BAILEY PAIRS AND INDEFINITE QUADRATIC FORMS, II. FALSE INDEFINITE THETA FUNCTIONS

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Abstract. We construct families of Bailey pairs (α_n, β_n) where the exponent of q in α_n is an indefinite quadratic form, but where the usual $(-1)^j$ is replaced by a sign function. This leads to identities involving "false" indefinite binary theta series. These closely resemble q-identities for mock theta functions or Maass waveforms, but the sign function prevents them from having the usual modular properties.

1. Introduction

Recall the standard q-series notation,

$$(a)_n := (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}), \tag{1.1}$$

valid for non-negative integers n as well as in the limit as $n \to \infty$. A pair of sequences (α_n, β_n) is called a Bailey pair relative to a if

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}}, \tag{1.2}$$

or, equivalently,

$$\alpha_n = \frac{1-aq^{2n}}{1-a} \sum_{j=0}^{n} \frac{(a)_{n+j}(-1)^{n-j}q^{\binom{n-j}{2}}}{(q)_{n-j}} \beta_j. \tag{1.3}$$

Bailey pairs are fundamental objects in the theory of basic hypergeometric series, and many of the most important results in the subject follow from their properties. For more background and further details on Bailey pairs, see [1, 2, 4, 5, 8, 16, 20].

This paper is a sequel to [17]. In that paper, we constructed families of Bailey pairs containing indefinite quadratic forms, motivated by the importance of such pairs in proving famous identities like

$$\psi_1(q) = \sum_{n\geq 0}(-q)_n q^{\binom{n+1}{2}} = \frac{(-q)_\infty}{(q)_\infty} \sum_{\substack{n\geq 0\\ |j|\leq n}} (-1)^j q^{n(5n+3)/2-j(3j+1)/2}(1-q^{2n+1}) \tag{1.4}$$

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or

$$\sigma(q) = \sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}}{(-q)_n} = \sum_{\substack{n\geq 0 \\ |j|\leq n}} (-1)^{n+j} q^{n(3n+1)/2-j^2}(1-q^{2n+1}). \tag{1.5}$$

The first of these identities, due to Andrews [3, Eq.(6.7)], relates Ramanujan's fifth order mock theta function $\psi_1(q)$ to indefinite binary theta functions, providing "the necessary first step in finding the transformation theory" of the fifth order mock theta functions [3, p. 114]. This transformation theory was later developed by Zwegers [21, 22]. The second identity, due to Andrews, Dyson, and Hickerson [7], relates Ramanujan's σ-function to the ring of integers of the real quadratic field $\mathbb{Q}(\sqrt{6})$ and the Fourier coefficients of a Maass waveform [7, 12].

The following theorem is just one example of the results in [17].

Theorem 1.1. [17, Theorem 1.1] *Suppose that $k, K \geq 1$, $0 \leq \ell < k$ and $0 \leq m < K$.*

(i) *The sequences $(\alpha_n^{(k,K,\ell)}, \beta_n^{(k,K,\ell)})$ form a Bailey pair relative to q, where*

$$\alpha_n^{(k,K,\ell)} = \frac{q^{(K+1)n^2+Kn}(1-q^{2n+1})}{(1-q)} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \tag{1.6}$$

and

$$\beta_n^{(k,K,\ell)} = \sum_{n\geq n_{k+K-1}\geq\cdots\geq n_1\geq 0} q^{\sum_{i=1}^{K-1} n_{k+i}(n_{k+i}+1)+\binom{n_k+1}{2}-\sum_{i=1}^{k-1} n_i n_{i+1}-\sum_{i=1}^{\ell} n_i} \times \frac{(-1)^{n_k}}{(q)_{n-n_{k+K-1}}(q)_{n_{k+K-1}-n_{k+K-2}}\cdots(q)_{n_2-n_1}(q)_{n_1}}. \tag{1.7}$$

(ii) *The sequences $(\alpha_n^{(k,K,\ell,m)}, \beta_n^{(k,K,\ell,m)})$ form a Bailey pair relative to 1, where*

$$\begin{aligned}\alpha_n^{(k,K,\ell,m)} &= q^{(K+1)n^2+(m+1)n} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \\ &\quad - \chi(n\neq 0) q^{(K+1)n^2-(m+1)n} \sum_{j=-n+1}^{n-1} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}\end{aligned} \tag{1.8}$$

and

$$\beta_n^{(k,K,\ell,m)} = \sum_{n\geq n_{k+K-1}\geq\cdots\geq n_1\geq 0} q^{\sum_{i=1}^{K-1} n_{k+i}^2+\sum_{i=1}^{m} n_{k+i}+\binom{n_k+1}{2}-\sum_{i=1}^{k-1} n_i n_{i+1}-\sum_{i=1}^{\ell} n_i} \times \frac{(-1)^{n_k}}{(q)_{n-n_{k+K-1}}(q)_{n_{k+K-1}-n_{k+K-2}}\cdots(q)_{n_2-n_1}(q)_{n_1}}. \tag{1.9}$$

These Bailey pairs turned out to have a number of interesting applications. Along the lines of (1.4) and (1.5), they were used to find families of q-hypergeometric mock theta functions [17] and to construct families of q-series whose coefficients are the Fourier coefficients of Maass waveforms [10]. Perhaps more surprisingly, they were key to finding expressions for the colored Jones polynomials of torus knots and the WRT invariants of special 3-manifolds [14, 15].

The proofs in [17] used a certain iterative process starting from two "seed" Bailey pairs. Given the applicability of the resulting pairs, we revisit this iterative process starting from two different seeds. Our choices are particularly motivated by the appearance of sums like

$$\sum_{r=-N}^{N-1} \operatorname{sgn}(r) q^{-kr^2-(k-1)r} \tag{1.10}$$

in expressions for colored Jones polynomials of torus links (see for example [13, Eq. (2.4), $q = 1/q$]) and Hecke-type identities for unimodal sequence generating functions [11, Lemma 4.17, $m = -m$]. Here $\operatorname{sgn}(j)$ is the modified sign function defined by

$$\operatorname{sgn}(j) = \begin{cases} 1, & \text{if } j \geq 0, \\ -1, & \text{if } j < 0. \end{cases} \tag{1.11}$$

Our first two main results are as follows. Note that the presence of the sign function $\operatorname{sgn}(j)$ in place of $(-1)^j$ is the only real difference between the α_n in (1.12) and (1.14) and those in (1.6) and (1.8). A similar comparison may be made between the α_n in (1.16) and (1.18) and those in [17, Theorem 1.2].

