OVERPARTITION PAIRS AND TWO CLASSES OF BASIC HYPERGEOMETRIC SERIES

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Abstract. We study the combinatorics of two classes of basic hypergeometric series. We first show that these series are the generating functions for certain overpartition pairs defined by frequency conditions on the parts. We then show that when specialized these series are also the generating functions for overpartition pairs with bounded successive ranks, overpartition pairs with conditions on their Durfee dissection, as well as certain lattice paths. When further specialized, the series become infinite products, leading to numerous identities for partitions, overpartitions, and overpartition pairs.

1. Statement of Results

In this paper we study two classes of basic hypergeometric series,

$$
\begin{aligned}
R_{k,i}(a,b;x;q) &= \frac{(-axq,-bxq)_\infty}{(xq,abxq)_\infty}\sum_{n\geq 0}\frac{(-ab)^n x^{kn} q^{kn^2+(k-i+1)n-\binom{n}{2}}(-1/a,-1/b)_n(xq)_n}{(q,-axq,-bxq)_n} \\
&\times \left(1-\frac{abx^i q^{(2n+1)i-2n}(1+q^n/a)(1+q^n/b)}{(1+axq^{n+1})(1+bxq^{n+1})}\right)
\end{aligned}
\tag{1.1}
$$

and

$$
\begin{aligned}
\tilde{R}_{k,i}(a,b;x;q) &= \frac{(-axq,-bxq)_\infty}{(xq,abxq)_\infty}\sum_{n\geq 0}\frac{(-ab)^n x^{(k-1)n} q^{kn^2+(k-i)n-2\binom{n}{2}}(-1/a,-1/b)_n(x^2q^2;q^2)_n}{(q^2;q^2)_n(-axq,-bxq)_n} \\
&\times \left(1-\frac{abx^i q^{(2n+1)i-2n}(1+q^n/a)(1+q^n/b)}{(1+axq^{n+1})(1+bxq^{n+1})}\right).
\end{aligned}
\tag{1.2}
$$

Here we have employed the standard q-series notation [29]

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j) \tag{1.3}$$

and

$$(a_1,a_2,...,a_k)_n = (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n. \tag{1.4}$$

In the first part of the paper we interpret the coefficient of $a^s b^t x^m q^n$ in (1.1) and (1.2) in terms of overpartition pairs. Recall that an overpartition is a partition in which the first occurrence of a number may be overlined. To speak concisely about the relevant overpartition pairs, we

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shall say that j occurs *unattached* in the overpartition pair (λ,μ) if it only occurs non-overlined and only in μ. For example, in the overpartition pair $((\overline{6},4,4,3),(6,\overline{4},4,\overline{2},2,1))$, only 1 occurs unattached. We also define the *valuation* of an overpartition pair (λ,μ) at j as

$$v_j((\lambda,\mu)) = f_j(\lambda) + f_{\overline{j}}(\lambda) + f_{\overline{j}}(\mu) + \chi(j \text{ occurs unattached in } (\lambda,\mu)), \tag{1.5}$$

where χ is the usual characteristic function and $f_j(\lambda)$ counts the number of occurrences of j in λ. We are now prepared to state our first two theorems. Here and throughout the paper we assume that $k \geq 2$ and $1 \leq i \leq k$, unless otherwise noted.

Theorem 1.1. *If*

$$R_{k,i}(a,b;x;q) = \sum_{s,t,m,n\geq 0} r_{k,i}(s,t,m,n)a^s b^t x^m q^n,$$

then $r_{k,i}(s,t,m,n)$ is equal to the number of overpartition pairs (λ,μ) of n with m parts, s of which are overlined and in λ or non-overlined and in μ, t of which are in μ, where (i) $v_1((\lambda,\mu)) \leq i-1$, and (ii) for each $j \geq 1$, $f_j(\lambda) + v_{j+1}((\lambda,\mu)) \leq k-1$.

Theorem 1.2. *If*

$$\tilde{R}_{k,i}(a,b;x;q) = \sum_{s,t,m,n\geq 0} \tilde{r}_{k,i}(s,t,m,n)a^s b^t x^m q^n,$$

then $\tilde{r}_{k,i}(s,t,m,n)$ is equal to the number of overpartition pairs (λ,μ) counted by $r_{k,i}(s,t,m,n)$ such that if there is equality in condition (ii) of Theorem 1.1 for some $j \geq 1$, then

$$jf_j(\lambda) + (j+1)v_{j+1}((\lambda,\mu)) \equiv i-1+\mathcal{O}_j(\lambda)+\mathcal{O}_j(\mu) \pmod 2, \tag{1.6}$$

where $\mathcal{O}_j(\cdot)$ denotes the number of overlined parts less than or equal to j.

These theorems unify and generalize many important families of partition identities, including Gordon's generalization of the Rogers-Ramanujan identities [30], Bressoud's Rogers-Ramanujan identities for even moduli [16], Gordon's theorems for overpartitions [32], Andrews' generalization of the Göllnitz-Gordon identities [5], their overpartition analogue [33], as well as some more general results of Corteel and the authors [25, 26]. How all of these results follow from Theorems 1.1 and 1.2 will be explained in Section 4, and several new families of identities will be presented.

In the second part of the paper we study three more classes of combinatorial objects counted by $R_{k,i}(a,b;1;q)$ and $\tilde{R}_{k,i}(a,b;1;q)$. It will be necessary to defer the definitions of these objects to later in the paper.

Theorem 1.3. *Let $B_{k,i}(s,t,n)$ denote the number of overpartition pairs which are counted by $r_{k,i}(s,t,m,n)$ for some m. Let $C_{k,i}(s,t,n)$ denote the number of overpartition pairs of n whose Frobenius representations have s non-overlined parts in their bottom rows and t non-overlined parts in their top rows, and whose successive ranks are in the interval $[-i+2, 2k-i-1]$. Let $D_{k,i}(s,t,n)$ denote the number of (k,i)-admissible overpartition pairs of n whose Frobenius representations have s non-overlined parts in their bottom rows and t non-overlined parts in their top rows. Let $E_{k,i}(s,t,n)$ denote the number of generalized Bressoud-Burge lattice paths of major index n satisfying the odd (k,i)-conditions, where the number of peaks marked by a (resp. marked by b) is s (resp. t). Then*

$$B_{k,i}(s,t,n) = C_{k,i}(s,t,n) = D_{k,i}(s,t,n) = E_{k,i}(s,t,n).$$

Theorem 1.4. *Let $\tilde{B}_{k,i}(s,t,n)$ denote the number of overpartition pairs which are counted by $\tilde{r}_{k,i}(s,t,m,n)$ for some m. Let $\tilde{C}_{k,i}(s,t,n)$ denote the number of overpartition pairs of n whose Frobenius representations have s non-overlined parts in their bottom rows and t non-overlined parts in their top rows, and whose successive ranks are in the interval $[-i+2, 2k-i-2]$. Let $\tilde{D}_{k,i}(s,t,n)$ denote the number of self-(k,i)-conjugate overpartition pairs of n whose Frobenius representations have s non-overlined parts in their bottom rows and t non-overlined parts in their top rows. Let $\tilde{E}_{k,i}(s,t,n)$ denote the number of generalized Bressoud-Burge lattice paths counted by $E_{k,i}(s,t,n)$ which also satisfy the even (k,i)-conditions. Then*

$$\tilde{B}_{k,i}(s,t,n) = \tilde{C}_{k,i}(s,t,n) = \tilde{D}_{k,i}(s,t,n) = \tilde{E}_{k,i}(s,t,n).$$

Theorems 1.3 and 1.4 extend overpartition-theoretic work of Corteel and the authors [25, 26], which had in turn generalized partition-theoretic work of Andrews, Bressoud, and Burge [8, 11, 18, 19, 20, 21].

The paper is organized as follows: Theorem 1.1 is proven in Section 2 using Andrews' q-difference equations for some families of basic hypergeometric series [6]. Theorem 1.2 is proven in Section 3 in the same way, except that we will have to develop the required q-difference equations from scratch. In Section 4, we present some of the many combinatorial identities which follow from Theorems 1.1 and 1.2. In Sections $5-7$ we define the combinatorial structures occurring in Theorems 1.3 and 1.4 and prove these theorems.

2. The $R_{k,i}(a,b;x;q)$

It was Andrews who first observed the combinatorial significance of series like (1.1) and (1.2). Selberg [36] had essentially proven q-difference equations for $R_{k,i}(0,0;x;q)$, and Andrews [2] showed how this could be used to prove Gordon's generalization of the Rogers-Ramanujan identities [30]. He then proceeded to develop a massive generalization of the $R_{k,i}(0,0;x;q)$ [6], and the combinatorics of these series has turned out to be one of the major areas of research in the theory of partitions over the last 40 years (e.g. [3, 4, 5, 7, 8, 9, 11, 15, 16, 17, 19, 20, 21, 14, 27, 26, 25, 32, 33, 34, 35]). The series also have direct applications to q-series identities (e.g. [6, 10]) and q-continued fractions (e.g. [6, 13]).

In terms of Andrews' series, called $J_{\lambda,k,i}(a_1, a_2, \ldots, a_\lambda; x; q)$ [6], we have

$$R_{k,i}(a,b;x;q) = \frac{1}{(abxq)_\infty} J_{2,k,i}(-1/a, -1/b; x; q). \tag{2.1}$$

Employing the q-difference equations for the $J_{2,k,i}$ (and related functions) [6, Eq. (2.1)-(2.4)], we may deduce that the $R_{k,i}(a,b;x;q)$ satisfy the following:

Lemma 2.1.

$$R_{k,1}(a,b;x;q) = R_{k,k}(a,b;xq;q), \tag{2.2}$$

$$\begin{aligned} R_{k,2}(a,b;x;q) - R_{k,1}(a,b;x;q) &= \frac{xq}{(1-abxq)} R_{k,k-1}(a,b;xq;q) \\ &+ \frac{axq}{(1-abxq)} R_{k,k}(a,b;xq;q) \\ &+ \frac{bxq}{(1-abxq)} R_{k,k}(a,b;xq;q) \\ &+ \frac{abxq}{(1-abxq)} R_{k,k}(a,b;xq;q), \end{aligned} \tag{2.3}$$

and for $3 \leq i \leq k$*,*

$$\begin{aligned} R_{k,i}(a,b;x;q) - R_{k,i-1}(a,b;x;q) &= \frac{(xq)^{i-1}}{(1-abxq)} R_{k,k-i+1}(a,b;xq;q) \\ &+ \frac{a(xq)^{i-1}}{(1-abxq)} R_{k,k-i+2}(a,b;xq;q) \\ &+ \frac{b(xq)^{i-1}}{(1-abxq)} R_{k,k-i+2}(a,b;xq;q) \\ &+ \frac{ab(xq)^{i-1}}{(1-abxq)} R_{k,k-i+3}(a,b;xq;q)). \end{aligned} \tag{2.4}$$

Using these, we may deduce Theorem 1.1.

Proof of Theorem 1.1. First, observe that the q-difference equations in Lemma 2.1 together with the fact that $R_{k,i}(a,b;0;q) = 1$ uniquely define the functions $R_{k,i}(a,b;x;q)$. Now, let

$$\widehat{R}_{k,i}(a,b;x;q) = \sum_{s,t,m,n \geq 0} r_{k,i}(s,t,m,n) a^s b^t x^m q^n.$$

We wish to show that $\widehat{R}_{k,i}(a,b;x;q) = R_{k,i}(a,b;x;q)$. We shall accomplish this by showing that the functions $\widehat{R}_{k,i}(a,b;x;q)$ satisfy the same q-difference equations as the $R_{k,i}(a,b;x;q)$ in Lemma 2.1. The fact that $\widehat{R}_{k,i}(a,b;0;q) = 1$ is obvious, since there are no overpartition pairs without any parts except for the empty one.

Observe that subtracting one from each part of an overpartition pair (and deleting the resulting zeros) that satisfies condition (ii) in Theorem 1.1 gives another overpartition pair that satisfies this condition. Similarly, adding one to each part of an overpartition pair that satisfies the condition gives another overpartition pair that satisfies the condition.

We begin with (2.2). An overpartition pair (λ, μ) counted by $r_{k,1}(s,t,m,n)$ has no ones whatsoever and hence has $v_2((\lambda,\mu)) \leq k-1$. By subtracting one from each part we see that $\widehat{R}_{k,1}(a,b;x;q) = \widehat{R}_{k,k}(a,b;xq;q)$.

