GORDON'S THEOREM FOR OVERPARTITIONS

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Abstract. We give overpartition-theoretic analogues of certain combinatorial generalizations of the Rogers-Ramanujan identities.

1. Introduction

In 1894 Rogers [7] established what would become known as the Rogers-Ramanujan identities,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}. \tag{1.2}$$

MacMahon [6, p.33-35] later observed that these identities have combinatorial interpretations in terms of partitions. Namely, for $i = 0$ or 1, the number of partitions of n whose parts differ by at least 2 and that have at most $1-i$ ones is equal to the number of partitions of n into parts congruent to $\pm(1+i)$ modulo 5. In 1961 Gordon [5] proved a two-parameter family of partition theorems which includes (1.1) and (1.2):

Let $B_{k,i}(n)$ denote the number of partitions of n of the form $y_1+y_2+\cdots+y_s$, where $y_j - y_{j+k-1} \geq 2$ and at most $i-1$ of the parts are equal to 1. Let $A_{k,i}(n)$ denote the number of partitions of n into parts not congruent to $0, \pm i$ modulo $2k+1$. Then $A_{k,i}(n) = B_{k,i}(n)$.

Our object is to investigate analogues of Gordon's theorem in the theory of overpartitions. An overpartition of n is a partition of n in which the final occurrence of a part may be overlined [3]. It turns out that the identities that play the roles of (1.1) and (1.2) in the theory of overpartitions are the cases $a = 1$ and $a = q$ of Lebesgue's identity [2, Cor. 2.7],

$$\sum_{n=0}^{\infty} \frac{(1+a)(1+aq)\cdots(1+aq^{n-1})q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{(1+aq^{2n-1})}{(1-q^{2n-1})}. \tag{1.3}$$

The relevant combinatorial interpretations of these two identities have been given in [1, 8]. The case $a = 1$ says that the number of overpartitions of n into distinct parts of the form $y_1 + y_2 + \cdots + y_s$, where $y_j - y_{j+1} \geq 2$ when y_{j+1} is not overlined, is equal to the number of overpartitions of n into odd parts. The case $a = q$ says that the number of overpartitions of n into distinct parts of the form $z_1 + z_2 + \cdots + z_s$, such that 1 can only occur as an overlined part, and where $z_j - z_{j+1} \geq 2$ if z_j is not overlined, is equal to the number of overpartitions of n where all non-overlined parts are congruent to 2 modulo 4.

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We shall prove analogues of the $i = k$ and $i = 1$ cases of Gordon's theorem, which embed the above results in families of overpartition theorems.

Theorem 1.1. *Let $\overline{B}_k(n)$ denote the number of overpartitions of n of the form $y_1 + y_2 + \cdots + y_s$, where $y_j - y_{j+k-1} \geq 1$ if y_{j+k-1} is overlined and $y_j - y_{j+k-1} \geq 2$ otherwise. Let $\overline{A}_k(n)$ denote the number of overpartitions of n into parts not divisible by k. Then $\overline{A}_k(n) = \overline{B}_k(n)$.*

Theorem 1.2. *Regard an overpartition as a partition in which the first occurrence of an integer may be overlined. Then let $\overline{D}_k(n)$ denote the number of overpartitions of n of the form $z_1 + z_2 + \cdots + z_s$, such that 1 cannot occur as a non-overlined part, and where $z_j - z_{j+k-1} \geq 1$ if z_j is overlined and $z_j - z_{j+k-1} \geq 2$ otherwise. Let $\overline{C}_k(n)$ denote the number of overpartitions of n whose non-overlined parts are not congruent to $0, \pm 1$ modulo $2k$. Then $\overline{C}_k(n) = \overline{D}_k(n)$.*

Gessel and Stanton [4], and later Andrews [1], proved an identity which is closely related to the $a = 1$ case of (1.3),

$$1 + \sum_{n=1}^{\infty} \frac{(1+q)(1+q^2)\cdots(1+q^{n-1})q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{(1+q^{16n-6})(1+q^{16n-10})(1-q^{16n})}{(1-q^{2n-1})(1-q^{4n})}. \quad (1.4)$$

As pointed out by Andrews [1], this implies an overpartition theorem for $\overline{B}_2(n)/2$. Although this is not the case for general $\overline{B}_k/2$, there are similar accidents which occur when $k = 3$ and 5. Notice that counting half of the overpartitions enumerated by $\overline{B}_k(n)$ corresponds to disallowing the overlining of the final occurrence of the largest part.