Theorem 1.2. *Let $1 \leq \ell < k$ and $0 \leq m \leq K$.*

(i) The sequences $(\alpha_n^{(k,K,\ell)}, \beta_n^{(k,K,\ell)})$ form a Bailey pair relative to q, where

$$\alpha_n^{(k,K,\ell)} = \frac{q^{(K+1)n^2+Kn}(1-q^{2n+1})}{1-q} \sum_{j=-n}^{n} \operatorname{sgn}(j) q^{-((2k-1)j^2+(2\ell-1)j)/2}, \tag{1.12}$$

$$\begin{aligned}\beta_n^{(k,K,\ell)} = 2 \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} & q^{\sum_{i=0}^{K-1}(n_{k+i}^2+n_{k+i}) - \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=2}^{\ell} n_i + \binom{n_1+1}{2}} \\ & \times \frac{(-1)^{n_k+n_1}}{(q)_{n-n_{k+K-1}} \cdots (q)_{n_2-n_1} (q)_{n_1}^2 (1+q^{n_1})}.\end{aligned} \tag{1.13}$$

(ii) The sequences $(\alpha_n^{(k,K,\ell,m)}, \beta_n^{(k,K,\ell,m)})$ form a Bailey pair relative to 1, where

$$\begin{aligned}\alpha_n^{(k,K,\ell,m)} &= q^{(K+1)n^2+mn} \sum_{j=-n}^{n} \operatorname{sgn}(j) q^{-((2k-1)j^2+(2\ell-1)j)/2} \\ &\quad - \chi(n \neq 0) q^{(K+1)n^2-mn} \sum_{j=-n+1}^{n-1} \operatorname{sgn}(j) q^{-((2k-1)j^2+(2\ell-1)j)/2},\end{aligned} \tag{1.14}$$

$$\begin{aligned}\beta_n^{(k,K,\ell,m)} = 2 \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} & q^{\sum_{i=0}^{K-1} n_{k+i}^2 + \sum_{i=0}^{m-1} n_{k+i} - \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=2}^{\ell} n_i + \binom{n_1+1}{2}} \\ & \times \frac{(-1)^{n_k+n_1}}{(q)_{n-n_{k+K-1}} \cdots (q)_{n_2-n_1} (q)_{n_1}^2 (1+q^{n_1})}.\end{aligned} \tag{1.15}$$

Theorem 1.3. *Let $1 \leq \ell < k$ and $0 \leq m \leq K$.*

(i) The sequences $(\alpha_n^{(k,K,\ell)}, \beta_n^{(k,K,\ell)})$ *form a Bailey pair relative to* q*, where*

$$\alpha_n^{(k,K,\ell)} = \frac{q^{(K+1)n^2+Kn}(1-q^{2n+1})}{1-q} \sum_{j=-n}^{n} \operatorname{sgn}(j) q^{-kj^2-\ell j}, \tag{1.16}$$

$$\begin{aligned} \beta_n^{(k,K,\ell)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} & q^{\sum_{i=0}^{K-1}(n_{k+i}^2+n_{k+i}) - \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i + \binom{n_1+1}{2}} \\ & \times \frac{(-1)^{n_k+n_1}}{(q)_{n-n_{k+K-1}} \cdots (q)_{n_2-n_1} (q)_{n_1}^2}. \end{aligned} \tag{1.17}$$

(ii) The sequences $(\alpha_n^{(k,K,\ell,m)}, \beta_n^{(k,K,\ell,m)})$ *form a Bailey pair relative to 1, where*

$$\begin{aligned} \alpha_n^{(k,K,\ell,m)} = {} & q^{(K+1)n^2+mn} \sum_{j=-n}^{n} \operatorname{sgn}(j) q^{-kj^2-\ell j} \\ & - \chi(n \neq 0) q^{(K+1)n^2-mn} \sum_{j=-n+1}^{n-1} \operatorname{sgn}(j) q^{-kj^2-\ell j}, \end{aligned} \tag{1.18}$$

$$\begin{aligned} \beta_n^{(k,K,\ell,m)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} & q^{\sum_{i=0}^{K-1} n_{k+i}^2 + \sum_{i=0}^{m-1} n_{k+i} - \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i + \binom{n_1+1}{2}} \\ & \times \frac{(-1)^{n_k+n_1}}{(q)_{n-n_{k+K-1}} \cdots (q)_{n_2-n_1} (q)_{n_1}^2}. \end{aligned} \tag{1.19}$$

For our third main result, we give variants of the case $m = 1$ of the second parts of Theorems 1.2 and 1.3. This type of variant was useful in applications of Theorem 1.1 [10, 14, 15].

Theorem 1.4. *Let* $1 \leq \ell < k$ *and* $K \geq 0$*.*

(i) The sequences $(\alpha_n^{(k,K,\ell)}, \beta_n^{(k,K,\ell)})$ *form a Bailey pair relative to 1, where*

$$\alpha_n^{(k,K,\ell)} = (1-q^{2n}) q^{(K+1)n^2-n} \sum_{j=-n}^{n-1} \operatorname{sgn}(j) q^{-((2k-1)j^2+(2\ell-1)j)/2}, \tag{1.20}$$

$$\begin{aligned} \beta_n^{(k,K,\ell)} = 2 \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} & q^{\sum_{i=0}^{K-1} n_{k+i}^2 - \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell-1} n_i + \binom{n_1+1}{2} + n_1} \\ & \times \frac{(-1)^{n_k+n_1}(1-q^{n_k-n_\ell})}{(q)_{n-n_{k+K-1}} \cdots (q)_{n_2-n_1} (q)_{n_1}^2 (1+q^{n_1})}. \end{aligned} \tag{1.21}$$

(ii) The sequences $(\alpha_n^{(k,K,\ell)}, \beta_n^{(k,K,\ell)})$ *form a Bailey pair relative to 1, where*

$$\alpha_n^{(k,K,\ell)} = (1-q^{2n}) q^{(K+1)n^2-n} \sum_{j=-n}^{n-1} \operatorname{sgn}(j) q^{-kj^2-\ell j}, \tag{1.22}$$

$$\beta_n^{(k,K,\ell)} = \sum_{n\geq n_{k+K-1}\geq\cdots\geq n_1\geq 0} q^{\sum_{i=0}^{K-1} n_{k+i}^2-\binom{n_k+1}{2}-\sum_{i=1}^{k-1} n_i n_{i+1}-\sum_{i=1}^{\ell-1} n_i+\binom{n_1+1}{2}} \times \frac{(-1)^{n_k+n_1}(1-q^{n_k-n_\ell})}{(q)_{n-n_{k+K-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}^2}. \tag{1.23}$$

As a first application of Theorems 1.2 – 1.4, we present some identities containing "false indefinite binary theta series" multiplied by infinite products. We emphasize that the right-hand sides of these identities are known mock theta functions if we replace $\mathrm{sgn}(j)$ by $(-1)^j$. Specifically, the case $k=2$ and $\ell=1$ of the series on the right-hand side of (1.24) corresponds to (1.4), and in general the right-hand side of (1.24) corresponds to the mock theta functions in [17, Theorem 1.6 (4)]. The series in (1.25) correspond to the mock theta functions in [17, Theorem 1.6 (1)], while the case $(k,\ell)=(t+1,t)$ of the series in (1.26) corresponds to [15, Theorem 1.3].