For (2.3), we observe that the function

$$\widehat{R}_{k,2}(a,b;x;q) - \widehat{R}_{k,1}(a,b;x;q)$$

is the generating function for those overpartition pairs (λ,μ) counted by $r_{k,2}(s,t,m,n)$ having $v_1((\lambda,\mu)) = 1$. We break these pairs into four disjoint classes: those having 1 as a part of λ, those having $\overline{1}$ as a part of λ, those having $\overline{1}$ as a part of μ, and those in which 1 occurs unattached. In the first of these four cases, $v_2((\lambda,\mu)) \leq k-2$. So, removing the 1 from λ along with any ones that may occur in μ, and then subtracting one from all of the remaining parts, we see that these overpartition pairs are generated by

$$\frac{xq}{(1-abxq)}\widehat{R}_{k,k-1}(a,b;xq;q).$$

In the second of these cases, where $\overline{1}$ occurs in λ, we have $v_2((\lambda,\mu)) \leq k-1$. So, removing the $\overline{1}$ from λ along with any ones that may occur in μ, and then subtracting one from all of the remaining parts, we see that these overpartition pairs are generated by

$$\frac{axq}{(1-abxq)}\widehat{R}_{k,k}(a,b;xq;q).$$

In exactly the same way we see that those pairs containing an $\overline{1}$ in μ are generated by

$$\frac{bxq}{(1-abxq)}\widehat{R}_{k,k}(a,b;xq;q).$$

For the final case, where 1 occurs unattached in the overpartition pair (λ,μ), again we have $v_2((\lambda,\mu) \leq k-1$. So, removing all of the ones from μ and subtracting one from all of the remaining parts, we see that these overpartition pairs are generated by

$$\frac{abxq}{(1-abxq)}\widehat{R}_{k,k}(a,b;xq;q).$$

Putting everything together gives (2.3).

We now turn to (2.4). As above, the function

$$\widehat{R}_{k,i}(a,b;x;q) - \widehat{R}_{k,i-1}(a,b;x;q)$$

is the generating function for those overpartition pairs which are counted by $r_{k,i}(s,t,m,n)$ and which have $v_1((\lambda,\mu)) = i-1$. And, as before, we consider four cases: $f_1(\lambda) = i-1$, $f_1(\lambda) = i-2$ and $f_{\overline{1}}(\lambda) = 1$, $f_1(\lambda) = i-2$ and $f_{\overline{1}}(\mu) = 1$, and $f_1(\lambda) = i-3$, $f_{\overline{1}}(\lambda) = 1$, and $f_{\overline{1}}(\mu) = 1$. Notice that since $i-1 \geq 2$ we cannot have an unattached occurrence of 1 in (λ,μ). Now, in the first of these cases, $v_2((\lambda,\mu)) \leq k-i$. So, removing the $i-1$ ones from λ as well as any non-overlined ones from μ, and then subtracting one from each remaining part, we see that these overpartition pairs are generated by

$$\frac{(xq)^{i-1}}{(1-abxq)}\widehat{R}_{k,k-i+1}(a,b;xq;q).$$

For the second case, where $f_1(\lambda) = i-2$ and $f_{\overline{1}}(\lambda) = 1$, we have $v_2((\lambda,\mu)) \leq k-i+1$. So, removing the $i-2$ ones and the $\overline{1}$ from λ, as well as any non-overlined ones from μ, and then subtracting one from each remaining part, we see that these overpartition pairs are generated by

$$\frac{axq(xq)^{i-2}}{(1-abxq)}\widehat{R}_{k,k-i+2}(a,b;xq;q).$$

Similarly, those overpartition pairs having $f_1(\lambda) = i-2$ and $f_{\overline{1}}(\mu) = 1$ are generated by

$$\frac{bxq(xq)^{i-2}}{(1-abxq)}\widehat{R}_{k,k-i+2}(a,b;xq;q).$$

Finally, if $f_1(\lambda) = i-3$, $f_{\overline{1}}(\lambda) = 1$, and $f_{\overline{1}}(\mu) = 1$, then $v_2((\lambda,\mu) \leq k-i+2$. So, removing the $i-3$ ones and the $\overline{1}$ from λ, the $\overline{1}$ and any non-overlined ones from μ, and then subtracting one from each remaining part, we see that these overpartition pairs are generated by

$$\frac{(axq)(bxq)(xq)^{i-3}}{(1-abxq)}\widehat{R}_{k,k-i+3}(a,b;xq;q).$$

Putting everything together gives (2.4) for the $\widehat{R}_{k,i}(a,b;x;q)$ and we may now conclude that $R_{k,i}(a,b;x;q) = \widehat{R}_{k,i}(a,b;x;q)$, establishing Theorem 1.1. □

3. The $\tilde{R}_{k,i}(a,b;x;q)$

Unlike the case for the $R_{k,i}(a,b;x;q)$, we will need to develop from scratch the theory of recurrences for the $\tilde{R}_{k,i}(a,b;x;q)$. In this endeavor we closely follow Andrews [6]. For $k \geq 1$ and $i \in \mathbb{Z}$, define

$$\tilde{H}_{2,k,i}(a,b;x;q) = \frac{(-axq,-bxq)_\infty}{(xq)_\infty}\sum_{n\geq 0}\frac{(-ab)^n x^{(k-1)n}q^{kn^2+n-in-2\binom{n}{2}}(-1/a,-1/b)_n(x^2;q^2)_n(1-x^iq^{2ni})}{(q^2;q^2)_n(-axq,-bxq)_n(1-x)}$$

and

$$\tilde{J}_{2,k,i}(a,b;x;q) = (abxq)_\infty \tilde{R}_{k,i}(a,b;x;q).$$

When $a=0$ or $b=0$, these functions simplify to the $\tilde{H}_{k,i}$ and $\tilde{J}_{k,i}$ studied in [25]. We shall establish the following facts:

Proposition 3.1. *We have*

$$\tilde{H}_{2,k,0}(a,b;x;q) = 0, \tag{3.1}$$

$$\tilde{H}_{2,k,-i}(a,b;x;q) = -x^{-i}\tilde{H}_{2,k,i}(a,b;x;q), \tag{3.2}$$

$$\tilde{H}_{2,k,i}(a,b;x;q) - \tilde{H}_{2,k,i-2}(a,b;x;q) = x^{i-2}(1+x)\tilde{J}_{2,k,k-i+1}(a,b;x;q), \tag{3.3}$$

and

$$\tilde{J}_{2,k,i}(a,b;x;q) = \tilde{H}_{2,k,i}(a,b;xq;q) + (axq+bxq)\tilde{H}_{2,k,i-1}(a,b;xq;q) + abx^2q^2\tilde{H}_{2,k,i-2}(a,b;xq;q). \tag{3.4}$$

Proof. Equations (3.1) and (3.2) are straightforward. For the other two, we introduce a little notation to simplify the calculations. We write

$$\tilde{C}_{2,k,i}(a,b;x;q) = \frac{(xq)_\infty}{(-axq,-bxq)_\infty}\tilde{H}_{2,k,i}(a,b;x;q) \tag{3.5}$$

and

$$\tilde{D}_{2,k,i}(a,b;x;q) = \frac{(xq)_\infty}{(-axq,-bxq)_\infty}\tilde{J}_{2,k,i}(a,b;x;q). \tag{3.6}$$

We also write

$$\tilde{M}_n(a,b;x;q) = \tilde{M}_n(x) = x^{(k-1)n}q^{(k-1)n^2+2n}(-ab)^n. \tag{3.7}$$

These three definitions are in analogy with Andrews' definitions in [6]. We note that

$$\tilde{M}_{n+1}(x) = -abx^{k-1}q^{2n(k-1)+k+1}\tilde{M}_n(x) \tag{3.8}$$

and

$$\tilde{M}_n(xq) = q^{(k-1)n}\tilde{M}_n(x). \tag{3.9}$$

We are now prepared to deal with (3.3). We have

$$
\begin{aligned}
\tilde{C}_{2,k,i}(a,b;x;q) \;&-\; \tilde{C}_{2,k,i-2}(a,b;x;q) \\
&= \sum_{n\geq 0} \frac{\tilde{M}_n(x)(x^2;q^2)_n(-1/a,-1/b)_n}{(1-x)(q^2;q^2)_n(-axq,-bxq)_n} \\
&\times \Big(q^{-in}(1-q^{2n}) + x^{i-2}q^{n(i-2)}(1-x^2q^{2n})\Big) \\
&= (1+x)\sum_{n\geq 1} \frac{\tilde{M}_n(x)q^{-in}(x^2q^2;q^2)_{n-1}(-1/a,-1/b)_n}{(q^2;q^2)_{n-1}(-axq,-bxq)_n} \\
&+ x^{i-2}(1+x)\sum_{n\geq 0} \frac{\tilde{M}_n(x)q^{n(i-2)}(x^2q^2;q^2)_n(-1/a,-1/b)_n}{(q^2;q^2)_n(-axq,-bxq)_n} \\
&= (1+x)\sum_{n\geq 0} \frac{\tilde{M}_{n+1}(x)q^{-i(n+1)}(x^2q^2;q^2)_n(-1/a,-1/b)_{n+1}}{(q^2;q^2)_n(-axq,-bxq)_{n+1}} \\
&+ x^{i-2}(1+x)\sum_{n\geq 0} \frac{\tilde{M}_n(x)q^{n(i-2)}(x^2q^2;q^2)_n(-1/a,-1/b)_n}{(q^2;q^2)_n(-axq,-bxq)_n} \\
&= -abx^{k-1}q^{k-i+1}(1+x)\sum_{n\geq 0} \frac{\tilde{M}_n(x)q^{n(2k-i-2)}(x^2q^2;q^2)_n(-1/a,-1/b)_{n+1}}{(q^2;q^2)_n(-axq,-bxq)_{n+1}} \\
&+ x^{i-2}(1+x)\sum_{n\geq 0} \frac{\tilde{M}_n(x)q^{n(i-2)}(x^2q^2;q^2)_n(-1/a,-1/b)_n}{(q^2;q^2)_n(-axq,-bxq)_n} \\
&= -abx^{k-1}q^{k-i+1}(1+x)\sum_{n\geq 0} \frac{\tilde{M}_n(xq)q^{n(k-i-1)}(x^2q^2;q^2)_n(-1/a,-1/b)_{n+1}}{(q^2;q^2)_n(-axq,-bxq)_{n+1}} \\
&+ x^{i-2}(1+x)\sum_{n\geq 0} \frac{\tilde{M}_n(xq)q^{n(-k+i-1)}(x^2q^2;q^2)_n(-1/a,-1/b)_n}{(q^2;q^2)_n(-axq,-bxq)_n} \\
&= x^{i-2}(1+x)\sum_{n\geq 0} \frac{\tilde{M}_n(xq)q^{-n(k-i+1)}(x^2q^2;q^2)_n(-1/a,-1/b)_n}{(q^2;q^2)_n(-axq,-bxq)_n} \\
&- abx^{i-2}(1+x)(xq)^{k-i+1} \\
&\qquad \times \sum_{n\geq 0} \frac{\tilde{M}_n(xq)q^{-n(k-i+1)+2n(k-i+1)-2n}(x^2q^2;q^2)_n(-1/a,-1/b)_{n+1}}{(q^2;q^2)_n(-axq,-bxq)_{n+1}} \\
&= x^{i-2}(1+x)\sum_{n\geq 0} \frac{\tilde{M}_n(xq)q^{-n(k-i+1)}(x^2q^2;q^2)_n(-1/a,-1/b)_n}{(q^2;q^2)_n(-axq,-bxq)_n} \\
&\times \left(1 - \frac{abx^{k-i+1}q^{(2n+1)(k-i+1)-2n}(1+q^n/a)(1+q^n/b)}{(1-axq^{n+1})(1-bxq^{n+1})}\right) \\
&= x^{i-2}(1+x)\tilde{D}_{2,k,k-i+1}(a,b;x;q).
\end{aligned}
$$

Multiplying the extremes of the above string of equations by $(-axq,-bxq)_\infty/(xq)_\infty$ yields (3.3).