Theorem 1.3. *Let $\overline{\mathfrak{B}}_k(n)$ denote the number of overpartitions counted by $\overline{B}_k(n)$ where the largest part cannot have its final occurrence overlined. Then*

(i) *$\overline{\mathfrak{B}}_3(n)$ equals the number of overpartitions of n where the overlined parts are equivalent to 5 or 7 modulo 12 and the non-overlined parts are not divisible by 12.*

(ii) *$\overline{\mathfrak{B}}_5(n)$ equals the number of overpartitions of n where the overlined parts are not equivalent to ± 1 or ± 2 modulo 10 and the non-overlined parts are not divisible by 10.*

The proofs of the theorems depend on foundations laid by Andrews [2, Ch. 7] for the study of q-difference equations involving certain well-poised basic hypergeometric series. These are reviewed in the following section, and the proofs are contained in §3.

2. q-Difference equations

We shall employ the standard q-series notation

$$(a_1, ..., a_r)_n = (a_1; q)_n \cdots (a_r; q)_n = \prod_{k=0}^{n-1}(1 - a_1 q^k) \cdots \prod_{k=0}^{n-1}(1 - a_r q^k).$$

Following Andrews [2], we define

$$H_{k,i}(a; x; q) = \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2+n-in} a^n (1 - x^i q^{2ni})(axq^{n+1})_\infty (1/a)_n}{(q)_n (xq^n)_\infty} \quad (2.5)$$

and

$$J_{k,i}(a; x; q) = H_{k,i}(a; xq; q) - axqH_{k,i-1}(a; xq; q). \quad (2.6)$$

A few lemmas concerning the functions $J_{k,i}(a;x;q)$ are required. The first lemma records the necessary q-difference equations from [2], while the second two indicate that special cases can be expressed as infinite products.

Lemma 2.4 ([2], p.106-107)**.**

$$J_{k,i}(a;x;q) - J_{k,i-1}(a;x;q) = (xq)^{i-1}J_{k,k-i+1}(a;xq;q) - a(xq)^{i-1}J_{k,k-i+2}(a;xq;q), \quad (2.7)$$

$$J_{k,1}(a;x;q) = J_{k,k}(a;xq;q). \quad (2.8)$$

Lemma 2.5.

$$J_{k,k}(-1;1;q) = \frac{(-q)_\infty (q^k;q^k)_\infty}{(q)_\infty(-q^k;q^k)_\infty}.$$

Proof.

$$\begin{aligned} J_{k,k}(-1;1;q) &= H_{k,k}(-1;q;q) + qH_{k,k-1}(-1;q;q) \\ &= \frac{(-1)_\infty}{(q)_\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^n q^{kn^2+n}(1-q^{k(2n+1)})}{(1+q^n)(1+q^{n+1})} + q\sum_{n=0}^{\infty}\frac{(-1)^n q^{kn^2+2n}(1-q^{(k-1)(2n+1)})}{(1+q^n)(1+q^{n+1})}\right) \\ &= \frac{(-1)_\infty}{(q)_\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^n q^{kn^2+n}(1+q^{n+1})}{(1+q^n)(1+q^{n+1})} - \sum_{n=0}^{\infty}\frac{(-1)^n q^{k(n+1)^2}(1+q^n)}{(1+q^n)(1+q^{n+1})}\right) \\ &= \frac{(-1)_\infty}{(q)_\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^n q^{kn^2+n}}{1+q^n} + \sum_{n=1}^{\infty}\frac{(-1)^n q^{kn^2}}{1+q^n}\right) \\ &= \frac{(-q)_\infty}{(q)_\infty}\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n q^{kn^2}(1+q^n)}{1+q^n}\right) \\ &= \frac{(-q)_\infty (q^k;q^k)_\infty}{(q)_\infty(-q^k;q^k)_\infty}, \end{aligned}$$

by Jacobi's triple product identity [2, Thm. 2.8],

$$\sum_{n\in\mathbb{Z}} x^n q^{n^2} = (-xq, -q/x, q^2; q^2)_\infty. \quad (2.9)$$