Corollary 1.5. *The following identities hold.*

(i) *For $1\leq\ell<k$ we have*

$$\begin{aligned}2\sum_{n_{2k-1}\geq\cdots\geq n_1\geq 0} & q^{\binom{n_{2k-1}+1}{2}+\sum_{i=0}^{k-2}(n_{k+i}^2+n_{k+i})-\binom{n_k+1}{2}-\sum_{i=1}^{k-1} n_i n_{i+1}-\sum_{i=2}^{\ell} n_i+\binom{n_1+1}{2}} \\ & \times\frac{(-q)_{n_{2k-1}}(-1)^{n_k+n_1}}{(q)_{n_{2k-1}-n_{2k-2}}\cdots(q)_{n_2-n_1}(q)_{n_1}^2(1+q^{n_1})} \\ = \frac{(-q)_\infty}{(q)_\infty} & \sum_{\substack{n\geq 0\\ |j|\leq n}} \mathrm{sgn}(j) q^{kn^2+(k-1)n+\binom{n+1}{2}-((2k-1)j^2+(2\ell-1)j)/2}(1-q^{2n+1}).\end{aligned} \tag{1.24}$$

(ii) *For $1\leq\ell<k$ and $0\leq m<k$ we have*

$$\begin{aligned}2\sum_{n_{2k-1}\geq\cdots\geq n_1\geq 0} & q^{n_{2k-1}^2+\sum_{i=0}^{k-2} n_{k+i}^2+\sum_{i=0}^{m-1} n_{k+i}-\binom{n_k+1}{2}-\sum_{i=1}^{k-1} n_i n_{i+1}-\sum_{i=2}^{\ell} n_i+\binom{n_1+1}{2}} \\ & \times\frac{(-1)^{n_k+n_1}}{(q)_{n_{2k-1}-n_{2k-2}}\cdots(q)_{n_2-n_1}(q)_{n_1}^2(1+q^{n_1})} \\ = \frac{1}{(q)_\infty} & \Bigg(\sum_{\substack{n\geq 0\\ |j|\leq n}} \mathrm{sgn}(j) q^{(k+1)n^2+mn-((2k-1)j^2+(2\ell-1)j)/2} \\ & - \sum_{\substack{n\geq 1\\ |j|\leq n-1}} \mathrm{sgn}(j) q^{(k+1)n^2-mn-((2k-1)j^2+(2\ell-1)j)/2}\Bigg).\end{aligned} \tag{1.25}$$

(iii) For $1 \leq \ell < k$ we have

$$2 \sum_{n_{2k-1} \geq \cdots \geq n_1 \geq 0} q^{n_{2k-1} + 2\sum_{i=0}^{k-2} n_{k+i}^2 - 2\binom{n_k+1}{2} - 2\sum_{i=1}^{k-1} n_i n_{i+1} - 2\sum_{i=1}^{\ell-1} n_i + 2\binom{n_1+1}{2} + 2n_1} \times \frac{(-q)_{n_{2k-1}-1} (-1)^{n_k+n_1} (1 - q^{2n_k - 2n_\ell})}{(q^2;q^2)_{n_{2k-1}-n_{2k-2}} \cdots (q^2;q^2)_{n_2-n_1} (q^2;q^2)_{n_1}^2 (1+q^{2n_1})} = \frac{(-q)_\infty}{(q)_\infty} \sum_{\substack{n \geq 0 \\ |j| \leq n}} \operatorname{sgn}(j) q^{2kn^2 - n - (2k-1)j^2 - (2\ell-1)j} (1 - q^{2n}). \tag{1.26}$$

For our second application of Theorems 1.2 – 1.4, we give identities for false indefinite binary theta series without infinite products. We note that the coefficients of the first two are the Fourier coefficients of Maass waveforms if we replace $\operatorname{sgn}(j)$ by $(-1)^j$ – see Proposition 3.2 of [10].

Corollary 1.6. *The following identities hold.*

(i) For $1 \leq \ell < k$ we have

$$2 \sum_{n_{2k-1} \geq \cdots \geq n_1 \geq 0} q^{\binom{n_{2k-1}+1}{2} + \sum_{i=0}^{k-2} n_{k+i}^2 - \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell-1} n_i + \binom{n_1+1}{2} + n_1} \times \frac{(q)_{n_{2k-1}-1} (-1)^{n_{2k-1}+n_k+n_1} (1 - q^{n_k - n_\ell})}{(q)_{n_{2k-1}-n_{2k-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}^2 (1+q^{n_1})} = \sum_{n \geq 1} \sum_{j=-n}^{n-1} (-1)^n \operatorname{sgn}(j) q^{kn^2 - n + \binom{n+1}{2} - ((2k-1)j^2 + (2\ell-1)j)/2} (1 + q^n). \tag{1.27}$$

(ii) For $1 \leq \ell < k$ we have

$$2 \sum_{n_{2k-1} \geq \cdots \geq n_1 \geq 0} q^{\binom{n_{2k-1}+1}{2} + \sum_{i=0}^{k-2} (n_{k+i}^2 + n_{k+i}) - \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=2}^{\ell} n_i + \binom{n_1+1}{2}} \times \frac{(q)_{n_{2k-1}} (-1)^{n_{2k-1}+n_k+n_1}}{(q)_{n_{2k-1}-n_{2k-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}^2 (1+q^{n_1})} = \sum_{\substack{n \geq 0 \\ |j| \leq n}} (-1)^n \operatorname{sgn}(j) q^{kn^2 + (k-1)n + \binom{n+1}{2} - ((2k-1)j^2 + (2\ell-1)j)/2} (1 - q^{2n+1}). \tag{1.28}$$

(iii) For $1 \leq \ell < k$ we have

$$\sum_{n_{2k-1} \geq \cdots \geq n_1 \geq 0} q^{\binom{n_{2k-1}+1}{2} + \sum_{i=0}^{k-2} (n_{k+i}^2 + n_{k+i}) - \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i + \binom{n_1+1}{2}} \times \frac{(q)_{n_{2k-1}} (-1)^{n_{2k-1}+n_k+n_1}}{(q)_{n_{2k-1}-n_{2k-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}^2} = \sum_{\substack{n \geq 0 \\ |j| \leq n}} (-1)^n \operatorname{sgn}(j) q^{kn^2 + (k-1)n + \binom{n+1}{2} - kj^2 - \ell j} (1 - q^{2n+1}). \tag{1.29}$$

The rest of the paper is organized as follows. In the next section we collect the necessary background on Bailey pairs, including one lemma that we prove in Section 3. In Section 4 we prove the main results. We conclude in Section 5 with some remarks.