We now turn to (3.4). By making a common denominator in the expression in parentheses in the definition of the $\tilde{J}_{2,k,i}(a,b;x;q)$, we have

$$\begin{aligned}
\tilde{D}_{2,k,i}(a,b;x;q) &= \sum_{n\geq 0}\frac{\tilde{M}_n(xq)q^{-in}(x^2q^2;q^2)_n(-1/a,-1/b)_n}{(q^2;q^2)_n(-axq,-bxq)_{n+1}}\\
&\times \Big(1+(a+b)xq^{n+1}+abx^2q^{2n+2}-x^iq^{(2n+1)i}(1+(a+b)q^{-n}+abq^{-2n})\Big)\\
&= \sum_{n\geq 0}\frac{\tilde{M}_n(xq)q^{-in}(x^2q^2;q^2)_n(-1/a,-1/b)_n}{(q^2;q^2)_n(-axq,-bxq)_{n+1}}\\
&\times \Big[(1-x^iq^{(2n+1)i})+(a+b)xq^{n+1}(1-x^{i-1}q^{(2n+1)(i-1)})\\
&+ abx^2q^{2n+2}(1-x^{i-2}q^{(2n+1)(i-2)})\Big]\\
&= \frac{(1-xq)}{(1+axq)(1+bxq)}\left[\sum_{n\geq 0}\frac{\tilde{M}_n(xq)q^{-in}((xq)^2;q^2)_n(-1/a,-1/b)_n(1-(xq)^iq^{2ni})}{(q^2;q^2)_n(-a(xq)q,-b(xq)q)_n(1-xq)}\right.\\
&+ (a+b)xq\sum_{n\geq 0}\frac{\tilde{M}_n(xq)q^{-(i-1)n}((xq)^2;q^2)_n(-1/a,-1/b)_n(1-(xq)^{i-1}q^{2n(i-1)})}{(q^2;q^2)_n(-a(xq)q,-b(xq)q)_n(1-xq)}\\
&+ \left. abx^2q^2\sum_{n\geq 0}\frac{\tilde{M}_n(xq)q^{-(i-2)n}((xq)^2;q^2)_n(-1/a,-1/b)_n(1-(xq)^{i-2}q^{2n(i-2)})}{(q^2;q^2)_n(-a(xq)q,-b(xq)q)_n(1-xq)}\right]\\
&= \frac{(1-xq)}{(1+axq)(1+bxq)}\Big[\tilde{C}_{2,k,i}(a,b;xq;q)\\
&+ (a+b)xq\tilde{C}_{2,k,i-1}(a,b;xq;q)+abx^2q^2\tilde{C}_{2,k,i-2}(a,b;xq;q)\Big].
\end{aligned}$$

Multiplying both sides of this string of equations by $(-axq,-bxq)_\infty/(xq)_\infty$ finishes the proof of (3.4). And this then completes the proof of Proposition 3.1. ☐

We now have the analogue of Lemma 2.1 for the $\tilde{R}_{k,i}(a,b;x;q)$ using Proposition 3.1.

Lemma 3.2.

$$\tilde{R}_{k,1}(a,b;x;q)=\tilde{R}_{k,k}(a,b;xq;q), \tag{3.10}$$

$$\begin{aligned}
\tilde{R}_{k,2}(a,b;x;q) &= \frac{1}{(1-abxq)}\tilde{R}_{k,k-1}(a,b;xq;q)\\
&+ \frac{xq}{(1-abxq)}\tilde{R}_{k,k-1}(a,b;xq;q)\\
&+ \frac{axq}{(1-abxq)}\tilde{R}_{k,k}(a,b;xq;q)\\
&+ \frac{bxq}{(1-abxq)}\tilde{R}_{k,k}(a,b;xq;q),
\end{aligned}\tag{3.11}$$

and, for $3 \leq i \leq k$,

$$\begin{aligned}\tilde{R}_{k,i}(a,b;x;q) - \tilde{R}_{k,i-2}(a,b;x;q) &= \frac{(xq)^{i-2}(1+xq)}{(1-abxq)}\tilde{R}_{k,k-i+1}(a,b;xq;q) \\ &+ \frac{a(xq)^{i-2}(1+xq)}{(1-abxq)}\tilde{R}_{k,k-i+2}(a,b;xq;q) \\ &+ \frac{b(xq)^{i-2}(1+xq)}{(1-abxq)}\tilde{R}_{k,k-i+2}(a,b;xq;q) \\ &+ \frac{ab(xq)^{i-2}(1+xq)}{(1-abxq)}\tilde{R}_{k,k-i+3}(a,b;xq;q).\end{aligned} \tag{3.12}$$

Proof of Theorem 1.2. We begin by observing that for an overpartition pair (λ, μ) counted by $\tilde{r}_{k,i}(s,t,m,n)$, if $(\lambda,\mu) - \vec{1}$ satisfies condition (ii) of Theorem 1.1 at j, then so does (λ,μ) at $j+1$, where $-\vec{1}$ is shorthand for subtracting 1 from each part and then deleting any zeros. Hence, we have

$$\begin{aligned}jf_j(\lambda - \vec{1}) + (j+1)v_{j+1}((\lambda,\mu)-\vec{1}) &= (j+1)f_{j+1}(\lambda) + (j+2)v_{j+2}((\lambda,\mu)) \\ &- (f_{j+1}(\lambda) + v_{j+2}((\lambda,\mu))) \\ &\equiv i - 1 + \mathcal{O}_{j+1}(\lambda) + \mathcal{O}_{j+1}(\mu) - (k-1) \pmod 2 \\ &\equiv k - i + \mathcal{O}_j(\lambda - \vec{1}) + \mathcal{O}_j(\mu - \vec{1}) \\ &+ \begin{cases} 0, & \text{if } \overline{1} \notin \lambda \text{ and } \overline{1} \notin \mu, \\ 0, & \text{if } \overline{1} \in \lambda \text{ and } \overline{1} \in \mu, \\ 1, & \text{otherwise} \end{cases} \pmod 2.\end{aligned} \tag{3.13}$$

We now proceed as in the proof of Theorem 1.1. Equation (3.13) will be used throughout the proof to ensure condition (1.6) in the overpartition pairs under consideration. This may not always be mentioned explicitly. We observe that since $\tilde{R}_{k,i}(a,b;0;q) = 1$, the q-difference equations in Lemma 3.2 uniquely define the functions $\tilde{R}_{k,i}(a,b;x;q)$. Now let

$$\tilde{S}_{k,i}(a,b;x;q) = \sum_{s,t,m,n \geq 0} \tilde{r}_{k,i}(s,t,m,n) a^s b^t x^m q^n.$$

Again, $\tilde{S}_{k,i}(a,b;0;q) = 1$ because there is only one overpartition pair without parts - the empty one. So, to prove Theorem 1.2 we need to show that the $\tilde{S}_{k,i}(a,b;x;q)$ satisfy the same q-difference equations as the $\tilde{R}_{k,i}(a,b;x;q)$ in Lemma 3.2.

We begin with (3.10). An overpartition pair counted by $\tilde{r}_{k,1}(s,t,m,n)$ has no ones and has $v_2((\lambda,\mu)) \leq k-1$. Subtracting one from each part and appealing to (3.13), we see that these overpartition pairs are generated by $\tilde{S}_{k,k}(a,b;xq;q)$.

For (3.11), an overpartition pair (λ,μ) counted by $\tilde{r}_{k,2}(s,t,m,n)$ has either no ones or $v_1((\lambda,\mu)) = 1$. If there are no ones, then $v_2((\lambda,\mu)) \leq k-2$. Notice that in this case we cannot have $v_2((\lambda,\mu)) = k-1$, for then we would have $1f_1((\lambda,\mu)) + 2v_2((\lambda,\mu)) \equiv 0 \pmod 2$, which violates the condition (1.6) defining the $\tilde{r}_{k,2}(s,t,m,n)$. Hence we have $v_2((\lambda,\mu) \leq k-2$. So, subtracting one from each part of (λ,μ) and appealing to (3.13), we see that these pairs are generated by

$$\tilde{S}_{k,k-1}(a,b;xq;q) \tag{3.14}$$

Now, if $v_1((\lambda,\mu)) = 1$, this may be for one of four reasons: 1 occurs in λ, $\overline{1}$ occurs in λ, $\overline{1}$ occurs in μ, or 1 occurs unattached. In the first case, we have $v_2((\lambda,\mu)) \leq k-2$. Subtracting one from each part, removing the 1 from λ as well as any non-overlined ones from μ, these pairs are seen to be generated by

$$\frac{xq}{(1-abxq)}\tilde{S}_{k,k-1}(a,b;xq;q). \tag{3.15}$$

In the second case, when $\overline{1}$ appears in λ, then we have $v_2((\lambda,\mu)) \leq k-1$. Notice that if $v_2((\lambda,\mu)) = k-1$, then we have $1f_1(\lambda) + 2v_2((\lambda,\mu)) \equiv 0 \pmod 2$, which is congruent to $2-1+\mathcal{O}_1(\lambda)+\mathcal{O}_1(\mu)$ modulo 2. Removing the $\overline{1}$ from μ, removing any non-overlined ones from μ, and then subtracting one from each remaining part, we find (keeping in mind (3.13)) that these pairs are generated by

$$\frac{axq}{(1-abxq)}\tilde{S}_{k,k}(a,b;xq;q). \tag{3.16}$$

The third case, when $\overline{1}$ occurs in μ, is analogous to the second case and these overpartition pairs are generated by

$$\frac{bxq}{(1-abxq)}\tilde{S}_{k,k}(a,b;xq;q). \tag{3.17}$$

Finally, we consider the case when 1 occurs unattached in the overpartition pair $((\lambda,\mu))$. If $v_2((\lambda,\mu)) = k-1$, then condition (1.6) in the definition of the $r_{k,2}(s,t,m,n)$ would be violated, so we have $v_2((\lambda,\mu)) \leq k-2$. Removing all of the unattached ones and subtracting one from each remaining part, we see that these pairs are generated by

$$\frac{abxq}{(1-abxq)}\tilde{S}_{k,k-1}(a,b;xq;q). \tag{3.18}$$

Adding (3.14) - (3.18) together now shows that the recurrence (3.11) is true for the $\tilde{S}_{k,i}(a,b;x;q)$.

We now turn to (3.12), proceeding much like before. The function $\tilde{S}_{k,i}(a,b;x;q)-\tilde{S}_{k,i-2}(a,b;x;q)$ is the generating function for those overpartition pairs (λ,μ) which are counted by $\tilde{r}_{k,i}(s,t,m,n)$ and which have either $v_1((\lambda,\mu)) = i-1$ or $v_1((\lambda,\mu)) = i-2$. We shall consider eight cases, the last of which has two subcases depending on whether $i > 3$.

In the first case, suppose that $v_1((\lambda,\mu)) = i-1$ and $f_1(\lambda) = i-1$. Then, $v_2((\lambda,\mu))$ can be as much as $k-i$. These pairs are generated by

$$\frac{(xq)^{i-1}}{(1-abxq)}\tilde{S}_{k,k-i+1}(a,b;xq;q). \tag{3.19}$$

In the second case, suppose that $v_1((\lambda,\mu)) = i-2$ and $f_1(\lambda) = i-2$. Then, $v_2((\lambda,\mu))$ can be as much as $k-i$, for if it were $k-i+1$ this would violate the condition (1.6). These pairs are generated by

$$\frac{(xq)^{i-2}}{(1-abxq)}\tilde{S}_{k,k-i+1}(a,b;xq;q). \tag{3.20}$$

In the third case, suppose that $v_1((\lambda,\mu)) = i-1$, $f_{\overline{1}}(\lambda) = 1$, and $f_1(\lambda) = i-2$. Then, $v_2((\lambda,\mu))$ can be as much as $k-i+1$. These pairs are generated by

$$\frac{a(xq)^{i-1}}{(1-abxq)}\tilde{S}_{k,k-i+2}(a,b;xq;q). \tag{3.21}$$

In the fourth case, suppose that $v_1((\lambda,\mu)) = i-2$, $f_{\overline{1}}(\lambda) = 1$, and $f_1(\lambda) = i-3$. Then, $v_2((\lambda,\mu))$ can be as much as $k-i+1$, for if it were $k-i+2$ this would violate the condition (1.6). These pairs are generated by

$$\frac{a(xq)^{i-2}}{(1-abxq)}\tilde{S}_{k,k-i+2}(a,b;xq;q). \tag{3.22}$$

The fifth and sixth cases are analogous to the third and fourth, respectively, where $f_{\overline{1}}(\lambda) = 1$ is replaced by $f_{\overline{1}}(\mu) = 1$. These pairs are generated by

$$\frac{b(xq)^{i-1} + b(xq)^{i-2}}{(1-abxq)}\tilde{S}_{k,k-i+2}(a,b;xq;q). \tag{3.23}$$

In the seventh case, suppose that $v_1((\lambda,\mu)) = i-1$, $f_{\overline{1}}(\lambda) = f_{\overline{1}}(\mu) = 1$, and $f_1(\lambda) = i-3$. Then $v_2((\lambda,\mu)$ could be as much as $k-i+2$. These pairs are generated by

$$\frac{ab(xq)^{i-1}}{(1-abxq)}\tilde{S}_{k,k-i+3}(a,b;xq;q). \tag{3.24}$$

For the eighth case, suppose that $v_1((\lambda,\mu)) = i-2$, $f_{\overline{1}}(\lambda) = f_{\overline{1}}(\mu) = 1$, and $f_1(\lambda) = i-4$. This requires the assumption that $i \geq 4$. Here $v_2((\lambda,\mu)) \leq k-i+2$, and these pairs are generated (for $i \geq 4$) by

$$\frac{ab(xq)^{i-2}}{(1-abxq)}\tilde{S}_{k,k-i+3}(a,b;xq;q). \tag{3.25}$$

Now, if $i = 3$ we cannot have $v_1((\lambda,\mu)) = i-2 = 1$ while at the same time having $\overline{1}$ occurring in both λ and μ. What we can have, however, is 1 occurring unattached. Then $v_2((\lambda,\mu)) \leq k-1$ $(= k-i+2)$, and so these pairs are generated by (3.25) when $i = 3$.

