AUTOMORPHIC PROPERTIES OF GENERATING FUNCTIONS FOR GENERALIZED RANK MOMENTS AND DURFEE SYMBOLS

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ABSTRACT. We define two-parameter generalizations of two combinatorial constructions of Andrews: the kth symmetrized rank moment and the k-marked Durfee symbol. We prove that three specializations of the associated generating functions are so-called quasimock theta functions, while a fourth specialization gives quasimodular forms. We then define a two-parameter generalization of Andrews' smallest parts function and note that this leads to quasimock theta functions as well. The automorphic properties are deduced using q-series identities relating the relevant generating functions to known mock theta functions.

1. INTRODUCTION

The series $\mathcal{N}_{2v}(0,0;q)$, defined for $v \geq 1$ by

$$\mathcal{N}_{2v}(0,0;q) := \prod_{k=1}^{\infty} \frac{1}{1-q^k} \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{(-1)^{n-1} q^{n(3n-1)/2+vn}}{(1-q^n)^{2v}},$$

have recently been the focus of several studies. These series are tied to some new and exciting partition-theoretic constructions [3] and their coefficients satisfy many elegant identities, congruences, and asymptotic properties [3, 7, 8]. From a theoretical standpoint, such results are ultimately due to the fact that the series $q^{-1}\mathcal{N}_{2v}(0,0;q^{24})$ enjoy the automorphic structure of *quasimock theta functions* (see below for the definition).

Quasimock theta functions were introduced in [7, 8] to describe functions like $\mathcal{N}_{2v}(0,0;q)$ which resemble Ramanujan's mock theta functions but involve additional quasimodular components. In the same way that the modularity of the generating function for partitions has many important consequences, the theory of quasimock theta functions can be used to prove important properties of the combinatorial objects encoded in their coefficients.

This paper lies at the intersection of two questions. First, are there meaningful generalizations of the combinatorial constructions associated with $\mathcal{N}_{2v}(0,0;q)$? Second, how can one find further examples of simple q-series which are quasimock theta functions? The answer to both of these questions will be found in the series $\mathcal{N}_{2v}(d,e;q)$, defined for $v \geq 1$ by

$$\mathcal{N}_{2v}(d,e;q) := \frac{(-dq,-eq)_\infty}{(q,deq)_\infty} \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{(-1)^{n-1} q^{\binom{n+1}{2}+vn} (de)^n \left(-1/d,-1/e\right)_n}{(1-q^n)^{2v} \left(-dq,-eq\right)_n}. \tag{1.1}$$

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Here we employ the standard q-series notation,

$$(a_1, a_2, \ldots, a_j)_n = (a_1, a_2, \ldots, a_j; q)_n := \frac{(a_1, a_2, \ldots, a_j)_\infty}{(a_1q^n, a_2q^n, \ldots, a_jq^n)_\infty},$$

where

$$(a_1, a_2, \ldots, a_j)_\infty = (a_1, a_2, \ldots, a_j; q)_\infty := \prod_{k=0}^{\infty} \left(1 - a_1q^k\right)\left(1 - a_2q^k\right)\cdots\left(1 - a_jq^k\right).$$

An intriguing and almost completely unsolved problem is to understand the overlap between classes of q-series and modular forms. This topic was one of the subjects of George Andrews' plenary address [2] at the Millennial Conference at the University of Illinois in 2000. A remarkable conjecture due to Werner Nahm (see [29] and [31]) relates the answer to this question in a very special case to algebraic k-theory and conformal field theory. In order to make progress in this general direction, it is natural to search for examples of q-series with automorphic properties.

In the first part of the paper we discuss two combinatorial interpretations of the series $\mathcal{N}_{2v}(d, e; q)$. The natural context is that of overpartition pairs [9, 24, 26]. In this setting we shall define a *generalized kth symmetrized rank moment* and a *generalized k-marked Durfee symbol*, each of which has $\mathcal{N}_{2v}(d, e; q)$ as its generating function (see Section 2). When $d = e = 0$ we recover the partition-theoretic work of Andrews [3].

In the second part of the paper we show that the series $\mathcal{N}_{2v}(1, 0; q)$, $\mathcal{N}_{2v}(1, 1/q; q^2)$, and $q^{-1}\mathcal{N}_{2v}(0, 1/q^8; q^{16})$ are quasimock theta functions, while the series $\mathcal{N}_{2v}(1, 1; q)$ are quasimodular forms. These four specializations correspond to four important "rank" functions (see Section 2).

Theorem 1.1. *The functions $\mathcal{N}_{2v}(1, 1; q)$ are quasimodular forms.*

Theorem 1.2. *The functions $\mathcal{N}_{2v}(1, 0; q)$, $\mathcal{N}_{2v}(1, 1/q; q^2)$, and $q^{-1}\mathcal{N}_{2v}(0, 1/q^8; q^{16})$ are quasimock theta functions.*

Let us now recall what it means to be a quasimock theta function. If $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, $z = x + iy$ with $x, y \in \mathbb{R}$, then the weight k hyperbolic Laplacian is given by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

If v is odd, then define ϵ_v by

$$\epsilon_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases}$$

Moreover we let χ be a Dirichlet character. A *(harmonic) weak Maass form of weight k with Nebentypus χ on a subgroup* $\Gamma \subset \Gamma_0(4)$ is any smooth function $g : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $z \in \mathbb{H}$, we have

$$g(Az) = \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \chi(d)\, (cz + d)^k\, g(z).$$

(2) We have that $\Delta_k g = 0$.
(3) The function $g(z)$ has at most linear exponential growth at all the cusps of Γ.