2. Bailey pairs

With the exception of Lemma 2.5, which will be proved in the next section, the key results on Bailey pairs needed for the proofs of the main theorems are the same as those in part I of this series [17, Sections 2 and 3]. For the benefit of the reader, we reproduce these results here. All of the results describe methods for producing new Bailey pairs from existing ones. We begin with the classical Bailey lemma.

Lemma 2.1. [2, p. 270] *If (α_n, β_n) is a Bailey pair relative to a, then so is (α_n', β_n'), where*

$$\alpha_n' = \frac{(b)_n(c)_n(aq/bc)^n}{(aq/b)_n(aq/c)_n}\alpha_n \tag{2.1}$$

and

$$\beta_n' = \sum_{k=0}^{n} \frac{(b)_k(c)_k(aq/bc)_{n-k}(aq/bc)^k}{(aq/b)_n(aq/c)_n(q)_{n-k}}\beta_k. \tag{2.2}$$

Repeated application of (2.1) and (2.2) is called "iterating along the Bailey chain." We will most often require the case $b, c \to \infty$, which gives the Bailey pair relative to a,

$$\alpha_n' = a^n q^{n^2}\alpha_n \tag{2.3}$$

and

$$\beta_n' = \sum_{k=0}^{n} \frac{a^k q^{k^2}}{(q)_{n-k}}\beta_k. \tag{2.4}$$

The second result follows from the "Bailey lattice" [1, p. 59], as detailed in [17, Section 3], and allows us to change a Bailey pair relative to q to one relative to 1 while fixing β_n.

Lemma 2.2. [17, Section 3] *If (α_n, β_n) is a Bailey pair relative to q, then (α_n', β_n') is a Bailey pair relative to 1, where $\alpha_0' = \alpha_0$ and for $n \geq 1$,*

$$\alpha_n' = \frac{1-q}{1-q^{2n+1}}\alpha_n - \frac{q^{2n-1}(1-q)}{1-q^{2n-1}}\alpha_{n-1} \tag{2.5}$$

and

$$\beta_n' = \beta_n. \tag{2.6}$$

The next lemma also allows us to fix the β_n, this time changing a to aq. This result will also be used in the proof of Lemma 2.5 below.

Lemma 2.3. [16, p. 1510] *If (α_n, β_n) is a Bailey pair relative to a, then (α_n', β_n') is a Bailey pair relative to aq, where*

$$\alpha_n' = \frac{(1-aq^{2n+1})(aq/b)_n(-b)^n q^{n(n-1)/2}}{(1-aq)(bq)_n} \sum_{r=0}^{n} \frac{(b)_r}{(aq/b)_r}(-b)^{-r}q^{-r(r-1)/2}\alpha_r \tag{2.7}$$

and

$$\beta_n' = \frac{(b)_n}{(bq)_n}\beta_n. \tag{2.8}$$

Observe that Lemma 2.3 with $a = 1$ and $b \to 0$ gives the Bailey pair relative to q,

$$\alpha'_n = \frac{q^{n^2}(1-q^{2n+1})}{1-q}\sum_{j=0}^{n} q^{-j^2}\alpha_j \tag{2.9}$$

and

$$\beta'_n = \beta_n. \tag{2.10}$$

The following lemma gives the so-called "dual" Bailey pair. It allows us to introduce negative powers of q in the α_n.

Lemma 2.4. [2, pp. 278-279] *If $(\alpha_n, \beta_n) = (\alpha_n(a, q), \beta_n(a, q))$ is a Bailey pair relative to a, then so is (α'_n, β'_n), where*

$$\alpha'_n = a^n q^{n^2}\alpha_n(a^{-1}, q^{-1}) \tag{2.11}$$

and

$$\beta'_n = a^{-n}q^{-n^2-n}\beta_n(a^{-1}, q^{-1}). \tag{2.12}$$

Our last lemma will be proved in the following section.

Lemma 2.5. *If (α_n, β_n) is a Bailey pair relative to 1, where*

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{An^2}(q^{(A-1)n} - q^{-(A-1)n}), & \text{otherwise,} \end{cases} \tag{2.13}$$

then so is (α'_n, β'_n), where

$$\alpha'_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{An^2}(q^{An} - q^{-An}), & \text{otherwise,} \end{cases} \tag{2.14}$$

and

$$\beta'_n = q^n\beta_n. \tag{2.15}$$

3. Proof of Lemma 2.5

In this section we establish Lemma 2.5 as a corollary of Lemmas 3.1 and 3.2 below. This might be done more directly, but the two auxiliary lemmas may be of independent interest – especially Lemma 3.2, which appeared in a preprint version of [16] but not in the published version.

Lemma 3.1. *If (α_n, β_n) is a Bailey pair relative to aq, then $(\gamma'_n - \gamma'_{n-1}, \beta'_n)$ is a Bailey pair relative to a, where $\gamma'_{-1} = 0$,*

$$\gamma'_n = \frac{(1-aq)q^n}{(1-aq^{2n+1})}\alpha_n, \tag{3.1}$$

and

$$\beta'_n = q^n\beta_n. \tag{3.2}$$

Proof. If (α_n, β_n) is a Bailey pair relative to aq, then by (1.3) we have

$$\frac{(1-aq)q^n}{(1-aq^{2n+1})}\alpha_n = \sum_{j=0}^{n} \frac{(aq)_{n+j}(-1)^{n-j}q^{\binom{n-j+1}{2}}}{(q)_{n-j}} q^j\beta_j.$$