Adding together (3.19) - (3.25) establishes (3.12) for the $\tilde{S}_{k,i}(a,b;x;q)$ and we may now conclude that $\tilde{S}_{k,i}(a,b;x;q) = \tilde{R}_{k,i}(a,b;x;q)$, finishing the proof of Theorem 1.2. □

4. Corollaries

Using the fact that

$$(a)_{-n} = \frac{(-1)^n q^{\binom{n+1}{2}}}{a^n (q/a)_n},$$

the following representations for the $R_{k,i}(a,b;1;q)$ and $\tilde{R}_{k,i}(a,b;1;q)$ can be deduced from (1.1) and (1.2):

$$R_{k,i}(a,b;1;q) = \frac{(-aq,-bq)_\infty}{(q,abq)_\infty}\sum_{n\in\mathbb{Z}} \frac{q^{kn^2+(k-i+1)n-\binom{n}{2}}(-1/a,-1/b)_n(-ab)^n}{(-aq,-bq)_n} \tag{4.1}$$

and

$$\tilde{R}_{k,i}(a,b;1;q) = \frac{(-aq,-bq)_\infty}{(q,abq)_\infty}\sum_{n\in\mathbb{Z}} \frac{q^{kn^2+(k-i)n-2\binom{n}{2}}(-1/a,-1/b)_n(-ab)^n}{(-aq,-bq)_n}. \tag{4.2}$$

Applying Jacobi's triple product identity,

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = (-zq,-q/z,q^2;q^2)_\infty, \tag{4.3}$$

one finds that many specializations of (4.1) and (4.2) are infinite products with nice combinatorial interpretations. Using Theorem 1.1, this leads to numerous identities for partitions, overpartitions, and overpartition pairs. For example, when $(a,b,q)=(1,1/q,q^2)$, we obtain an infinite product in (4.1) when $i=k$:

$$R_{k,k}(1,1/q;1;q^2)=\frac{(-q)_\infty(q^{2k-1};q^{2k-1})_\infty}{(q)_\infty(-q^{2k-1};q^{2k-1})_\infty}.$$

In the $r_{k,k}(s,t,m,n)$ of Theorem 1.1, the notion of an unattached part then transfers to overpartitions by saying that an odd part $2j-1$ occurs unattached if $2j,\overline{2j}$, and $\overline{2j-1}$ do not occur. The valuation $v_j((\lambda,\mu))$ becomes a valuation defined on overpartitions at even numbers, $v^1_{2j}(\lambda)=f_{2j}(\lambda)+f_{\overline{2j-1}}(\lambda)+f_{\overline{2j}}(\lambda)+\chi(2j-1 \text{ occurs unattached in } \lambda)$. The corresponding theorem is the overpartition analogue of the Andrews-Gordon-Göllnitz identities mentioned in the introduction:

Corollary 4.1 (Lovejoy, [33])**.** *Let $A^1_k(n)$ denote the number of overpartitions of n into parts not divisible by $2k-1$. Let $B^1_k(n)$ denote the number of overpartitions λ of n such that (i) $v^1_2(\lambda)\le i-1$ and (ii) for all $j\ge 1$ we have $f_{2j}(\lambda)+v^1_{2j+2}(\lambda)\le k-1$. Then $A^1_k(n)=B^1_k(n)$.*

All of the other results mentioned in the introduction (under Theorem 1.2) follow in the same way. Gordon's generalization of the Rogers-Ramanujan identities [30] corresponds to the case $R_{k,i}(0,0;1;q)$, Bressoud's Rogers-Ramanujan identities for even moduli [16] is the case $\tilde{R}_{k,i}(0,0;1;q)$, Andrews' generalization of the Göllnitz-Gordon identities [5] is the case $R_{k,i}(0,1/q;1;q^2)$, and the Gordon's theorems for overpartitions [32] are the cases $R_{k,k}(0,1;1;q)$ and $R_{k,1}(0,1/q;1;q)$. The reader may work out the details, or consult [25, 26], where these are discussed along with several other families of identities coming from the case $a=0$.

As another example when neither a nor b is 0, let us take $a=\sqrt{-1}$ and $b=-\sqrt{-1}$ in (4.2). We obtain an infinite product when $i=k-1$,

$$\tilde{R}_{k,k-1}(\sqrt{-1},-\sqrt{-1};1;q)=\frac{(-q)_\infty(-q^2;q^2)_\infty(q^{k-1};q^{k-1})_\infty}{(q)_\infty(q^2;q^2)_\infty(-q^{k-1};q^{k-1})_\infty}. \tag{4.4}$$

Applying Theorem 1.2 and splitting the generating functions into real and imaginary parts, we obtain two weighted identities, one of which is the following:

Corollary 4.2. *Let $A^2_k(n)$ denote the number of overpartition pairs where the parts of μ are even and the parts of λ are not divisible by $k-1$. Let $B^2_k(n)$ denote the number of overpartition pairs of n satisfying the conditions of Theorem 1.2 for $i=k-1$ (i.e., conditions (i) and (ii) of Theorem 1.1 and (1.6) if there is equality in condition (ii)), having an even number of overlined parts, and weighted by*

$$(\sqrt{-1})^{\#\ \textit{overlined parts in}\ \lambda}(-\sqrt{-1})^{\#\ \textit{overlined parts in}\ \mu}. \tag{4.5}$$

Then $A^2_k(n)=B^2_k(n)$.

It is interesting to note that $A^2_k(n)$ is a non-weighted counting function, while $B^2_k(n)$ is weighted. Other non-weighted interpretations of $A^2_k(n)$ may be found in [34].

For our last example, we consider the case $abq=1$. The apparent problem is that in this case there may be an unlimited number of non-overlined zeros in μ. Indeed, the term $1/(abxq)_\infty$ tends to infinity when $x = 1$. To remedy this, we shall not consider $R_{k,i}(a, b; 1; q)$ or $\tilde{R}_{k,i}(a, b; 1; q)$ as before, but the limits

$$\lim_{x\to 1}(1-x)R_{k,i}(a, b; x; q) \tag{4.6}$$

and

$$\lim_{x\to 1}(1-x)\tilde{R}_{k,i}(a, b; x; q). \tag{4.7}$$

These limits follow easily from (4.3):

$$\lim_{x\to 1}(1-x)R_{k,i}(a, 1/aq; x; q) = \frac{(-aq, -1/a)_\infty(q^{i-1}, q^{2k-i}, q^{2k-1}; q^{2k-1})_\infty}{(q)_\infty^2}. \tag{4.8}$$

$$\lim_{x\to 1}(1-x)\tilde{R}_{k,i}(a, 1/aq; x; q) = \frac{(-aq, -1/a)_\infty(q^{i-1}, q^{2k-i-1}, q^{2k-2}; q^{2k-2})_\infty}{(q)_\infty^2}. \tag{4.9}$$

On the other hand, recalling Abel's lemma, which states that

$$\lim_{x\to 1}(1-x)\sum_{n\geq 0} A_n x^n = \lim_{n\to\infty} A_n,$$

the limits (4.8) and (4.9) may be interpreted as the generating functions for those overpartition pairs counted by $r_{k,i}(s, t, \infty, n)$ and $\tilde{r}_{k,i}(s, t, \infty, n)$, respectively, the infinitude of the number of parts corresponding to an infinite number of non-overlined zeros in μ.

For example, if we take $(a, b, q) = (1/q, 1/q, q^2)$, then the infinite product in (4.8) is

$$\frac{(-q; q^2)_\infty^2(q^{2i-2}, q^{4k-2i}, q^{4k-2}; q^{4k-2})_\infty}{(q^2; q^2)_\infty^2}.$$

For the overpartition pairs of Theorem 1.1, those parts j in λ become $2j$, those parts $\overline{j}$ in λ or μ become $2j-1$, and those parts j in μ become $2j-2$. Given that the overlined parts are necessarily odd, the overlining becomes redundant and we can talk simply about partition pairs without repeated odd parts. The definition of unattached changes to: an even part $2j$ of μ is said to be unattached in the partition pair (λ, μ) if $2j+1$ doesn't occur in λ or μ and $2j+2$ does not occur in μ. The valuation function becomes a valuation at even numbers,

$$v_{2j}^3((\lambda, \mu)) = f_{2j}(\lambda) + f_{2j-1}(\lambda) + f_{2j-1}(\mu) + \chi(2j-2 \text{ occurs unattached in } \mu).$$

We may then state:

Corollary 4.3. *For $i \geq 2$ let $A_{k,i}^3(n)$ denote the number of partition pairs (λ, μ) of n such that the odd parts cannot be repeated, and the even parts of μ are not congruent to 0 or $\pm(2i-2)$ modulo $4k-2$. Let $B_{k,i}^3(n)$ denote the number of partition pairs (λ, μ) of n without repeated odd parts, where (i) $f_1(\lambda)+f_2(\lambda)+f_1(\mu) \leq i-1$ and (ii) for each $j \geq 1$ we have $f_{2j}(\lambda)+v_{2j+2}^3((\lambda, \mu)) \leq k-1$. Then $A_{k,i}^3(n) = B_{k,i}^3(n)$.*

Of course, a similar result holds for $(a, b, q) = (1/q, 1/q, q^2)$ using (4.9) and Theorem 1.2. This is left to the interested reader.

5. Lattice paths

In the next three sections, we shall prove Theorems 1.3 and 1.4. We begin in this section by defining the lattice paths counted by $E_{k,i}(s,t,n)$ and $\tilde{E}_{k,i}(s,t,n)$ and showing that their generating functions are (4.1) and (4.2), respectively.

We study paths in the first quadrant that use five kinds of unitary steps,

- North-East (NE): $(x,y) \to (x+1,y+1)$,
- South-East (SE): $(x,y) \to (x+1,y-1)$,
- South (S): $(x,y) \to (x,y-1)$,
- South-West (SW): $(x,y) \to (x-1,y-1)$, and
- East (E): $(x,0) \to (x+1,0)$,

with the following additional restrictions:

- A South or South-West step can only appear after a North-East step,
- An East step can only appear at height 0,
- The paths must start on the y-axis and end on the x-axis.

The most important notion associated with these paths is the peak. A *peak* is a vertex preceded by a North-East step and followed by a South step (in which case it can be labelled by a or b and called an a-peak or a b-peak, respectively), by a South-West step (in which case it is called an ab-peak) or by a South-East step (in which case it will be called a 1-peak). We say that a peak is *marked by a* if it is an a-peak or an ab-peak, and that it is *marked by b* if it is a b-peak or an ab-peak.

The *major index* of a path is the sum of the x-coordinates of its peaks. To avoid ambiguity in the graphical representation of a path, we add a label to the a-peaks and b-peaks and we may add a number above a vertex to indicate the presence of an otherwise indistinguishable ab-peak (see Figures 1 and 2 for examples).

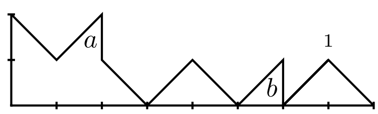

FIGURE 1. Example of a path. There are two 1-peaks (located at $(4,1)$ and $(7,1)$), an a-peak (located at $(2,2)$), a b-peak (located at $(6,1)$), and an ab-peak (located at $(7,1)$). The sequence of steps is SE-NE-S-SE-NE-SE-NE-S-NE-SW-NE-SE. The major index is $2+4+6+7+7=26$.

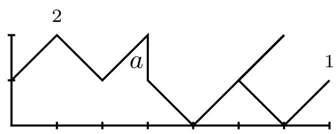

FIGURE 2. Another example. There is a 1-peak (located at $(1,2)$), an a-peak (located at $(3,2)$) and four ab-peaks (located at $(1,2)$, $(1,2)$, $(6,2)$ and $(7,1)$). The sequence of steps is NE-SW-NE-SW-NE-SE-NE-S-SE-NE-NE-SW-SE-NE-SW. The major index is $1+1+1+3+6+7=19$.

We call these lattice paths *generalized Bressoud-Burge lattice paths*, for they are generalizations of some lattice paths studied by Bressoud, [19], based on work of Burge [20, 21]. We might note that when there are no peaks marked by b, we recover the paths studied in [25] and [26].

We now define the (k,i)-conditions that appear in Theorems 1.3 and 1.4.

Definition 5.1. *We say that a path satisfies the* odd (k,i)-conditions *if it starts at height $k-i$ and its height is always less than k. We say that a path satisfies the* even (k,i)-conditions *if it satisfies the odd (k,i)-conditions and for each peak of coordinates $(x,k-1)$, we have $x-u+v \equiv i-1 \pmod 2$ where u is the number of a-peaks to the left of the peak and v is the number of b-peaks to the left of the peak.*

We first consider the paths satisfying the odd (k,i)-conditions. Let $E_{k,i}(n,s,t,N)$ be the number of paths of major index n with N peaks, s of which are marked by a and t of which are marked by b, which satisfy the odd (k,i)-conditions. For $0 < i \leq k$, let $\mathcal{E}_{k,i}(N)$ be the generating function for these paths, that is $\mathcal{E}_{k,i}(N) = \mathcal{E}_{k,i}(N,a,b,q) = \sum_{s,t,n} E_{k,i}(n,s,t,N)a^s b^t q^n$. Moreover, for $0 \leq i < k$, let $\Gamma_{k,i}(N)$ be the generating function for the paths obtained by deleting the first NE step of a path which is counted by $\mathcal{E}_{k,i+1}(N)$ and begins with a NE step.