□

Lemma 2.6.

$$J_{k,1}(-1/q;1;q) = \frac{(q, q^{2k-1}, q^{2k}; q^{2k})_\infty(-q)_\infty}{(q;q)_\infty}.$$

Proof.

$$\begin{aligned}
J_{k,1}(-1/q;1;q) &= H_{k,1}(-1/q;q;q) \\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{kn^2+(k-1)n}(1-q^{2n+1}) \\
&= \frac{(-q)_\infty}{(q)_\infty}\sum_{n\in\mathbb{Z}}(-1)^n q^{kn^2+(k-1)n} \\
&= \frac{(q,q^{2k-1},q^{2k};q^{2k})_\infty(-q)_\infty}{(q;q)_\infty},
\end{aligned}$$

by (2.9).

Unfortunately the functions $J_{k,i}(-1;1;q)$ and $J_{k,i}(-1/q;1;q)$, for $i \neq k$ and $i \neq 1$ respectively, do not appear to be expressible as single infinite products.

3. Proofs of the Main Theorems

Proof of Theorem 1.1. From now on, let $1 \leq i \leq k$. Define

$$J_{k,i}(-1;x;q) = \sum_{m,n\geq 0} b_{k,i}(m,n)x^m q^n.$$

For $i \geq 2$, the recurrence (2.7) implies that

$$b_{k,i}(m,n) - b_{k,i-1}(m,n) = b_{k,k-i+1}(m-i+1,n-m) + b_{k,k-i+2}(m-i+1,n-m), \tag{3.10}$$

while equation (2.8) implies that

$$b_{k,1}(m,n) = b_{k,k}(m,n-m). \tag{3.11}$$

We also have

$$J_{k,i}(-1;0;q) = J_{k,i}(-1;x;0) = 1,$$

so that

$$b_{k,i}(m,n) = \begin{cases} 1, & (m,n)=(0,0), \\ 0, & m \leq 0 \text{ or } n \leq 0, \text{ but } (m,n) \neq (0,0). \end{cases} \tag{3.12}$$

The conditions (3.10), (3.11), and (3.12) uniquely determine the sequence $b_{k,i}(m,n)$.

Let $\overline{b}_{k,i}(m,n)$ denote the number of overpartitions $y_1 + y_2 + \cdots + y_m$ of n having at most $i-1$ ones (overlined or not), and satisfying $y_j - y_{j+k-1} \geq 2$ when y_{j+k-1} is not overlined and $y_j - y_{j+k-1} \geq 1$ otherwise. We shall determine that the $\overline{b}_{k,i}(m,n)$ satisfy the same conditions

(3.10), (3.11), and (3.12). In that case we'll have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{B}_k(n)q^n &= \sum_{m,n\geq 0}\overline{b}_{k,k}(m,n)q^n \\ &= \sum_{m,n\geq 0}b_{k,k}(m,n)q^n \\ &= J_{k,k}(-1;1;q) \\ &= \sum_{n=0}^{\infty}\overline{A}_k(n)q^n,\end{aligned}$$

by Lemma 2.5.

So, consider the overpartitions enumerated by $\overline{b}_{k,i}(m,n)-\overline{b}_{k,i-1}(m,n)$, for $i\geq 2$. These are overpartitions counted by $\overline{b}_{k,i}(m,n)$ where there are exactly $i-1$ ones. They may be broken up into two classes: those whose final occurrence of 1 is not overlined and those whose final occurrence of 1 is overlined. In each case, we transform the overpartitions by deleting the $i-1$ ones and removing one from each remaining part. The resulting overpartitions of $n-m$ have $m-i+1$ parts. In the first case, the total number of appearances of 1 and 2 could not have exceeded $k-1$ due to the difference condition. Hence the new overpartition has at most $k-i$ ones. In the second case, the total number of appearances of 1 and 2 could have been as big as k. Hence the new overpartition may have as many as $k-i+1$ ones. Since these transformations preserve differences between successive parts and are easily inverted, we have

$$\overline{b}_{k,i}(m,n)-\overline{b}_{k,i-1}(m,n)=\overline{b}_{k,k-i+1}(m-i+1,n-m)+\overline{b}_{k,k-i+2}(m-i+1,n-m),$$

which is condition (3.10). When $i=1$ we may remove one from each part of an overpartition counted by $\overline{b}_{k,1}(m,n)$ to get an overpartition of $n-m$ into m parts, at most $k-1$ of which are equal to 1. This implies that

$$\overline{b}_{k,1}(m,n)=\overline{b}_{k,k}(m,n-m),$$

which is (3.11). The conditions in (3.12) are standard boundary conditions for partitions. □