In light of recent work of the first author and Ono [13, 14] combined with work of Zwegers [32], [33] we now know that what Ramanujan called mock theta functions in his last letter to Hardy [30] are actually "holomorphic parts" of weak Maass forms. In turn the holomorphic part of any weak Maass form may be called a mock theta function. In analogy with quasimodular forms [21], a *quasiweak Maass form* is defined to be any linear combination of derivatives of weak Maass forms. A function $f(q)$ is called a *quasimock theta function* if there is a quasimodular form $h(q)$ such that $f(q)+h(q)$ is a linear combination of derivatives of holomorphic parts of weak Maass forms. As usual, $q := e^{2\pi i z}$. Notice that taking derivatives preserves the space of quasimock theta functions.

Our approach to Theorem 1.2 highlights the role that q-series identities can play in the study of weak Maass forms. Typically (see [7], for example) one requires lengthy and delicate analytic calculations to determine transformation properties. However, we shall use q-series identities to circumvent these calculations. We proceed in the same manner for each of the three cases. First, we use a generalized Lambert series identity to establish the case $v = 1$ by relating the relevant function to known weak Maass forms studied in [10, 11]. Then, following the lead of [8], we prove a partial differential equation and use it to establish the case $v \geq 2$ by induction. These partial differential equations are of independent interest [5], [12], [15].

The functions $\mathcal{N}_{2v}(1, 0; q)$ are treated in Section 4, the $\mathcal{N}_{2v}(1, 1/q; q^2)$ in Section 5, and the $q^{-1}\mathcal{N}_{2v}(0, 1/q^8; q^{16})$ in Section 6. The proof of Theorem 1.1 is more straightforward, following from a certain infinite product associated with overpartition pairs. This is discussed in Section 3.

Finally in Section 7 we take a closer look at the case $v = 1$ of (1.1). Andrews observed that the function $Spt(0, 0; q)$, where

$$Spt(d, e; q) := \frac{(-dq, -eq)_\infty}{(q, deq)_\infty} \sum_{n\geq 1} \frac{nq^n}{(1-q^n)} - \mathcal{N}_2(d, e; q), \tag{1.2}$$

has an elegant combinatorial interpretation and satisfies some nice congruence properties [4]. Further congruence properties were found by Folsom-Ono [18] and Garvan [19]. Again these ultimately arise from the fact that $q^{-1}Spt(0, 0; q^{24})$ is a quasimock theta function. We shall give a combinatorial interpretation of $Spt(d, e; q)$ that reduces to Andrews' when $d = e = 0$. Since specializations of the first term on the right hand side of (1.2) are quasimodular forms, the following corollary is immediate from Theorem 1.2:

Corollary 1.3. *The series $Spt(1, 0; q)$, $Spt(1, 1/q; q^2)$, and $q^{-1}Spt(0, 1/q^8; q^{16})$ are quasimock theta functions.*

Remark 1.4. *As for the case $d = e = 1$, it turns out that $Spt(1, 1; q)$ easily simplifies. We have*

$$Spt(1, 1; q) = -1/4 + (-q)_\infty^2/4(q)_\infty^2,$$

which is (essentially) a modular form.

Theorems 1.1 and 1.2 along with Corollary 1.3 provide the theoretical framework necessary to prove any specific number-theoretic fact about these functions. The types of results obtainable and the methods to be employed are well-documented in [3, 4, 7, 8, 18, 19], and so we shall not pursue this here.

2. Rank moments and marked Durfee symbols

2.1. **A generalized kth symmetrized rank moment.** Recall that Dyson's *rank* of a partition is the largest part minus the number of parts [17]. Atkin and Garvan [5] initiated the study of rank moments, the *kth rank moment* $N_k(n)$ being defined by

$$N_k(n) := \sum_{m\in\mathbb{Z}} m^k N(m,n).$$

Here $N(m,n)$ denotes the number of partitions of n with rank m. Following their lead, Andrews [3] defined the *kth symmetrized rank moment* by

$$\eta_k(n) := \sum_{m\in\mathbb{Z}} \binom{m+\lfloor\frac{k-1}{2}\rfloor}{k} N(m,n).$$

Evidently, the symmetrized rank moments can be expressed in terms of the ordinary rank moments, and vice versa. One reason to consider the symmetrized rank moment is its simple generating function. Namely, we have (see Theorem 2 in [3])

$$\sum_{n\geq 0} \eta_{2v}(n)q^n = \mathcal{N}_{2v}(0,0;q).$$

When k is odd, the relation $N(m,n) = N(-m,n)$ implies that both of the moments $\eta_k(n)$ and $N_k(n)$ are 0.