Proposition 5.2. *We have the following:*

$$\mathcal{E}_{k,i}(N) = q^N\Gamma_{k,i-1}(N) + q^N\mathcal{E}_{k,i+1}(N) \qquad (0 < i < k), \tag{5.1}$$
$$\Gamma_{k,i}(N) = q^N\Gamma_{k,i-1}(N) + (a+b+q^{N-1}+abq^{1-N})\mathcal{E}_{k,i+1}(N-1) \qquad (0 < i < k), \tag{5.2}$$
$$\mathcal{E}_{k,k}(N) = q^N\Gamma_{k,k-1}(N) + q^N\mathcal{E}_{k,k}(N), \tag{5.3}$$
$$\mathcal{E}_{k,i}(0) = 1, \tag{5.4}$$
$$\Gamma_{k,0}(N) = 0. \tag{5.5}$$

Proof. The defining conditions of the generalized Bressoud-Burge paths imply that the path has no peaks if and only if $N = 0$. Hence $\mathcal{E}_{k,i}(0) = 1$ (corresponding to the path that starts at $(0,k-i)$ and descends with SE steps to $(k-i,0)$). This is (5.4). If the path has at least one peak, then we take off its first step and shift the path one unit to the left. If $0 < i < k$, then a path counted by $\mathcal{E}_{k,i}(N)$ starts with a North-East step (corresponding to $q^N\Gamma_{k,i-1}(N)$) or a South-East step (corresponding to $q^N\mathcal{E}_{k,i+1}(N)$). This gives (5.1). For (5.2), $\Gamma_{k,i}(N)$ is the generating function for the paths counted by $\mathcal{E}_{k,i+1}(N)$ where the first North-East step was deleted. These paths can start with a North-East step ($q^N\Gamma_{k,i-1}(N)$), a South step ($(a+b)\mathcal{E}_{k,i+1}(N-1)$), a South-East step ($q^{N-1}\mathcal{E}_{k,i+1}(N-1)$) or a South-West step ($abq^{1-N}\mathcal{E}_{k,i+1}(N-1)$). If $i = k$ then a path counted by $\mathcal{E}_{k,k}(N)$ starts with a North-East ($q^N\Gamma_{k,k-1}(N)$) or an East step ($q^N\mathcal{E}_{k,k}(N)$). The height of the paths is less than k, therefore no path which starts at height $k-1$ can start with a North-East step and so $\Gamma_{k,0}(N) = 0$. □

Notice that the recurrences and initial conditions above uniquely define the generating functions $\mathcal{E}_{k,i}(N)$ and $\Gamma_{k,i}(N)$. We shall exploit this fact to prove the following generating functions:

Theorem 5.3.

$$\begin{aligned}
\mathcal{E}_{k,i}(N) &= (ab)^N(-1/a,-1/b)_N q^N \sum_{n=-N}^{N} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-1)n}}{(q)_{N-n}(q)_{N+n}} \\
\Gamma_{k,i}(N) &= (ab)^N(-1/a,-1/b)_N \sum_{n=-N}^{N-1} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-2)n}}{(q)_{N-n-1}(q)_{N+n}}
\end{aligned}$$

Proof. Let

$$\begin{aligned}
\mathcal{E}'_{k,i}(N) &= (ab)^N(-1/a,-1/b)_N q^N \sum_{n=-N}^{N} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-1)n}}{(q)_{N-n}(q)_{N+n}} \\
\Gamma'_{k,i}(N) &= (ab)^N(-1/a,-1/b)_N \sum_{n=-N}^{N-1} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-2)n}}{(q)_{N-n-1}(q)_{N+n}}.
\end{aligned}$$

We will prove that these functions satisfy the five equations in Proposition 5.2. We begin with (5.1):

$$\begin{aligned}
& q^N\mathcal{E}'_{k,i+1}(N) + q^N\Gamma'_{k,i-1}(N) \\
= \; & (ab)^N(-1/a,-1/b)_N q^N \sum_{n=-N}^{N} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-2)n}}{(q)_{N-n}(q)_{N+n}} q^N \\
+ \; & (ab)^N(-1/a,-1/b)_N \sum_{n=-N}^{N-1} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-1)n}}{(q)_{N-n-1}(q)_{N+n}} q^N \\
= \; & (ab)^N(-1/a,-1/b)_N q^N \sum_{n=-N}^{N-1} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-1)n}}{(q)_{N-n}(q)_{N+n}} \left(q^{N-n} + (1-q^{N-n})\right) \\
+ \; & (ab)^N(-1/a,-1/b)^N q^N \frac{(-1)^N q^{N((2k-1)N+3)/2+(k-i-1)N}}{(q)_0(q)_{2N}} \\
= \; & (ab)^N(-1/a,-1/b)_N q^N \sum_{n=-N}^{N} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-1)n}}{(q)_{N-n}(q)_{N+n}} \\
= \; & \mathcal{E}'_{k,i}(N).
\end{aligned}$$

This gives (5.1). Notice that the above string of equations holds for $i = k$. Next, we establish (5.2):

$$\begin{aligned}
& q^N\Gamma'_{k,i-1}(N) + (a+b+q^{N-1}+q^{1-N}ab)\mathcal{E}'_{k,i+1}(N-1) \\
= \;& (ab)^N(-1/a,-1/b)_N \sum_{n=-N}^{N-1} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-1)n}}{(q)_{N-n-1}(q)_{N+n}} q^N \\
+ \;& (ab)^{N-1}(-1/a,-1/b)_{N-1} \sum_{n=-N+1}^{N-1} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-2)n}}{(q)_{N-n-1}(q)_{N+n-1}} (a+b+q^{N-1}+q^{1-N}ab) \\
= \;& (ab)^N(-1/a,-1/b)_N \sum_{n=-N}^{N-1} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-2)n}}{(q)_{N-n-1}(q)_{N+n}} q^{N+n} \\
+ \;& (ab)^{N-1}(-1/a,-1/b)_{N-1} \sum_{n=-N+1}^{N-1} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-2)n}}{(q)_{N-n-1}(q)_{N+n-1}} \\
& \quad \times abq^{1-N}(1+a^{-1}q^{N-1})(1+b^{-1}q^{N-1}) \\
= \;& (ab)^N(-1/a,-1/b)_N \sum_{n=-N+1}^{N-1} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-2)n}}{(q)_{N-n-1}(q)_{N+n}} \left(q^{N+n} + (1-q^{N+n})\right) \\
+ \;& (ab)^N(-1/a,-1/b)_N \frac{(-1)^{-N} q^{-N((2k-1)(-N)+3)/2+(k-i-2)(-N)}}{(q)_{2N-1}(q)_0} \\
= \;& (ab)^N(-1/a,-1/b)_N \sum_{n=-N}^{N-1} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-2)n}}{(q)_{N-n-1}(q)_{N+n}} \\
= \;& \Gamma'_{k,i}(N).
\end{aligned}$$

For (5.3), we prove that $\mathcal{E}'_{k,k+1}(N) = \mathcal{E}'_{k,k}(N)$ and then combine this with the fact that the $\mathcal{E}'_{k,i}(N)$ satisfy (5.1) for $i = k$.

$$\begin{aligned}
\mathcal{E}'_{k,k+1}(N) &= (ab)^N(-1/a,-1/b)_N q^N \sum_{n=-N}^{N} \frac{(-1)^n q^{n((2k-1)n+3)/2-2n}}{(q)_{N-n}(q)_{N+n}} \\
&= (ab)^N(-1/a,-1/b)_N q^N \sum_{n=-N}^{N} \frac{(-1)^n q^{-n((2k-1)(-n)+3)/2+2n}}{(q)_{N+n}(q)_{N-n}} \qquad \text{(replacing } n \text{ by } -n) \\
&= (ab)^N(-1/a,-1/b)_N q^N \sum_{n=-N}^{N} \frac{(-1)^n q^{n((2k-1)n+3)/2-n}}{(q)_{N+n}(q)_{N-n}} \\
&= \mathcal{E}'_{k,k}(N)
\end{aligned}$$

Hence we have, using (5.1):

$$\mathcal{E}'_{k,k}(N) = q^N\mathcal{E}'_{k,k}(N) + q^N\Gamma'_{k,k-1}(N).$$

Notice that (5.4) is immediate. Finally, for (5.5), we have

$$\begin{aligned}
&\Gamma'_{k,0}(N)\\
=\ &(ab)^N(-1/a,-1/b)_N \sum_{n=-N}^{N-1} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-2)n}}{(q)_{N-n-1}(q)_{N+n}}\\
=\ &(ab)^N(-1/a,-1/b)_N \left(\sum_{n=0}^{N-1} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-2)n}}{(q)_{N-n-1}(q)_{N+n}} + \sum_{n=-N}^{-1} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-2)n}}{(q)_{N-n-1}(q)_{N+n}}\right).
\end{aligned}$$

Replacing n by $-n-1$ in the second sum and simplifying gives the negative of the first sum, which shows that $\Gamma'_{k,0}(N) = 0$. Now, since $\mathcal{E}'_{k,i}(N)$ and $\Gamma'_{k,i}(N)$ satisfy the same defining recurrences and initial conditions as $\mathcal{E}_{k,i}(N)$ and $\Gamma_{k,i}(N)$, we have $\mathcal{E}_{k,i}(N) = \mathcal{E}'_{k,i}(N)$ and $\Gamma_{k,i}(N) = \Gamma'_{k,i}(N)$, which completes the proof. □

We are almost ready to prove that $E(s,t,n) = B(s,t,n)$ in Theorem 1.3. We just need a q-series lemma.

Lemma 5.4. *For any integer n, we have*

$$\sum_{N\geq |n|} \frac{(-q^n/a,-q^n/b)_{N-n}(abq)^{N-n}(-aq,-bq)_n}{(q)_{N+n}(q)_{N-n}} = \frac{(-aq,-bq)_\infty}{(q,abq)_\infty} \tag{5.6}$$

Proof. We only prove the case $n \geq 0$. The case $n < 0$ is identical, as one may compute that

$$\frac{(-q^{-n}/a,-q^{-n}/b)_{N+n}(abq)^{N+n}(-aq,-bq)_{-n}}{(q)_{N-n}(q)_{N+n}} = \frac{(-q^n/a,-q^n/b)_{N-n}(abq)^{N-n}(-aq,-bq)_n}{(q)_{N+n}(q)_{N-n}}.$$

We have

$$\begin{aligned}
&\sum_{N\geq n} \frac{(-q^n/a,-q^n/b)_{N-n}(abq)^{N-n}(-aq,-bq)_n}{(q)_{N+n}(q)_{N-n}}\\
=\ &\sum_{N\geq 0} \frac{(-q^n/a,-q^n/b)_{N}(abq)^{N}(-aq,-bq)_n}{(q)_{N+2n}(q)_{N}}\\
=\ &\frac{(-aq,-bq)_n}{(q)_{2n}} \sum_{N\geq 0} \frac{(-q^n/a,-q^n/b)_N(abq)^N}{(q,q^{2n+1})_N}\\
=\ &\frac{(-aq,-bq)_n}{(q)_{2n}} \frac{(-aq^{n+1},-bq^{n+1})_\infty}{(q^{2n+1},abq)_\infty}\\
&\text{by Corollary 2.4 of [12] with } n\to N,\ a\to -q^n/a,\ b\to -q^n/b \text{ and } c\to q^{2n+1}\\
=\ &\frac{(-aq,-bq)_\infty}{(q,abq)_\infty}.
\end{aligned}$$

□

Proof of the case $B_{k,i}(s,t,n) = E_{k,i}(s,t,n)$ of Theorem 1.3. Using the generating function from Theorem 5.3 and summing on N using Lemma 5.4, we have

$$\begin{aligned}
\sum_{s,t,n\geq 0} E_{k,i}(s,t,n)a^s b^t q^n &= \sum_{N\geq 0} \mathcal{E}_{k,i}(N) \\
&= \sum_{N\geq 0} (ab)^N (-1/a,-1/b)_N q^N \sum_{n=-N}^{N} \frac{(-1)^n q^{n((2k-1)n+3)/2+(k-i-1)n}}{(q)_{N-n}(q)_{N+n}} \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{n((2k-1)n+3)/2+(k-i-1)n} \sum_{N\geq |n|} \frac{(ab)^N(-1/a,-1/b)_N q^N}{(q)_{N-n}(q)_{N+n}} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-ab)^n(-1/a,-1/b)_n q^{n((2k-1)n+3)/2+(k-i)n}}{(-aq)_n(-bq)_n} \times \\
&\qquad \sum_{N\geq |n|} \frac{(abq)^{N-n}(-q^n/a,-q^n/b)_{N-n}}{(q)_{N-n}(q)_{N+n}} \\
&= \frac{(-aq)_\infty(-bq)_\infty}{(q)_\infty(abq)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-ab)^n(-1/a,-1/b)_n q^{n((2k-1)n+3)/2+(k-i)n}}{(-aq)_n(-bq)_n} \\
&= R_{k,i}(a,b;1;q) \quad \text{(by (4.1))} \\
&= \sum_{s,t,n\geq 0} B_{k,i}(s,t,n)a^s b^t q^n \quad \text{(by Theorem 1.1).}
\end{aligned}$$

Hence we have

$$E_{k,i}(s,t,n) = B_{k,i}(s,t,n).$$

□

We now treat the paths satisfying the even (k,i)-conditions. Many of the arguments are similar to those for the paths satisfying the odd (k,i)-conditions. Hence, we shall not be as verbose with details. Let $\tilde{E}_{k,i}(n,s,t,N)$ be the number of paths of major index n with N peaks, s of which are marked by a and t of which are marked by b, which satisfy the even (k,i)-conditions. Let $\tilde{\mathcal{E}}_{k,i}(N)$ and $\tilde{\Gamma}_{k,i}(N)$ be the even analogues of $\mathcal{E}_{k,i}(N)$ and $\Gamma_{k,i}(N)$.