Using the notation in (3.1), we then have

$$\begin{aligned}\gamma'_n - \gamma'_{n-1} &= \sum_{j=0}^{n} \frac{(aq)_{n+j}(-1)^{n-j}q^{\binom{n-j+1}{2}}}{(q)_{n-j}} q^j\beta_j - \sum_{j=0}^{n-1} \frac{(aq)_{n+j-1}(-1)^{n-j-1}q^{\binom{n-j}{2}}}{(q)_{n-j-1}} q^j\beta_j \\ &= \sum_{j=0}^{n} \frac{(aq)_{n+j-1}(-1)^{n-j}q^{\binom{n-j}{2}}}{(q)_{n-j}} q^j\beta_j \left((1-aq^{n+j})q^{n-j} + (1-q^{n-j})\right) \\ &= \sum_{j=0}^{n} \frac{(aq)_{n+j-1}(-1)^{n-j}q^{\binom{n-j}{2}}}{(q)_{n-j}} q^j\beta_j(1-aq^{2n}) \\ &= \frac{1-aq^{2n}}{1-a} \sum_{j=0}^{n} \frac{(a)_{n+j}(-1)^{n-j}q^{\binom{n-j}{2}}}{(q)_{n-j}} q^j\beta_j.\end{aligned}$$

The result now follows from the definition of a Bailey pair in (1.3). □

Lemma 3.2. *If (α_n, β_n) is a Bailey pair relative to a, then so is $(\gamma'_n - \gamma'_{n-1}, \beta'_n)$, where $\gamma'_{-1} = 0$,*

$$\gamma'_n = \frac{(aq/b)_n(-b)^n q^{n(n+1)/2}}{(bq)_n} \sum_{r=0}^{n} \frac{(b)_r(-b)^{-r}q^{-r(r-1)/2}\alpha_r}{(aq/b)_r}, \tag{3.3}$$

and

$$\beta'_n = \frac{(b)_n q^n}{(bq)_n}\beta_n. \tag{3.4}$$

Proof. This follows from an application of Lemma 3.1 to Lemma 2.3. □

We are now ready to prove Lemma 2.5.

Proof of Lemma 2.5. Setting $b = 0$ and $a = 1$ in Lemma 3.2, equations (3.3) and (3.4) read

$$\begin{aligned}\gamma'_n &= q^{n^2+n}\sum_{r=0}^{n} q^{-r^2}\alpha_r, \\ \beta'_n &= q^n\beta_n.\end{aligned}$$

The second equation above is (2.15). To obtain (2.14), we use (2.13) in the first equation above and find that for $n \geq 1$,

$$\begin{aligned}\alpha'_n &= q^{n^2+n}\left(\sum_{r=0}^{n} q^{(A-1)r^2+(A-1)r} - \sum_{r=1}^{n} q^{(A-1)r^2-(A-1)r}\right) \\ &\quad - q^{n^2-n}\left(\sum_{r=0}^{n-1} q^{(A-1)r^2+(A-1)r} - \sum_{r=1}^{n-1} q^{(A-1)r^2-(A-1)r}\right) \\ &= q^{n^2+n}q^{(A-1)n^2+(A-1)n} - q^{n^2-n}q^{(A-1)(n-1)^2+(A-1)(n-1)} \\ &= q^{An^2}(q^{An} - q^{-An}),\end{aligned}$$

as desired. □

4. Proofs of the main results

In this section we prove the main results. We begin with proofs of Theorems 1.2 and 1.3. The step-by-step process is similar to the one described in Section 3 of [17].

Proof of Theorem 1.2. **Step 1.** Begin with the Bailey pair relative to 1 [19, p. 468],

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{\frac{n^2}{2}} \left(q^{\frac{n}{2}} - q^{-\frac{n}{2}} \right), & \text{if } n > 0 \end{cases} \tag{4.1}$$

and

$$\beta_n = \frac{2q^n}{(q)_n^2 (1+q^n)}. \tag{4.2}$$

Step 2. Alternately applying (2.3) and (2.4) (with $a = 1$) and Lemma 2.5 $\ell - 1$ times, for $\ell \geq 1$, we obtain the Bailey pair

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{((2\ell-1)n^2 + (2\ell-1)n)/2} - q^{((2\ell-1)n^2 - (2\ell-1)n)/2}, & \text{if } n > 0, \end{cases} \tag{4.3}$$

and

$$\beta_n = \beta_{n_\ell} = 2 \sum_{n_\ell \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{\ell-1} n_i^2 + \sum_{i=1}^{\ell} n_i}}{(q)_{n_\ell - n_{\ell-1}} \cdots (q)_{n_2 - n_1} (q)_{n_1}^2 (1 + q^{n_1})}. \tag{4.4}$$

Step 3. Next we apply (2.3) and (2.4) $k - \ell$ times with $a = 1$, for $k > \ell$, to obtain

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{((2k-1)n^2 + (2\ell-1)n)/2} - q^{((2k-1)n^2 - (2\ell-1)n)/2}, & \text{if } n > 1, \end{cases} \tag{4.5}$$

and

$$\beta_n = \beta_{n_k} = 2 \sum_{n_k \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{k-1} n_i^2 + \sum_{i=1}^{\ell} n_i}}{(q)_{n_k - n_{k-1}} \cdots (q)_{n_2 - n_1} (q)_{n_1}^2 (1 + q^{n_1})}. \tag{4.6}$$

Step 4. Using Lemma 2.4 to compute the dual Bailey pair gives

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{(-(2k-3)n^2 - (2\ell-1)n)/2} - q^{(-(2k-3)n^2 + (2\ell-1)n)/2}, & \text{if } n > 0, \end{cases} \tag{4.7}$$

and

$$\beta_n = \beta_{n_k} = 2(-1)^{n_k} q^{-\binom{n_k+1}{2}} \sum_{n_k \geq \cdots \geq n_1 \geq 0} \frac{q^{-\sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=2}^{\ell} n_i + \binom{n_1+1}{2}} (-1)^{n_1}}{(q)_{n_k - n_{k-1}} \cdots (q)_{n_2 - n_1} (q)_{n_1}^2 (1 + q^{n_1})}. \tag{4.8}$$

Step 5. An application of (2.9) and (2.10) gives a Bailey pair relative to q,

$$\alpha_n = \frac{q^{n^2}(1 - q^{2n+1})}{1 - q} \sum_{j=-n}^{n} \operatorname{sgn}(j) q^{-((2k-1)j^2 + (2\ell-1)j)/2}, \tag{4.9}$$

$$\beta_n = \beta_{n_k} = 2(-1)^{n_k} q^{-\binom{n_k+1}{2}} \sum_{n_k \geq \cdots \geq n_1 \geq 0} \frac{q^{-\sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=2}^{\ell} n_i + \binom{n_1+1}{2}} (-1)^{n_1}}{(q)_{n_k - n_{k-1}} \cdots (q)_{n_2 - n_1} (q)_{n_1}^2 (1 + q^{n_1})}. \tag{4.10}$$

Step 6. Now apply (2.3) and (2.4) m times with $a = q$, for $m \geq 0$, to obtain

$$\alpha_n = \frac{q^{(m+1)n^2+mn}(1-q^{2n+1})}{1-q} \sum_{j=-n}^{n} \operatorname{sgn}(j) q^{-((2k-1)j^2+(2\ell-1)j)/2}, \tag{4.11}$$

$$\begin{aligned}\beta_n = \beta_{n_{k+m}} = 2 \sum_{n_{k+m}\geq\cdots\geq n_1\geq 0} & q^{\sum_{i=0}^{m-1}(n_{k+i}^2+n_{k+i})-\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=2}^{\ell}n_i+\binom{n_1+1}{2}} \\ & \times \frac{(-1)^{n_k+n_1}}{(q)_{n_{k+m}-n_{k+m-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}^2(1+q^{n_1})}.\end{aligned} \tag{4.12}$$

For $m = K$ this gives part (i) of Theorem 1.2.