Proof of Theorem 1.2. The proof here is analogous to the one given above. In this case, let

$$J_{k,i}(-1/q;x;q)=\sum_{m,n\geq 0}d_{k,i}(m,n)x^mq^n.$$

Lemma 2.4 implies that

$$d_{k,i}(m,n)-d_{k,i-1}(m,n)=d_{k,k-i+1}(m-i+1,n-m)+d_{k,k-i+2}(m-i+1,n-m+1) \quad (3.13)$$

for $i\geq 2$ and

$$d_{k,1}(m,n)=d_{k,k}(m,n-m). \quad (3.14)$$

Since

$$J_{k,i}(-1/q;0;q)=1$$

and

$$J_{k,i}(-1/q;x;0)=\begin{cases}1+x, & i\geq 2\\ 1, & i=1,\end{cases}$$

we have

$$d_{k,i}(m,n) = \begin{cases} 1, & (m,n) = (0,0), \\ 1, & (m,n) = (1,0), 2 \leq i \leq k, \\ 0, & (m,n) = (1,0), i = 1, \\ 0, & m \leq 0, (m,n) \neq (0,0), \\ 0, & n \leq 0, (m,n) \neq (1,0), (0,0). \end{cases} \tag{3.15}$$

We again make the observation that (3.13), (3.14), and (3.15) uniquely determine the sequence $d_{k,i}(m,n)$ for $1 \leq i \leq k$.

Let $\overline{d}(m,n)$ denote the number of overpartitions $z_1 + z_2 + \cdots + z_m$ of n where $\overline{0}$ may appear as a part. This time we will allow the overlining of the first occurrence of a part instead of the last. Let $\overline{d}_{k,i}(m,n)$ be the number of those overpartitions counted by $\overline{d}(m,n)$ which satisfy $z_j - z_{j+k-1} \geq 2$ when z_j is not overlined, $z_j - z_{j+k-1} \geq 1$ when z_j is overlined, and which have at most $i-1$ occurrences of 1 and $\overline{0}$. Here we distinguish between 1 and $\overline{1}$ when counting ones. We shall determine that the $\overline{d}_{k,i}(m,n)$ satisfy the same conditions (3.13), (3.14), and (3.15) as the $d_{k,i}(m,n)$, so that we'll have

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{D}_k(n) q^n &= \sum_{m,n \geq 0} \overline{d}_{k,1}(m,n) q^n \\ &= \sum_{m,n \geq 0} d_{k,1}(m,n) q^n \\ &= J_{k,1}(-1/q; 1; q) \\ &= \sum_{n=0}^{\infty} \overline{C}_k(n) q^n, \end{aligned}$$

by Lemma 2.6.

So, consider the overpartitions generated by $\overline{d}_{k,i}(m,n) - \overline{d}_{k,i-1}(m,n)$. for $i \geq 2$. These are overpartitions counted by $\overline{d}_{k,i}(m,n)$ where there are exactly $i-1$ occurrences of 1 and $\overline{0}$. They may be broken up into two classes: those with a $\overline{0}$ and those without. In the first case, we transform the overpartitions by removing the $i-2$ occurrences of 1, removing the $\overline{0}$, and subtracting one from each of the remaining parts. The resulting overpartitions of $n-m+1$ have $m-i+1$ parts, and the total number of appearances of 1 and $\overline{0}$ is at most $k-i+1$. In the second case, we remove the $i-1$ ones and subtract one from each part to get overpartitions of $n-m$ with $m-i+1$ parts. These have at most $k-i$ occurrences of 1 and $\overline{0}$. Since the differences between parts is unaffected by such a transformation, the resulting partitions are counted by $\overline{d}_{k,k-i+2}(m-i+1, n-m+1)$ in the first case and by $\overline{d}_{k,k-i+1}(m-i+1, n-m)$ in the second case. These transformations are invertible, so the recurrence (3.13) is proven for the $\overline{d}_{k,i}(m,n)$. When $i = 1$, there are no occurrences of 1 or $\overline{0}$, so by removing one from each part we see that $\overline{d}_{k,1}(m,n) = \overline{d}_{k,k}(m, n-m)$, which is (3.14). The boundary conditions (3.15) are again the standard ones, only here we allow for the overpartition of 0 consisting of one part, $\overline{0}$. ∎