Proposition 5.5.

$$\tilde{\mathcal{E}}_{k,i}(N) = q^N\tilde{\Gamma}_{k,i-1}(N) + q^N\tilde{\mathcal{E}}_{k,i+1}(N) \qquad (0 < i < k), \tag{5.7}$$

$$\tilde{\Gamma}_{k,i}(N) = q^N\tilde{\Gamma}_{k,i-1}(N) + (a+b+q^{N-1}+abq^{1-N})\tilde{\mathcal{E}}_{k,i+1}(N-1) \qquad (0 < i < k), \tag{5.8}$$

$$\tilde{\mathcal{E}}_{k,k}(N) = q^N\tilde{\mathcal{E}}_{k,k-1}(N) + q^N\tilde{\Gamma}_{k,k-1}(N), \tag{5.9}$$

$$\tilde{\mathcal{E}}_{k,i}(0) = 1, \tag{5.10}$$

$$\tilde{\Gamma}_{k,0}(N) = 0. \tag{5.11}$$

Proof. If $i < k$, we proceed just as in the proof of the Proposition 5.2. If the path is not empty, then taking off its first step increases or decreases i by 1 and thus changes the parity of $i-1$. Moreover, all the peaks are shifted by 1, so the parity of $x-u+v$ is not changed (for the recurrence for $\tilde{\Gamma}_{k,i}(N)$, if the step we remove is a South step, the peaks are not shifted but u or v decreases by 1 for all peaks, so the result is the same).

The case $i = k$ needs further explanation. The paths counted by $\tilde{\mathcal{E}}_{k,k}(N)$ begin with either an East or a North-East step. Those that begin with a North-East step where this step is deleted are the paths counted by $\tilde{\Gamma}_{k,k-1}(N)$. Shifting these one unit to the left contributes the term $q^N\tilde{\Gamma}_{k,k-1}(N)$.

For the paths that begin with an East step, first observe that the fact that every peak of coordinates $(x, k-1)$ satisfies $x - u + v \equiv k - 1 \pmod 2$ is equivalent to the fact that every peak of coordinates $(x, k-1)$ has an even number of East steps to its left. We now consider two cases for the paths counted in $\tilde{\mathcal{E}}_{k,k}(N)$ that start with an East step where this step has been deleted. If the path does not have any other East step, then there is no peak of height $k - 1$ and so we may shift the path upward, i.e. each vertex of the path (x, y) is changed to $(x, y+1)$. Shifting to the left then creates a path in $\tilde{\mathcal{E}}_{k,k-1}(N)$ that does not have any vertex of the form $(x, 0)$. If the path does contain another East step, then the path before the first of these other East steps is shifted up, the East step is changed to a South-East step and the rest of the path is not changed. Shifting to the left creates a path in $\tilde{\mathcal{E}}_{k,k-1}(N)$ that has at least one vertex of the form $(x, 0)$. This gives the term $q^N\tilde{\mathcal{E}}_{k,k-1}(N)$. □

As in the odd case, the recurrences and initial conditions above uniquely define the functions $\tilde{\mathcal{E}}_{k,i}(N)$ and $\tilde{\Gamma}_{k,i}(N)$. In this case, we have

Theorem 5.6.

$$\begin{aligned}\tilde{\mathcal{E}}_{k,i}(N) &= (ab)^N(-1/a,-1/b)_N q^N \sum_{n=-N}^{N} (-1)^n \frac{q^{kn^2+(k-i-1)n-2\binom{n}{2}}}{(q)_{N-n}(q)_{N+n}} \\ \tilde{\Gamma}_{k,i}(N) &= (ab)^N(-1/a,-1/b)_N \sum_{n=-N}^{N-1} (-1)^n \frac{q^{kn^2+(k-i-2)n-2\binom{n}{2}}}{(q)_{N-n-1}(q)_{N+n}}\end{aligned}$$

The proof is omitted since it is very similar to that of Theorem 5.3.

Proof of the case $\tilde{B}_{k,i}(s,t,n) = \tilde{E}_{k,i}(s,t,n)$ of Theorem 1.4. This is identical to the case of $B_{k,i}(s,t,n) = E_{k,i}(s,t,n)$ proven above. Summing the generating function for $\tilde{\mathcal{E}}_{k,i}(N)$ over N in Theorem 5.6, changing the order of summation and using Lemma 5.4 we get

$$\begin{aligned}\sum_{s,t,n\geq 0} \tilde{E}_{k,i}(s,t,n)a^s b^t q^n &= \tilde{R}_{k,i}(a,b;1;q) \\ &= \sum_{s,t,n\geq 0} \tilde{B}_{k,i}(s,t,n)a^s b^t q^n,\end{aligned}$$

and we conclude that

$$\tilde{E}_{k,i}(s,t,n) = \tilde{B}_{k,i}(s,t,n).$$

□

6. Successive Ranks

In this section we turn to the overpartition pairs counted by $C_{k,i}(s,t,n)$ and $\tilde{C}_{k,i}(s,t,n)$. We construct a bijection between the relevant pairs and the lattice paths of the previous section, which will establish the equality of $C_{k,i}(s,t,n)$ and $E_{k,i}(s,t,n)$ (resp. $\tilde{C}_{k,i}(s,t,n)$ and $\tilde{E}_{k,i}(s,t,n)$). This is a generalization of overpartition-theoretic work in [25] and [26].

The Frobenius representation of an overpartition pair [23, 24, 34] of n is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_N \\ b_1 & b_2 & \dots & b_N \end{pmatrix}$$

where $(a_1, \dots, a_N)$ and $(b_1, \dots, b_N)$ are overpartitions into nonnegative parts where $N + \sum(a_i + b_i) = n$. This is called the Frobenius representation of an overpartition pair because these arrays are in bijection with overpartition pairs of n [23, 38].

We now define the successive ranks of an overpartition pair using the Frobenius representation.

Definition 6.1. *If an overpartition pair has Frobenius representation*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ b_1 & b_2 & \cdots & b_N \end{pmatrix}$$

then its ith successive rank r_i is $a_i - b_i$ minus the number of non-overlined parts in $\{b_{i+1}, \dots, b_N\}$ plus the number of non-overlined parts in $\{a_{i+1}, \dots, a_N\}$.

For example, the successive ranks of $\begin{pmatrix} \overline{7} & 4 & \overline{2} & 0 \\ \overline{3} & 3 & 1 & \overline{0} \end{pmatrix}$ are $(4, 1, 2, 0)$.

We shall prove the following:

Proposition 6.2. *There exists a one-to-one correspondence between the paths of major index n counted by $E_{k,i}(s, t, n)$ and the overpartition pairs of n counted by $C_{k,i}(s, t, n)$. This correspondence is such that the paths have N peaks if and only if the Frobenius representation of the overpartition pair has N columns.*

Proof. We prove this proposition by a direct mapping which is a generalization of a mapping in [26]. Given a lattice path counted by $E_{k,i}(s, t, n)$, which starts at $(0, k - i)$, and a peak (x, y), let u (resp. v) be the number of a-peaks to the left of the peak (resp. the number of b-peaks to the left of the peak). Starting on the left of the path, we construct a two-rowed array from the right by mapping this peak to a column $\begin{pmatrix} p \\ q \end{pmatrix}$, where

$$p = (x + k - i - y + u - v)/2 \tag{6.1}$$

and

$$q = (x - k + i + y - 2 - u + v)/2, \tag{6.2}$$

if there are an even number of East steps to the left of the peak, and

$$p = (x + k - i + y - 1 + u - v)/2 \tag{6.3}$$

and

$$q = (x - k + i - y - 1 - u + v)/2, \tag{6.4}$$

if there are an odd number of East steps to the left of the peak. Moreover, we overline the corresponding parts as follows:

- if the peak is a 1-peak, we overline p and q,
- if the peak is an a-peak, we overline p, and
- if the peak is a b-peak, we overline q.

For example, the path counted by $E_{5,3}(3,4,115)$ in Figure 3 below maps to the overpartition pair counted by $C_{5,3}(3,4,115)$ whose Frobenius representation is

$$\begin{pmatrix} 14 & \overline{12} & 12 & 8 & \overline{7} & \overline{4} & \overline{3} & 2 \\ \overline{9} & \overline{8} & 8 & \overline{7} & \overline{5} & \overline{4} & 3 & 1 \end{pmatrix}.$$

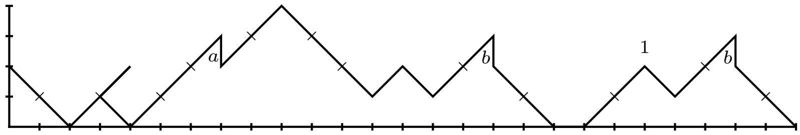

FIGURE 3. A path counted by $E_{5,3}(3,4,115)$.

To establish the proposition, we must show that the result of this mapping is indeed the Frobenius representation of an overpartition pair counted by $C_{k,i}(s,t,n)$ and that the mapping is invertible. This is a somewhat tedious argument, involving a few pages of small calculations and observations. Ultimately we will omit some details where these are similar to previous ones.

First, it may not even be clear that p and q defined above are integers. To see this, note that at the starting point $(x,y)=(0,k-i)$ of the path, the quantities p and q in (6.1) and (6.2) are integers. The parities of $x-y$, $x+y$, $u-v$, and $u+v$ are preserved by NE, SE, and SW steps, while a S step changes the parity of each of these. The only problem is with an E step, which changes the parity of $x-y$ and $x+y$. This gives rise to the two cases for the definition in p and q, which guarantees that the two-rowed array contains integer entries.

Next, it is clear that the number of peaks in the path is equal to the number of columns in the corresponding array. It is also clear that if the path contains s (resp. t) peaks marked by a (resp. b), then the two-rowed array has s (resp. t) non-overlined parts in the bottom (resp. top) row. Regarding n, in either definition of p and q above, we have

$$p+q+1=x. \tag{6.5}$$

Hence, if n is the major index of the path, then n is the sum of all entries of the corresponding array and the number of columns.

Applying Definition 6.1, we compute the successive ranks of the two-rowed array. The peaks all have height at least one, thus for a peak (x,y) which is preceded by an even number of East steps, we have:

$$\begin{aligned} & 1 \le y = k-i+1+q-p+u-v \le k-1 \\ \Leftrightarrow\quad & -i+2 \le p-q-u+v \le k-i \\ \Leftrightarrow\quad & \text{the corresponding successive rank is } \ge -i+2 \text{ and } \le k-i, \end{aligned} \tag{6.6}$$

and if the peak is preceded by an odd number of East steps, we have:

$$\begin{aligned} & 1 \le y = p-q-u+v-k+i \le k-1 \\ \Leftrightarrow\quad & k-i+1 \le p-q-u+v \le 2k-i-1 \\ \Leftrightarrow\quad & \text{the corresponding successive rank is } \ge k-i+1 \text{ and } \le 2k-i-1. \end{aligned} \tag{6.7}$$

Hence, the successive ranks of the two-rowed array are all in the interval $[-i+2, 2k-i-1]$.

Finally, we need to prove that the two-rowed array we constructed has an overpartition into non-negative parts in each row. In what follows, let (x_j, y_j) be the coordinates of the jth peak from the right and $\binom{p_j}{q_j}$ be the corresponding column, the jth column from the left.

First, we show that $p_N \geq 0$. If the leftmost peak has an even number of East steps to its left, then $p_N = (x_N + k - i - y_N)/2$. It is obvious that any vertex has a greater (or equal) value of $x - y$ than the previous vertex in the path. Since the path begins at $(0, k-i)$, we have $x - y = -k + i$ at the beginning of the path and thus we have $x - y \geq -k + i$ for all vertices and in particular for the leftmost peak. Now if that peak has an odd number of East steps to its left, then $p_N = (x_N + y_N + k - i - 1)/2$. Since $x_N \geq 1$ and $y_N \geq 1$, we get that $p_N \geq 0$.