Step 7. Using (2.5) and (2.6) we have

$$\begin{aligned}\alpha_n = {} & q^{(m+1)n^2+mn} \sum_{j=-n}^{n} \operatorname{sgn}(j) q^{-((2k-1)j^2+(2\ell-1)j)/2} \\ & - \chi(n\neq 0) q^{(m+1)n^2-mn} \sum_{j=-n+1}^{n-1} \operatorname{sgn}(j) q^{-((2k-1)j^2+(2\ell-1)j)/2},\end{aligned} \tag{4.13}$$

$$\begin{aligned}\beta_n = \beta_{n_{k+m}} = 2 \sum_{n_{k+m}\geq\cdots\geq n_1\geq 0} & q^{\sum_{i=0}^{m-1}(n_{k+i}^2+n_{k+i})-\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=2}^{\ell}n_i+\binom{n_1+1}{2}} \\ & \times \frac{(-1)^{n_k+n_1}}{(q)_{n_{k+m}-n_{k+m-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}^2(1+q^{n_1})}.\end{aligned} \tag{4.14}$$

Step 8. Finally we use (2.3) and (2.4) $K - m$ times with $a = 1$, yielding

$$\begin{aligned}\alpha_n = {} & q^{(K+1)n^2+mn} \sum_{j=-n}^{n} \operatorname{sgn}(j) q^{-((2k-1)j^2+(2\ell-1)j)/2} \\ & - \chi(n\neq 0) q^{(K+1)n^2-mn} \sum_{j=-n+1}^{n-1} \operatorname{sgn}(j) q^{-((2k-1)j^2+(2\ell-1)j)/2},\end{aligned} \tag{4.15}$$

$$\begin{aligned}\beta_n = \beta_{n_{k+K}} = 2 \sum_{n_{k+K}\geq\cdots\geq n_1\geq 0} & q^{\sum_{i=0}^{K-1}n_{k+i}^2+\sum_{i=0}^{m-1}n_{k+i}-\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=2}^{\ell}n_i+\binom{n_1+1}{2}} \\ & \times \frac{(-1)^{n_k+n_1}}{(q)_{n_{k+K}-n_{k+K-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}^2(1+q^{n_1})}.\end{aligned} \tag{4.16}$$

This completes the proof. □

Proof of Theorem 1.3. The proof of Theorem 1.3 follows the same steps as above, except that we start with the Bailey pair relative to 1 [19, p. 468],

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{n^2}(q^n - q^{-n}), & \text{if } n > 0 \end{cases} \tag{4.17}$$

and

$$\beta_n = \frac{q^n}{(q)_n^2}. \tag{4.18}$$

We follow Steps 2–3 as before and then at Step 4 the dual Bailey pair relative to 1 is

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{-(k-1)n^2-\ell n} - q^{-(k-1)n^2+\ell n}, & \text{if } n > 0, \end{cases} \tag{4.19}$$

and

$$\beta_n = \beta_{n_k} = (-1)^{n_k} q^{-\binom{n_k+1}{2}} \sum_{n_k \geq \cdots \geq n_1 \geq 0} \frac{q^{-\sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i + \binom{n_1+1}{2}} (-1)^{n_1}}{(q)_{n_k - n_{k-1}} \cdots (q)_{n_2-n_1} (q)_{n_1}^2}, \tag{4.20}$$

and at Step 5 the Bailey pair relative to q is

$$\alpha_n = \frac{q^{n^2}(1-q^{2n+1})}{1-q} \sum_{j=-n}^{n} \operatorname{sgn}(j) q^{-kj^2-\ell j} \tag{4.21}$$

and

$$\beta_n = \beta_{n_k} = (-1)^{n_k} q^{-\binom{n_k+1}{2}} \sum_{n_k \geq \cdots \geq n_1 \geq 0} \frac{q^{-\sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i + \binom{n_1+1}{2}} (-1)^{n_1}}{(q)_{n_k - n_{k-1}} \cdots (q)_{n_2-n_1} (q)_{n_1}^2}. \tag{4.22}$$

Following Steps 6–8 as before gives the result. □

We now turn to the proof of Theorem 1.4. For this, we need a lemma.

Lemma 4.1. *We have the following Bailey pairs.*

(*i*) *Let* $1 \leq \ell < k$ *and* $K \geq 0$. *The sequences* (α_n, β_n) *form a Bailey pair relative to* 1, *where*

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{(K+1)n^2 - \frac{(2k-1)}{2}n^2 - \frac{(2\ell-3)}{2}n} - q^{(K+1)n^2 - \frac{(2k-1)}{2}n^2 + \frac{(2\ell-3)}{2}n}, & \text{if } n > 0, \end{cases} \tag{4.23}$$

and

$$\beta_n = 2 \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=0}^{K-1} n_{k+i}^2 - \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell-1} n_i + \binom{n_1+1}{2} + n_1} (-1)^{n_k+n_1}}{(q)_{n-n_{k+K-1}} \cdots (q)_{n_2-n_1} (q)_{n_1}^2 (1+q^{n_1})}. \tag{4.24}$$

(*ii*) *Let* $1 \leq \ell < k$ *and* $K \geq 0$. *The sequences* (α_n, β_n) *form a Bailey pair relative to* 1, *where*

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{(K+1)n^2 - kn^2 - (\ell-1)n} - q^{(K+1)n^2 - kn^2 + (\ell-1)n}, & \text{if } n > 0, \end{cases} \tag{4.25}$$

and

$$\beta_n = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=0}^{K-1} n_{k+i}^2 - \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell-1} n_i + \binom{n_1+1}{2}} (-1)^{n_k+n_1}}{(q)_{n-n_{k+K-1}} \cdots (q)_{n_2-n_1} (q)_{n_1}^2}. \tag{4.26}$$

Proof. We begin with part (*i*). First suppose that $\ell \geq 2$, and start with the Bailey pair relative to 1 in (4.7) and (4.8), with ℓ replaced by $\ell - 1$. Iterating this K times along the Bailey chain in (2.3) and (2.4) gives the required Bailey pair in (4.23) and (4.24).

