q^2, q^3, q^9, q^{10}; q^{12})_\infty$, the generating function for $D_1(n)$. On the left hand side, uncolored parts $k$ become $3k$, and parts colored $a$ ($1/a$) become $3k+1$ ($3k-1$). Hence there are no ones. The difference conditions from Theorem 1.1 become the difference conditions in the definition of $C_1(n)$ in Corollary 1.3. □

For the next three corollaries, observe that the proof of Theorem 1.2 reveals that $(-q)_\infty(-aq, -q/a; q^2)_\infty$ is the generating function for partitions $\lambda$ with difference two at distance two, weighted by $w(\lambda) = (1 + a + 1/a)^{i(\lambda)}(2 + a + 1/a)^{s(\lambda)}$.

For Corollary 1.4, we let $a = e^{2\pi i/3}$. Then the product $(-q)_\infty(-aq, -q/a; q^2)_\infty$ becomes $(-q^2; q^2)_\infty(-q^3; q^6)_\infty$, the generating function for $D_2(n)$. On the other hand, $1 + a + 1/a = 0$ and $2 + a + 1/a = 1$, so the weight on $\lambda$ just means that we exclude partitions with isolated parts. □

For Corollary 1.5, we let $a = i$. Then the product is $(-q^2; q^4)_\infty/(q; q^2)_\infty$, the generating function for $D_3(n)$. Since $a + 1/a = 0$, the weight is $2^{s(\lambda)}$. □

For Corollary 1.6, let $a = e^{2\pi i/6}$. Then $a + 1/a = 1$, so the weight on $\lambda$ is $2^{i(\lambda)}3^{s(\lambda)}$. The product becomes

$$\frac{(-q)_\infty (q^3;q^6)_\infty}{(q;q^2)_\infty}$$

$$= \prod_{n\equiv 0 \pmod 2} (1+q^n) \prod_{n\equiv 3 \pmod 6} (1+q^n) \prod_{\substack{n\equiv 1 \pmod 2 \\ n\not\equiv 3 \pmod 6}} \frac{(1+q^n)}{(1-q^n)}.$$

Since

$$\frac{(1+x)}{(1-x)} = 1 + 2x + 2x^2 + 2x^3 + \cdots,$$

the product is indeed the generating function for $D_4(n)$ □

We now turn to Theorem 1.7. The term $(-zq)_\infty/(z^2q)_\infty$ is the generating function for jagged partitions $\lambda$, where parts are non-overlined, and $(-zq)_\infty$ is the generating function for partitions $\mu$ into distinct overlined parts. Given a pair of such partitions $\lambda$ and $\mu$, we insert each part $\overline{k}$ of $\mu$ into $\lambda$ to the left of the level $k$. Adding a staircase $0, 1, \ldots, n-1$ gives partitions into overlined and non-overlined parts, where $\lambda_i - \lambda_{i+2} \geq 2$ and where

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 2, & \lambda_{i+1} \text{ overlined} \\ 1, & \lambda_{i+1} \text{ non-overlined and } \lambda_i \text{ overlined} \\ 0, & \text{otherwise} \end{cases}$$

□

It is worth noting that $A_2(n)$ may also be regarded as a weighted count of the partitions of $n$ with difference two at distance two, the weight being

$$2^{\#\text{ of sequences } + \text{ }\#\text{ of isolated parts}}.$$

We would also like to remark that, given the second Rogers-Ramanujan identity,

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}, \tag{3.2}$$

there are a couple of other constant term identities that look very much like (1.5) and have similar partition-theoretic interpretations. For example, following the proof of (1.5) and employing (3.2), one may show that

$$[z^0]\frac{(-zq^2,-zq,-zq,-1/z,q)_\infty}{(z^2q^2)_\infty} = \frac{1}{(q,q^2,q^3,q^4;q^5)_\infty}. \tag{3.3}$$

Arguing as above, this implies the following theorem:

**Theorem 3.2.** *Let $A_2'(n)$ denote the number of partitions of $n$ counted by $A_2(n)$ where, additionally, $\overline{1}$ cannot occur, $1$ cannot be repeated, and any sequence beginning with $1$ cannot end in an overlined part. Let $B_2'(n)$ denote the number of partitions of $n$ into parts not divisible by $5$. Then $A_2'(n) = B_2'(n)$.*

As with $A_2(n)$, the function $A_2'(n)$ may be regarded as a weighted count, this time of the partitions of $n$ with distance two at distance two without a repeated 1. The weight is

$$2^{\#\text{ of sequences not beginning with 1 } + \text{ }\#\text{ of isolated parts } \neq 1}.$$

Next, we treat Theorem 1.8. The term $(-zq)_\infty/(z^2q;q^2)_\infty$ is the generating function for jagged partitions where the level $k$ has at most one $k$ on the left, and where the exponent of $z$ counts the number of parts. Hence the constant term on the left hand side of (1.6), by Lemma 3.1, is the generating function for those partitions obtained by adding a staircase $0, 1, \ldots, n-1$ to such a jagged partition. But these are precisely partitions with difference two at distance two having no sequences, which are counted by $A_3(n)$. The right hand side of (1.6) clearly is the generating function for $B_3^+(n) - B_3^-(n)$. □