Next, we show that $q_N \geq 0$. This can be proven similarly. If the leftmost peak has an even number of East steps to the left, then $q_N = (x_N + y_N - (k-i) - 2)/2$. The path begins at $(0, k-i)$, the only steps allowed before the first peak do not decrease $x+y$, and there must be one NE step before the first peak, which increases $x+y$ by 2. Hence $x_N + y_N - (k-i) - 2 \geq 0$. If the leftmost peak has an odd number of East steps to the left, then $q_N = (x_N - y_N - (k-i) - 1)/2$. Here the path passes through the point $(k-i+1, 0)$, and since $x - y$ never decreases we have $q_N \geq 0$.

Having shown that all entries of the two-rowed array are non-negative, we now argue that the sequences $\{p_j\}$ and $\{q_j\}$ are overpartitions, i.e., that $p_j \geq p_{j+1}$ (resp. $q_j \geq q_{j+1}$) with strict inequality if p_{j+1} (resp. q_{j+1}) is overlined. Let us show first that $p_j \geq p_{j+1}$. We consider four cases. If the jth peak and the $j+1$th peak both have an even number of East steps to their left, then $p_j - p_{j+1} = (x_j - x_{j+1} - y_j + y_{j+1} + u_j - u_{j+1} - v_j + v_{j+1})/2$. We always have $x_j - x_{j+1} \geq y_j - y_{j+1}$. We can only have $u_j - u_{j+1} - v_j + v_{j+1} < 0$ if the $j+1$th peak is a b-peak, but in that case we have $x_j - x_{j+1} > y_j - y_{j+1}$. If the jth peak and the $j+1$th peak both have an odd number of East steps to their left, the proof is identical. If the jth peak has an odd number of East steps to its left and the $j+1$th peak has an even number of East steps to its left, the result is easily shown using the fact that $x_j - x_{j+1} \geq 2$ since there is at least an East step between the two peaks. In the final case, where the jth peak has an even number of East steps to the left and the $j+1$th peak has an odd number of East steps to its left, we have $p_j - p_{j+1} = (x_j - x_{j+1} - y_j - y_{j+1} + 1 + u_j - u_{j+1} - v_j + v_{j+1})/2$. Since there is at least one East step between the jth peak and the $j+1$th peak, we have $x_j - x_{j+1} \geq y_j + y_{j+1}$ unless the $j+1$th peak is an ab-peak (see Figure 4). Since $u_j - u_{j+1}$ can only be equal to 0 or 1 (the same holds for $v_j - v_{j+1}$), we have $1 + u_j - u_{j+1} - v_j + v_{j+1} \geq 0$ and therefore, $p_j - p_{j+1} \geq 0$. If the $j+1$th peak is an ab-peak, we have $x_j - x_{j+1} \geq y_j + y_{j+1} - 1$, $u_j = u_{j+1}$ and $v_j = v_{j+1}$. Thus, we also have $p_j - p_{j+1} \geq 0$.

So, we have seen that in all cases we have $p_j \geq p_{j+1}$. If p_{j+1} is overlined, then the $j+1$th peak is a 1-peak or an a-peak. Going back through the above arguments, one finds that the inequality $p_j \geq p_{j+1}$ is strict when the $j+1$th peak is a 1-peak or a a-peak. The proof for the $\{q_j\}$ is quite similar to the case of the $\{p_j\}$ above, so we omit the details.

That the mapping is invertible is rather straightforward. Beginning at the left of the Frobenius representation of an overpartition pair, at any column $\binom{p}{q}$ the values of u and v are determined by the overlined parts in the columns to the right. Hence to recover the location (x, y) of a peak in the path from the column, we need to solve either equations (6.1) and (6.2) or equations (6.3) and (6.4) for x and y. Only one set of these equations can be solved for positive x and y.

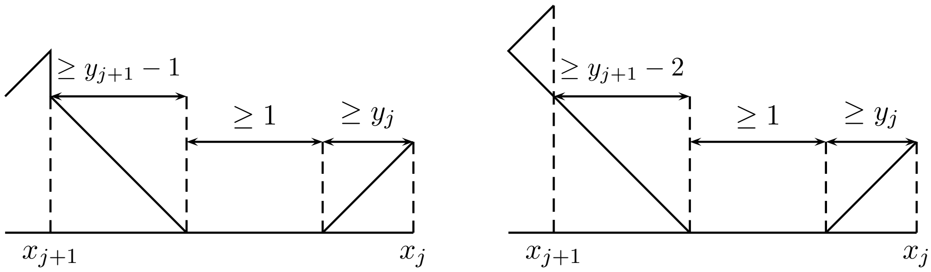

FIGURE 4. If the $j+1$th peak is not an ab-peak (left), we have $x_j - x_{j+1} \geq y_j + y_{j+1}$. If the $j+1$th peak is an ab-peak (right), we only have $x_j - x_{j+1} \geq y_j + y_{j+1} - 1$ but since $u_j = u_{j+1}$ and $v_j = v_{j+1}$, $p_j - p_{j+1}$ is indeed nonnegative.

Equation (6.5) shows that x does not increase as we proceed. There is a unique way to fill in the steps between the peaks, and the computations in (6.6) and (6.7) show that the path never goes above height $k-1$. This completes the proof of the proposition. □

We will conclude this section by stating and proving the analogue of Proposition 6.2 for the functions $\tilde{C}_{k,i}(s,t,n)$ and $\tilde{E}_{k,i}(s,t,n)$.

Proposition 6.3. *There exists a one-to-one correspondence between the paths of major index n counted by $\tilde{E}_{k,i}(s,t,n)$ and the overpartition pairs of n counted by $\tilde{C}_{k,i}(s,t,n)$. This correspondence is such that the paths have N peaks if and only if the Frobenius representation of the overpartition pair has N columns.*

Proof. A path counted by $\tilde{E}_{k,i}(s,t,n)$ is also counted by $E_{k,i}(s,t,n)$ and an overpartition pair counted by $\tilde{C}_{k,i}(s,t,n)$ is also counted by $C_{k,i}(s,t,n)$. Hence we may apply the bijection used in the proof of Proposition 6.2 to a path counted by $\tilde{E}_{k,i}(s,t,n)$. We must then show that such paths correspond to overpartition pairs where no successive rank can be equal to $2k-i-1$. Indeed, if this was the case, we would have $p-q-u+v = 2k-i-1$ and from the map we know that $p-q-u+v = k-i-y+1$ or $k-i+y$. The first case is impossible when $k \geq 2$. The second case implies that $y = k-1$ and $p = (x+u-v+2k-i-2)/2$. As p is an integer, we have $x-u+v \equiv i \pmod 2$. This is forbidden by the last condition of the definition of $\tilde{E}_{k,i}(s,t,n)$. □

7. THE DURFEE DISSECTION AND A FAMILY OF CONJUGATIONS FOR OVERPARTITION PAIRS

In this section we discuss the overpartition pairs counted by $D_{k,i}(s,t,n)$ in Theorem 1.3 and by $\tilde{D}_{k,i}(s,t,n)$ in Theorem 1.4. We complete our proof of these two theorems using generating function identities to show that these quantities are equal to $B_{k,i}(s,t,n)$ and $\tilde{B}_{k,i}(s,t,n)$, respectively. The idea is to extend work of Andrews [11] and Garvan [28] to overpartition pairs via the Frobenius representation.

We begin by recalling a useful little bijection for overpartitions, called the Joichi-Stanton algorithm [31]. From an overpartition α into N nonnegative parts, we obtain a partition λ into N nonnegative parts and a partition μ into distinct nonnegative parts less than N as follows: First, we initialize λ to α. Then, if the mth part of α is overlined, we remove the overlining of the mth part of λ, we decrease the $m-1$ first parts of λ by one and we add a part $m-1$ to μ.

Definition 7.1. *We say that λ is the* associated partition *of α.*

Thus, given an overpartition pair we may decompose its Frobenius representation into four partitions λ_1, μ_1, λ_2, μ_2, where λ_1 and μ_1 (resp. λ_2 and μ_2) are obtained by applying the Joichi-Stanton algorithm to the top (resp. bottom) row. For example, the overpartition pair whose Frobenius representation is

$$\pi = \begin{pmatrix} 12 & 12 & \overline{8} & 7 & 6 & \overline{3} & 2 & \overline{1} \\ 14 & 12 & \overline{10} & \overline{8} & 6 & 5 & \overline{3} & 2 \end{pmatrix}$$

gives $\lambda_1 = (9,9,6,5,4,2,1,1)$, $\mu_1 = (7,5,2)$, $\lambda_2 = (11,9,8,7,5,4,3,2)$ and $\mu_2 = (6,3,2)$.

Next, we describe the notion of a (k,i)-admissible overpartition pair occurring in the statement of Theorem 1.3. This is similar to, but not exactly the same as, the concept of (k,i)-admissibility in [11]. Recall that the Durfee square of a partition is the largest upper-left-justified square that fits inside the Ferrers diagram of the partition [6]. Below such a square, there is another partition and one may identify its Durfee square, and so on, to obtain a sequence of successive Durfee squares.

Definition 7.2. *We say that an overpartition pair is (k,i)-admissible if the conjugate, λ_2', of the associated partition λ_2 of the bottom row of its Frobenius representation is obtained from a partition ν into non-negative parts with at most $k-2$ Durfee squares by inserting a part of size n_j into ν for each j with $i \leq j \leq k-1$. Here n_j is the size of the $j-1$th Durfee square of ν, where the size of the 0th Durfee square is taken to be the number of columns in the Frobenius representation of the overpartition pair.*

Proposition 7.3. *Recall the definition of $D_{k,i}(s,t,n)$ from Theorem 1.3. We have the following generating function:*

$$\sum_{s,t,n\geq 0} D_{k,i}(s,t,n)a^s b^t q^n = \sum_{n_1\geq\cdots\geq n_{k-1}\geq 0} \frac{q^{n_1+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}(-1/a,-1/b)_{n_1}a^{n_1}b^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}}. \quad (7.1)$$

Proof. Consider an overpartition pair counted by $D_{k,i}(s,t,n)$ whose Frobenius representation has n_1 columns. By using the Joichi-Stanton algorithm on each row, we can decompose our overpartion pair in the following way:

- the top row, which is counted by
$$\frac{(-1/b)_{n_1}b^{n_1}}{(q)_{n_1}},$$
- the partition μ_2 into n_1 nonnegative parts coming from the bottom row, which is counted by $(-1/a)_{n_1}a^{n_1}$,
- the n_1 columns, which are counted by q^{n_1},
- the at most $k-2$ Durfee squares of a partition ν, which are counted by $q^{n_2^2+\cdots+n_{k-1}^2}$,
- the regions between the Durfee squares, which are counted by
$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q,$$
where
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k(q)_{n-k}}$$

is the generating function for partitions whose Ferrers diagrams fit inside a $(n-k)\times k$ rectangle, and

- the inserted parts, counted by

$$q^{n_i+\cdots+n_{k-1}}.$$

These last three together make up the conjugate λ_2' of the associated partition of the bottom row. Summing on $n_1,\ldots,n_{k-1}$, we get the generating function:

$$\sum_{n_1\geq n_2\geq\cdots\geq n_{k-1}\geq 0}\frac{(-1/b)_{n_1}b^{n_1}}{(q)_{n_1}}(-1/a)_{n_1}a^{n_1}q^{n_1}q^{n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}\begin{bmatrix}n_1\\n_2\end{bmatrix}_q\cdots\begin{bmatrix}n_{k-2}\\n_{k-1}\end{bmatrix}_q$$

$$=\sum_{n_1\geq n_2\geq\cdots\geq n_{k-1}\geq 0}\frac{q^{n_1+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}(-1/a)_{n_1}a^{n_1}(-1/b)_{n_1}b^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}}.$$

□

To incorporate $D_{k,i}(s,t,n)$ into Theorem 1.3, we use the Bailey lattice structure from [1] to transform the generating function above to (4.1). Recall that a pair of sequences (α_n,β_n) form a Bailey pair with respect to a if for all $n\geq 0$ we have

$$\beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}.$$

We shall employ the following lemma:

Lemma 7.4. *If (α_n,β_n) is a Bailey pair with respect to q, then for all $0\leq i\leq k$ we have*

$$\begin{aligned}\frac{(abq)_\infty}{(q,-aq,-bq)_\infty}\times&\sum_{n_1\geq\cdots\geq n_k\geq 0}\frac{q^{n_1+n_2^2+\cdots n_k^2+n_{i+1}+\cdots n_k}(-1/a,-1/b)_{n_1}(ab)^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}}\beta_{n_k}\\ =&\ \frac{\alpha_0}{(q)_\infty^2}+\frac{1}{(q)_\infty^2}\sum_{n\geq 1}\frac{(-1/a,-1/b)_n(ab)^nq^{(n^2-n)(i-1)+in}(1-q)}{(-aq,-bq)_n}\\ \times&\left(\frac{q^{(n^2+n)(k-i)}}{(1-q^{2n+1})}\alpha_n-\frac{q^{((n-1)^2+(n-1))(k-i)+2n-1}}{(1-q^{2n-1})}\alpha_{n-1}\right)\end{aligned}\tag{7.2}$$