For the case $\ell = 1$ we start with the Bailey pair relative to 1 [19, p. 468],

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{\frac{n^2}{2}} \left(q^{-\frac{n}{2}} - q^{\frac{n}{2}} \right), & \text{if } n > 0 \end{cases} \tag{4.27}$$

and

$$\beta_n = \frac{2}{(q)_n^2 (1 + q^n)}. \tag{4.28}$$

We iterate $k-1$ times using (2.3) and (2.4) and then compute the dual Bailey pair to obtain

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ q^{-\frac{(2k-3)}{2} n^2} \left(q^{\frac{n}{2}} - q^{-\frac{n}{2}} \right), & \text{if } n > 0, \end{cases} \tag{4.29}$$

and

$$\beta_n = \beta_{n_k} = 2(-1)^{n_k} q^{-\binom{n_k+1}{2}} \sum_{n_k \geq \cdots \geq n_1 \geq 0} \frac{q^{-\sum_{i=1}^{k-1} n_i n_{i+1} + \binom{n_1+1}{2} + n_1} (-1)^{n_1}}{(q)_{n_k - n_{k-1}} \cdots (q)_{n_2 - n_1} (q)_{n_1}^2 (1 + q^{n_1})}. \tag{4.30}$$

Iterating K more times using (2.3) and (2.4) gives (4.23) and (4.24) when $\ell = 1$.

For part (ii) we again begin by assuming that $\ell \geq 2$. Taking the Bailey pair in (4.19) and (4.20) with ℓ replaced by $\ell - 1$ and then iterating K times along the Bailey chain using (2.3) and (2.4) gives (4.25) and (4.26) for $\ell \geq 2$.

For $\ell = 1$ note that in (4.25) we have $\alpha_n = \chi(n = 0)$ independently of K and k. This corresponds to [19, p. 468]

$$\beta_n = \frac{1}{(q)_n^2}. \tag{4.31}$$

In order to write β_n in the form of (4.26) we use the Bailey chain and the dual Bailey pair as above. Namely, we iterate $k-1$ times using (2.3) and (2.4) and then compute the dual Bailey pair to obtain

$$\beta_n = \beta_{n_k} = (-1)^{n_k} q^{-\binom{n_k+1}{2}} \sum_{n_k \geq \cdots \geq n_1 \geq 0} \frac{q^{-\sum_{i=1}^{k-1} n_i n_{i+1} + \binom{n_1+1}{2}} (-1)^{n_1}}{(q)_{n_k - n_{k-1}} \cdots (q)_{n_2 - n_1} (q)_{n_1}^2}. \tag{4.32}$$

Then we iterate K more times using (2.3) and (2.4), giving (4.25) and (4.26) when $\ell = 1$. (Note that β_n never actually changed here, only the form in which it was written.) □

Proof of Theorem 1.4. The case $m = 1$ of the second part of Theorem 1.2 gives

$$\begin{aligned} \beta_n = 2 \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} & q^{\sum_{i=0}^{K-1} n_{k+i}^2 - \binom{n_k}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i + \binom{n_1+1}{2} + n_1} \\ & \times \frac{(-1)^{n_k + n_1}}{(q)_{n - n_{k+K-1}} \cdots (q)_{n_2 - n_1} (q)_{n_1}^2 (1 + q^{n_1})} \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} \alpha_n = & \, q^{(K+1)n^2 + n} \sum_{j=-n}^{n} \operatorname{sgn}(j) q^{-((2k-1)j^2 + (2\ell-1)j)/2} \\ & - \chi(n \neq 0) q^{(K+1)n^2 - n} \sum_{j=-n+1}^{n-1} \operatorname{sgn}(j) q^{-((2k-1)j^2 + (2\ell-1)j)/2} \end{aligned}$$

$$= -(1-q^{2n})q^{(K+1)n^2-n}\sum_{j=-n}^{n-1}\operatorname{sgn}(j)q^{-((2k-1)j^2+(2\ell-1)j)/2}$$
$$+\begin{cases}1, & \text{if } n=0,\\ q^{(K+1)n^2-\frac{(2k-1)}{2}n^2-\frac{(2\ell-3)}{2}n}-q^{(K+1)n^2-\frac{(2k-1)}{2}n^2+\frac{(2\ell-3)}{2}n}, & \text{if } n>0.\end{cases} \tag{4.34}$$

Comparing equations (4.33) and (4.34) with part (i) of Lemma (4.1) and using the linearity of Bailey pairs, we have that (α_n,β_n) is a Bailey pair relative to 1, where

$$\alpha_n = (1-q^{2n})q^{(K+1)n^2-n}\sum_{j=-n}^{n-1}\operatorname{sgn}(j)q^{-((2k-1)j^2+(2\ell-1)j)/2} \tag{4.35}$$

and

$$\begin{aligned}\beta_n &= 2\sum_{n\geq n_{k+K-1}\geq\cdots\geq n_1\geq 0} q^{\sum_{i=0}^{K-1}n_{k+i}^2-\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell-1}n_i+\binom{n_1+1}{2}+n_1}\\ &\qquad\times\frac{(-1)^{n_k+n_1}}{(q)_{n-n_{k+K-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}^2(1+q^{n_1})}\\ &\quad -2\sum_{n\geq n_{k+K-1}\geq\cdots\geq n_1\geq 0} q^{\sum_{i=0}^{K-1}n_{k+i}^2-\binom{n_k}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell}n_i+\binom{n_1+1}{2}+n_1}\\ &\qquad\times\frac{(-1)^{n_k+n_1}}{(q)_{n-n_{k+K-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}^2(1+q^{n_1})}\\ &= 2\sum_{n\geq n_{k+K-1}\geq\cdots\geq n_1\geq 0} q^{\sum_{i=0}^{K-1}n_{k+i}^2-\binom{n_k+1}{2}-\sum_{i=1}^{k-1}n_in_{i+1}-\sum_{i=1}^{\ell-1}n_i+\binom{n_1+1}{2}+n_1}\\ &\qquad\times\frac{(-1)^{n_k+n_1}(1-q^{n_k-n_\ell})}{(q)_{n-n_{k+K-1}}\cdots(q)_{n_2-n_1}(q)_{n_1}^2(1+q^{n_1})}.\end{aligned} \tag{4.36}$$

This gives the first part of the theorem. The second part follows in exactly the same way, using the case $m=1$ of part (ii) of Theorem 1.3 along with part (ii) of Lemma 4.1. The details are omitted. □

We are now ready to prove Corollaries 1.5 and 1.6. We begin by collecting some standard consequences of the Bailey lemma.