Proof of Theorem 1.3. From the case $k = 3$ of Theorem 1.1 we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{\mathfrak{B}}_3(n)q^n &= \frac{1}{2} + \frac{1}{2}\frac{(-q,-q^2;q^3)_\infty}{(q,q^2;q^3)_\infty} \\
&= \frac{(q,q^2,q^3;q^3)_\infty + (-q,-q^2,q^3;q^3)_\infty}{2(q)_\infty} \\
&= \frac{1}{2(q)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2} + \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2}\right) \\
&= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty} q^{n(6n+1)} \\
&= \frac{(-q^5,-q^7,q^{12};q^{12})_\infty}{(q)_\infty}.
\end{aligned}$$

Similarly, the case $k = 5$ of Theorem 1.1 says that

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{\mathfrak{B}}_5(n)q^n &= \frac{1}{2} + \frac{1}{2}\frac{(-q,-q^2,-q^3,-q^4;q^5)_\infty}{(q,q^2,q^3,q^4;q^5)_\infty} \\
&= \frac{(q,q^2,q^3,q^4,q^5,q^5;q^5)_\infty + (-q,-q^2,-q^3,-q^4,q^5,q^5;q^5)_\infty}{2(q)_\infty(q^5;q^5)_\infty} \\
&= \frac{1}{2(q)_\infty(q^5;q^5)_\infty}\left(\sum_{n,m\in\mathbb{Z}}(-1)^{n+m}q^{n(5n+1)/2+m(5m+3)/2} + q^{n(5n+1)/2+m(5m+3)/2}\right) \\
&= \frac{1}{(q)_\infty(q^5;q^5)_\infty}\left(\sum_{n+m\in 2\mathbb{Z}} q^{n(5n+1)/2+m(5m+3)/2}\right) \\
&= \frac{1}{(q)_\infty(q^5;q^5)_\infty}\left(\sum_{n\in\mathbb{Z}} q^{n(5n+1)/2}\sum_{k\in\mathbb{Z}} q^{(n+2k)(5n+10k+3)/2}\right) \\
&= \frac{1}{(q)_\infty(q^5;q^5)_\infty}\left(\sum_{k\in\mathbb{Z}} q^{10k^2+3k}\sum_{n\in\mathbb{Z}} q^{5n^2+2n+10nk}\right) \\
&= \frac{1}{(q)_\infty(q^5;q^5)_\infty}\left(\sum_{k\in\mathbb{Z}} q^{5k^2+k}\sum_{n\in\mathbb{Z}} q^{5n^2+2n}\right) \\
&= \frac{(-q^3,-q^4,-q^5,-q^6,-q^7,-q^{10},q^{10};q^{10})_\infty}{(q)_\infty}.
\end{aligned}$$

□

References

[1] G.E. Andrews, Rogers-Ramanujan identities for two-color partitions, *Indian J. Math.* **29** (1987), 117-125.
[2] G.E. Andrews, The Theory of Partitions, Cambridge Univ. Press, Cambridge, 1998.
[3] S. Corteel and J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.*, to appear.

[4] I. Gessel and D. Stanton, Applications of q-Lagrange inversion to basic hypergeometric series, *Trans. Amer. Math. Soc.* **277** (1983), 173-201.
[5] B. Gordon, A combinatorial generalization of the Rogers-Ramanujan identities, *Amer. J. Math.* **83** (1961), 393-399.
[6] P.A. MacMahon, Combinatory Analysis, vol. 2, Cambridge Univ. Press, Cambridge, 1916.
[7] L.J. Rogers, Second memoir on the expansion of certain infinite products, *Proc. London Math. Soc.* **25** (1894), 318-343.
[8] J.P.O. Santos and D.V. Sills, q-Pell sequences and two identities of V.A. Lebesgue, *Disc. Math.* **257** (2002), 125 - 143.

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