For Theorem 1.9, the term $(-azq)_\infty/(-az^2q;q^2)_\infty$ is the generating function for jagged partitions where the level $k$ has at most one $k$ on the left, where the exponent of $z$ counts the number of parts, where the exponent $m$ of $a$ counts the number of pairs $(k-1,k)$ plus the number of levels $k$ that have a $k$ on the left, and where each jagged partition is weighted by $-1$ to the number of pairs $(k-1,k)$. Hence the left hand side of (1.7), by Lemma 3.1, is the generating function for those partitions obtained by adding a staircase to such a jagged partition. These are counted by $A_4^+(m,n) - A_4^-(m,n)$. The right hand side of (1.7) is clearly the generating function for $B_4(m,n)$. □

Finally, we have Theorem 1.10. Here the term

$$\sum_{s\geq 0} \frac{q^{\binom{s+1}{2}}(-z)^s}{(q^2;q^2)_s} / (z^2q;q^2)_\infty$$

generates certain jagged partitions. As in the previous two theorems, there is at most one $k$ on the left of a level $k$. However, the partition into distinct parts formed by these $k$'s on the left is not unrestricted. The series above is the generating function for odd-even partitions, i.e., partitions whose smallest part is odd and whose parts alternate in parity. Moreover, we have a weight of $(-1)$ to the number of levels $k$ having a $k$ on the left. Adding a staircase to such a jagged partition, we obtain a partition with difference two at distance two, with no sequences, where isolated parts are odd, weighted by $(-1)$ to the number of isolated parts. These are counted by $A_5^+(n) - A_5^-(n)$. It is easy to determine that the product side of (1.8) generates the partitions claimed. □

## 4. The Frobenius symbol interpretation of the constant terms

Let $P_{A,B}(n)$ denote the number of generalized Frobenius partitions of $n$, i.e., the number of two-rowed arrays,

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{pmatrix}, \tag{4.1}$$

in which the top (bottom) row is a partition from a set $A$ ($B$), and such that $\sum(a_i+b_i)+m=n$ [4]. The classical example is the case $P_{D,D}(n)$, where $D$ is the set of partitions into distinct non-negative parts. Frobenius observed that these objects are in one-to-one correspondence with the ordinary partitions of $n$, giving

$$\sum_{n=0}^{\infty} P_{D,D}(n)q^n = \frac{1}{(q)_\infty}. \tag{4.2}$$

Given a set $A$ of partitions we denote by $P_A(n,k)$ the number of partitions of $n$ from the set $A$ having $k$ parts. It was Andrews [4] who first studied the ramifications of the following observation:

**Lemma 4.1.** *The generating function for Frobenius partitions is given by*

$$\sum_{n=0}^{\infty} P_{A,B}(n)q^n = [z^0]\sum_{n,k} P_A(n,k)q^n(zq)^k \sum_{n,k} P_B(n,k)q^n z^{-k}. \tag{4.3}$$

Using this fact we can deduce, from constant terms identities like those in Theorem 1.1, generating functions for Frobenius partitions. Briefly put, the terms with a positive exponent of $z$ contribute to the top row and terms with a negative exponent of $z$ contribute to the bottom row. We state these generating functions for the first three identities (1.4) - (1.6) and leave the other two to the interested reader. We use the notation $J$ for the set of 01-jagged partitions whose final part is non-negative and $\tilde{J}$ for the set of such 01-jagged partitions that in addition contain no instances of $(i,i)$ for any $i$. We also use the notation $AB$ for the set of vector partitions $(\lambda_A, \lambda_B) \in A \times B$.

**Theorem 4.2.** *We have*

(i)

$$\sum_{n\geq 0} P_{DDDJ,D}(n)q^n = \frac{(-q)_\infty(-q;q^2)_\infty^2}{(q)_\infty}$$

(ii)

$$\sum_{n\geq 0} P_{DDJ,D}(n)q^n = \frac{1}{(q)_\infty(q,q^4;q^5)_\infty^2}$$

(iii)

$$\sum_{n\geq 0} P_{DD\tilde{J},D}(n)q^n = \frac{(q^3;q^3)_\infty}{(q)_\infty^2(-q^3;q^3)_\infty}$$

## 5. Concluding Remarks

We have seen that 01-jagged partitions are useful in the study of partition theorems arising from constant term identities because the presence of a $z^2$ in the generating functions affords new opportunities for proving identities and because the 01-jagged partitions are well-behaved with respect to adding a staircase. An obvious suggestion for further study would be to prove and interpret more constant term identities involving jagged partitions. In addition to the jagged partitions used here, one could try using some of the other types of jagged partitions that have nice generating functions [7]. We should point out that our original motivation came from the fact (whose details we do not present here) that a "weighted word" generalization of the Capparelli theorem due to Alladi, Andrews, and Gordon [1, Theorem 3] is equivalent to the constant term identity

$$[z^0]\frac{(-zq,-1/z,q,-azq^2,-bzq^2)_\infty}{(zq)_\infty(abz^2q^3;q^2)_\infty} = (-q)_\infty(-aq^2,-bq^2;q^2)_\infty, \tag{5.1}$$

where

$$(-azq^2,-bzq^2)_\infty/(abz^2q^3;q^2)_\infty$$

can be interpreted as a type of jagged partition. Finally, it is hoped that at least some of the results presented here have elegant bijective proofs.

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