Lemma 4.2. *We have the following identities.*

(i) *If (α_n,β_n) is a Bailey pair relative to 1, then*

$$\sum_{n\geq 0}q^{n^2}\beta_n = \frac{1}{(q)_\infty}\sum_{n\geq 0}q^{n^2}\alpha_n, \tag{4.37}$$

$$\sum_{n\geq 0}(-1)_{2n}q^n\beta_n(q^2) = \frac{(-1)_\infty}{(q)_\infty}\sum_{n\geq 0}\frac{q^n}{1+q^{2n}}\alpha_n(q^2), \tag{4.38}$$

$$\sum_{n\geq 1}(q)_{n-1}(-1)^nq^{\binom{n+1}{2}}\beta_n = \sum_{n\geq 1}\frac{(-1)^nq^{\binom{n+1}{2}}}{1-q^n}\alpha_n. \tag{4.39}$$

(ii) If (α_n, β_n) is a Bailey pair relative to q, then

$$\sum_{n\geq 0} (-q)_n q^{\binom{n+1}{2}} \beta_n = \frac{(-q)_\infty}{(q^2)_\infty} \sum_{n\geq 0} q^{\binom{n+1}{2}} \alpha_n, \tag{4.40}$$

$$\sum_{n\geq 0} (q)_n (-1)^n q^{\binom{n+1}{2}} \beta_n = (1-q) \sum_{n\geq 0} (-1)^n q^{\binom{n+1}{2}} \alpha_n. \tag{4.41}$$

Proof. Using (2.1) and (2.2) in (1.2) and letting $n \to \infty$, we have that if (α_n, β_n) is a Bailey pair relative to a, then

$$\sum_{n\geq 0} (b)_n (c)_n (aq/bc)^n \beta_n = \frac{(aq/b)_\infty (aq/c)_\infty}{(aq)_\infty (aq/bc)_\infty} \sum_{n\geq 0} \frac{(b)_n (c)_n (aq/bc)^n}{(aq/b)_n (aq/c)_n} \alpha_n. \tag{4.42}$$

Now, with $a = 1$, equation (4.37) follows from taking $b, c \to \infty$ in (4.42), equation (4.38) by setting $(q, b, c) = (q^2, -1, -q)$, and equation (4.39) by taking the derivative $\frac{\partial}{\partial b}|_{b=1}$ and then letting $c \to \infty$. With $a = q$, equations (4.40) and (4.41) follow from letting $b \to \infty$ and taking $c = \mp q$. □

Proof of Corollaries 1.5 and 1.6. The identities in Corollaries 1.5 and 1.6 now follow readily upon substituting one of the Bailey pairs from Theorems 1.2 – 1.4 into one of the equations in Lemma 4.2. We have restricted to $K = k - 1$ for simplicity. Specifically, in Corollary 1.5, equation (1.24) follows from substituting the Bailey pair in (1.12) and (1.13) into (4.40), equation (1.25) follows from substituting (1.14) and (1.15) into (4.37), and equation (1.26) follows from substituting (1.20) and (1.21) into (4.38). In Corollary 1.6, equation (1.27) follows from substituting the Bailey pair in (1.20) and (1.21) into (4.39), equation (1.28) follows from substituting (1.12) and (1.13) into (4.41), and equation (1.29) follows from substituting (1.16) and (1.17) into (4.41). □

5. Concluding Remarks

We close with several remarks. First, while Theorems 1.2 – 1.4 involve only quadratic forms of the type

$$(K+1)n^2 + mn - ((2k-1)j^2 + (2\ell-1)j)/2 \tag{5.1}$$

and

$$(K+1)n^2 + mn - kj^2 - \ell j, \tag{5.2}$$

Bailey pairs with quadratic forms of the type

$$((2K+3)n^2 + (2m+1)n)/2 - ((2k-1)j^2 + (2\ell-1)j)/2 \tag{5.3}$$

and

$$((2K+3)n^2 + (2m+1)n)/2 - kj^2 - \ell j \tag{5.4}$$

can be easily constructed as well. The simplest way to do this is to just apply the Bailey lemma to Bailey pairs containing the quadratic forms in (5.1) or (5.2). For example, if (α_n, β_n) is a Bailey pair relative to q, then (2.1) and (2.2) with $b = -q$ and $c \to \infty$ gives

$$\alpha'_n = q^{\binom{n+1}{2}} \alpha_n.$$

Second, in the special case $K = k$ of Theorem 1.1, the β_n admit a simplification from $(2k-1)$-fold sums to $(k-1)$-fold sums, as detailed in [14, Sections 3 and 5] or [10, Section 3.2]. Indeed, this simplification revealed that those β_n are polynomials, a fact which played an important role in connection with Maass waveforms [10] and knot theory [14]. Some kind of simplification of certain instances of the β_n in Theorems 1.2 – 1.4 may also be possible, but we have not pursued this.

Third, there are expressions for mock theta functions in terms of indefinite binary theta functions which do not appear to be covered by the Bailey pairs in [17]. For instance, if $\psi(q)$ denotes Ramanujan's third order mock theta function

$$\psi(q) = \sum_{n \geq 1} \frac{q^{n^2}}{(q;q^2)_n}, \tag{5.5}$$

Andrews [6] showed that

$$1 + \psi(q) = \frac{1}{(q)_\infty} \sum_{n \geq 0} (-1)^n q^{2n^2+n}(1 - q^{6n+6}) \sum_{j=0}^{n} q^{-\binom{j+1}{2}} \tag{5.6}$$

and Mortenson [18] showed that

$$1 + 2\psi(q) = \frac{1}{(q)_\infty} \sum_{n \geq 0} (-1)^n q^{2n^2+n}(1 + q^{2n+1}) \sum_{j=-n}^{n} q^{-\binom{j+1}{2}}. \tag{5.7}$$

Neither sum on j in these two equations figures in the results in [17], suggesting that it may be worthwhile to further explore our iterative method using seeds from [19] other than the ones used here or in [17].

Finally, given the similarities between indefinite binary theta series related to mock theta functions and Maass waveforms and the false indefinite binary theta series in Corollaries 1.5 and 1.6, one wonders about the possibility of any modular properties in the false case. It is known that modular completions can be constructed for certain false binary theta functions when the quadratic form is positive definite – see [9], for example. It remains to be seen if the same is true for indefinite quadratic forms.

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