Proof. This a special case of identity (3.8) in [1]. Specifically, we let $a=q$, $\rho_1=-1/a$, $\sigma_1=-1/b$, and then let n as well as all remaining ρ_i and σ_i tend to ∞ in that identity to obtain (7.2). □

Proof of the case $B_{k,i}(s,t,n)=D_{k,i}(s,t,n)$ of Theorem 1.3. We use the Bailey pair with respect to q [37, p.468, (B3)],

$$\beta_n=\frac{1}{(q)_n}\qquad\text{and}\qquad\alpha_n=\frac{(-1)^nq^{n(3n+1)/2}(1-q^{2n+1})}{(1-q)}.$$

Substituting into Lemma 7.4 and simplifying, we obtain

$$\sum_{n_1\geq\cdots\geq n_k\geq 0}\frac{q^{n_1+n_2^2+\cdots+n_k^2+n_{i+1}+\cdots+n_k}(-1/a,-1/b)_{n_1}a^{n_1}b^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{n_k}}$$

$$=\frac{(-aq,-bq)_\infty}{(q,abq)_\infty}\left(1+\sum_{n\geq 1}\frac{q^{kn^2+(k-i+1)n+n(n+1)/2}(-ab)^n(-1/a,-1/b)_n}{(-aq,-bq)_n}\right.$$

$$\left.+\sum_{n\geq 1}\frac{q^{kn^2-(k-i)n+n(n+1)/2}(-ab)^n(-1/a,-1/b)_n}{(-aq,-bq)_n}\right).$$

Replacing n by $-n$ in the second sum, simplifying, and then replacing k by $k-1$ and i by $i-1$ gives (4.1). □

Now we turn to the function $\tilde{D}_{k,i}(s,t,n)$. We define an operation on overpartition pairs, called k-conjugation, again using the Frobenius representation. Recall the decomposition of such a representation described after Definition 7.1. Let λ_1' (resp. λ_2') be the conjugate of λ_1 (resp. of λ_2). Thus, λ_1' and λ_2' are partitions into parts less than or equal to n_1, where n_1 is the number of columns in the Frobenius representation. We consider two regions. As above, we say that the 0th Durfee square of a partition has size n_1. The first region G_2 we consider is the portion of λ_2' below its $(k-2)$-th Durfee square. The second region G_1 consists of the parts of λ_1' which are less than or equal to the size of the $(k-2)$-th Durfee square of λ_2'.

Definition 7.5. *For the* k-conjugation *of an overpartition pair, we first interchange these two regions G_1 and G_2 of λ_1' and λ_2' to get two new partitions λ_1'' and λ_2''. Next, we conjugate these to get λ_1''' and λ_2'''. Finally, we use the Joichi-Stanton algorithm to assemble λ_1''' and μ_1 into the top row and λ_2''' and μ_2 into the bottom row.*

We remark that if λ_2' has less than $k-2$ Durfee squares, the k-conjugation is the identity. Note that this k-conjugation is a generalization of the k-conjugation for overpartitions defined by Corteel and the present authors in [25] (which in turn was a generalization of Garvan's k-conjugation for partitions [28]).

Continuing with the example from after Definition 7.1, it is easy to see that we have $\lambda_1' = (8,6,5,5,4,3,2,2,2,2)$ and $\lambda_2' = (8,8,7,6,5,4,4,3,2,1,1)$. For $k=4$, if we interchange the two regions defined above, we get $\lambda_1'' = (8,6,5,5,4,2,1,1)$ and $\lambda_2'' = (8,8,7,6,5,4,4,3,3,2,2,2)$ (see Figure 5). Conjugating, we get $\lambda_1''' = (8,6,5,5,4,2,1,1)$ and $\lambda_2''' = (12,12,9,7,5,4,3,2)$. By applying the Joichi-Stanton algorithm in reverse (remember that $\mu_1 = (7,5,2)$ and $\mu_2 = (6,3,2)$), we see that the 4-conjugate of π is

$$\pi^{(4)} = \begin{pmatrix} 11 & 9 & \overline{7} & 7 & 6 & \overline{3} & 2 & \overline{1} \\ 15 & 15 & \overline{11} & \overline{8} & 6 & 5 & \overline{3} & 2 \end{pmatrix}.$$

Definition 7.6. *We say that an overpartition pair is* self-k-conjugate *if it is fixed by k-conjugation.*

Proposition 7.7. *The generating function for self-k-conjugate overpartition pairs is*

$$\sum_{n_1\geq n_2\geq\cdots\geq n_{k-1}\geq 0}\frac{q^{n_1+n_2^2+\cdots+n_{k-1}^2}(-1/a)_{n_1}a^{n_1}(-1/b)_{n_1}b^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q^2;q^2)_{n_{k-1}}}, \tag{7.3}$$

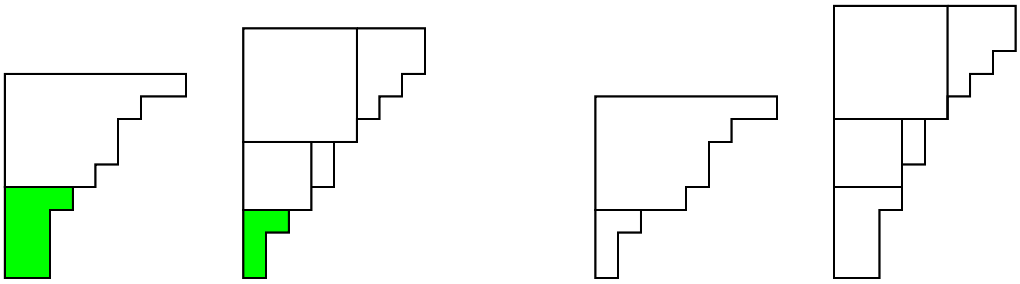

FIGURE 5. Illustration of the 4-conjugation. For the initial overpartition π, we have $\lambda_1' = (8,6,5,5,4,3,2,2,2)$ and $\lambda_2' = (8,8,7,6,5,4,4,3,2,1,1)$. The regions highlighted are interchanged by 4-conjugation, which gives $\lambda_1' = (8,6,5,5,4,2,1,1)$ and $\lambda_2' = (8,8,7,6,5,4,4,3,3,2,2,2)$ for $\pi^{(4)}$, the 4-conjugate of π.

where n_1 is the number of columns of the Frobenius symbol and $n_2,\dots,n_{k-1}$ are the sizes of the $k-2$ first successive Durfee squares of λ_2'.

Proof. The decomposition of a self-k-conjugate overpartition pair is similar to the decomposition of a (k,k)-admissible overpartition pair. We have the following pieces:

- μ_1, which is counted by $(-1/b)_{n_1} b^{n_1}$,
- μ_2, which is counted by $(-1/a)_{n_1} a^{n_1}$,
- The n_1 columns, which are counted by q^{n_1},
- the $k-2$ Durfee squares of λ_2', which are counted by $q^{n_2^2+\cdots+n_{k-1}^2}$,
- the regions between the Durfee squares of λ_2', which are counted by
$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q,$$
- the parts in λ_1' which are $> n_{k-1}$ and of course $\le n_1$: they are counted by
$$\frac{1}{(1-q^{n_{k-1}+1})\cdots(1-q^{n_1})} = \frac{(q)_{n_{k-1}}}{(q)_{n_1}},$$
- the two identical regions G_1 and G_2, which are counted by
$$\frac{1}{(q^2;q^2)_{n_{k-1}}}.$$

For example, in Figure 5 we do not have a self-4-conjugate overpartition pair because the shaded regions are not identical.

Summing on $n_1, n_2, \dots, n_{k-1}$, we get the generating function:
$$\sum_{n_1 \ge n_2 \ge \cdots \ge n_{k-1} \ge 0} (-1/b)_{n_1} b^{n_1} (-1/a)_{n_1} a^{n_1} q^{n_1} q^{n_2^2+\cdots+n_{k-1}^2} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \frac{(q)_{n_{k-1}}}{(q)_{n_1}} \frac{1}{(q^2;q^2)_{n_{k-1}}}$$
$$= \sum_{n_1 \ge n_2 \ge \cdots \ge n_{k-1} \ge 0} \frac{q^{n_1+n_2^2+\cdots+n_{k-1}^2} (-1/a)_{n_1} a^{n_1} (-1/b)_{n_1} b^{n_1}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-2}-n_{k-1}} (q^2;q^2)_{n_{k-1}}}.$$

□

Definition 7.8. *We say that an overpartition pair is* self-(k,i)-conjugate *if it is obtained by taking a self-k-conjugate overpartition pair and adding a part n_j (n_j is the size of the $(j-1)$-th successive Durfee square of λ_2') to λ_2' for $i \leq j \leq k-1$.*

Remember that we denote by $\tilde{D}_{k,i}(s,t,n)$ the number of self-(k,i)-conjugate overpartition pairs of n whose Frobenius representations have s non-overlined parts in their bottom rows and t non-overlined parts in their top rows. We may now complete the proof of Theorem 1.4.

Proof of the case $\tilde{D}_{k,i}(s,t,n) = \tilde{B}_{k,i}(s,t,n)$ of Theorem 1.4. It is obvious from Proposition 7.7 and Definition 7.8 that

$$\sum_{s,t,n\geq 0} \tilde{D}_{k,i}(s,t,n)a^s b^t q^n = \sum_{n_1\geq n_2\geq\cdots\geq n_{k-1}\geq 0} \frac{q^{n_1+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}(-1/a,-1/b)_{n_1}a^{n_1}b^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q^2;q^2)_{n_{k-1}}}. \tag{7.4}$$

Consider the Bailey pair with respect to q [37, p.468, (E3)],

$$\beta_n = \frac{1}{(q^2;q^2)_n} \qquad \text{and} \qquad \alpha_n = \frac{(-1)^n q^{n^2}(1-q^{2n+1})}{(1-q)}.$$

Substituting into Lemma 7.4 and arguing in the case of $D_{k,i}(s,t,n)$ above shows that (7.4) is equal to (4.2).

□

8. Concluding Remarks

We wish to close with a look at some possible future research topics. First, several authors [3, 16, 34] have derived combinatorial identities from Andrews' $J_{1,k,i}(a;x;q)$ when k and/or i are half-integers. Can this idea be applied to the $R_{k,i}(a,b;x;q)$ or $\tilde{R}_{k,i}(a,b;x;q)$? For example, we might mention that the $R_{k,3/2}(1/q,1/q^2;1;q^2)$ are expressible as infinite products.

Second, it would be worthwhile to develop the recurrences for a "tilde version" of Andrews' $J_{\lambda,k,i}$ for all λ and see if there is perhaps an analogue of Andrews' general Rogers-Ramanujan theorem [9]. Such a theorem was in fact predicted by Bressoud [17, p.19]. Moreover, there are other nice applications of Andrews' functions besides proving combinatorial theorems. For instance, they have been used to prove q-series identities of the Rogers-Ramanujan type [6, 10] and in the study of q-continued fractions [6, 13]. The tilde analogues would, no doubt, be equally fruitful.

Finally, we now know that Andrews' functions $J_{0,k,i}(-;x;q)$ are generating functions for certain partitions, the $J_{1,k,i}(-1/a;x;q)$ are generating functions for certain overpartitions, and the $J_{2,k,i}(-1/a,-1/b;x;q)/(abxq)_\infty$ are generating functions for certain overpartition pairs. The natural question, of course, is what happens to all of the combinatorial objects considered in this paper when we pass to $\lambda = 3$? There are a number of barriers that make it unclear how to go beyond overpartition pairs. First, from the perspective of generating functions, passing from partitions to overpartitions to overpartition pairs involves passing from q^{n^2} to $(-1/a)_n a^n q^{n(n+1)/2}$ to $(-1/a,-1/b)_n(abq)^n$. What would be next? Second, in terms of Frobenius symbols, we pass from symbols with partitions into distinct parts in both rows to symbols with an overpartition in one row and a partition into distinct parts in the other to symbols with overpartitions in both rows. Again, what would be next? Third, in terms of the lattice paths, a peak can be open when dealing with partitions, half-open when we allow overpartitions, or closed when we pass to overpartition pairs. What would happen to these peaks and paths in the next case?

In terms of q-series identities, we know that the correspondence between an overpartition pair and its Frobenius symbol is the essence of two famous identities, the q-Gauss summation and the ${}_1\psi_1$ summation [22, 23, 38]. Perhaps a clue to going beyond overpartition pairs lies in finding a natural bijective proof of some generalization of the q-Gauss summation. The ${}_6\phi_5$ summation, for example, would be a good candidate. Some work toward a bijective proof of this identity is presented in [22].

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