Here we will interpret the series $\mathcal{N}_{2v}(d,e;q)$ in terms of rank moments as well, but using the rank of an overpartition pair [9, 26]. Recall that an *overpartition* λ of n is a partition of n in which the first occurrence of a number may be overlined. An *overpartition pair* (λ,μ) of n is a pair of overpartitions where the sum of all of the parts is n. To define the rank of an overpartition pair we use the notations $\ell(\cdot)$ and $n(\cdot)$ for the largest part and the number of parts of an object. Overlining these functions indicates that we are only considering the overlined parts. We order the parts of (λ,μ) by stipulating that for a number k,

$$\overline{k}_\lambda > k_\lambda > \overline{k}_\mu > k_\mu,$$

where the subscript indicates to which of the two overpartitions the part belongs. The *rank of an overpartition pair* (λ,μ) is

$$\ell((\lambda,\mu)) - n(\lambda) - \overline{n}(\mu) - \chi((\lambda,\mu)),$$

where $\chi((\lambda,\mu))$ is defined to be 1 if the largest part of (λ,μ) is non-overlined and in μ, and 0 otherwise. For example, the rank of the overpartition pair $((\overline{6},6,5,4,4,4,\overline{3},\overline{1}),(7,7,\overline{5},2,2,2))$ is $7-8-1-1=-3$, while the rank of the overpartition pair $((4,\overline{3},3,\overline{2},1),(4,4,4,\overline{1}))$ is $4-5-1-0=-2$.

Let $N(r,s,m,n)$ denote the number of overpartition pairs of n having rank m, such that r is the number of overlined parts in λ plus the number of non-overlined parts in μ and s is the number of parts in μ. Appealing to the case $(b,q) = (q^{1/2},q^{1/2})$ of [25, Thm 1.2], we have the generating function

$$N(d,e,x;q) := \sum_{\substack{r,s,n\geq 0\\ m\in\mathbb{Z}}} N(r,s,m,n)d^r e^s x^m q^n = \sum_{n\geq 0} \frac{(-1/d,-1/e)_n(deq)^n}{(xq,q/x)_n}. \tag{2.1}$$

This includes generating functions for several important "ranks". When $e=0$ and $d=1$ we recover the generating function for Dyson's rank for overpartitions [23], and when both d and $e=0$ we recover the generating function for Dyson's rank for partitions. When $q=q^2$, $d=1$, and $e=1/q$, we have the M_2-rank for overpartitions [25], and when $q=q^2$, $d=0$, and $e=1/q$, we have the M_2-rank for partitions without repeated odd parts [6, 28]. Note that the invariance of the right hand side of (2.1) under $x \leftrightarrow 1/x$ implies that $N(r,s,m,n) = N(r,s,-m,n)$.

We are now prepared to define the general kth symmetrized rank moment for overpartition pairs. It will be useful to also define the ordinary *kth rank moment for overpartition pairs*. It is

$$N_k(r,s,n) := \sum_{m\in\mathbb{Z}} m^k N(r,s,m,n), \tag{2.2}$$

and we denote its generating function by $M_k(d,e;q)$,

$$M_k(d,e;q) := \sum_{r,s,n\geq 0} N_k(r,s,n) d^r e^s q^n. \tag{2.3}$$

The *generalized kth symmetrized rank moment* is

$$\eta_k(r,s,n) := \sum_{m\in\mathbb{Z}} \binom{m+\lfloor\frac{k-1}{2}\rfloor}{k} N(r,s,m,n). \tag{2.4}$$

Theorem 2.1. *We have*

$$\sum_{r,s,n\geq 0} \eta_k(r,s,n) d^r e^s q^n = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \mathcal{N}_{2v}(d,e;q) & \text{if } k = 2v. \end{cases}$$

Proof. Since the proof is similar to the proof of [3, Theorem 2], we omit most of the details. From a limiting case of the Watson-Whipple transformation [20],

$$\sum_{n=0}^{\infty} \frac{(aq/bc,d,e)_n \left(\frac{aq}{de}\right)^n}{(q,aq/b,aq/c)_n} = \frac{(aq/d,aq/e)_\infty}{(aq,aq/de)_\infty} \sum_{n=0}^{\infty} \frac{(a,\sqrt{a}q,-\sqrt{a}q,b,c,d,e)_n (aq)^{2n}(-1)^n q^{n(n-1)/2}}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e)_n (bcde)^n},$$

applied to (2.1) with $(a,b,c,d,e) = (1,x,1/x,-1/d,-1/e)$, one may deduce the following alternative form for the generating function for $N(r,s,m,n)$:

$$\sum_{\substack{r,s,n\geq 0 \\ m\in\mathbb{Z}}} N(r,s,m,n) d^r e^s x^m q^n = \frac{(-dq,-eq)_\infty (1-x)}{(q,deq)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-de)^n q^{n(n+3)/2}(-1/d,-1/e)_n}{(-dq,-eq)_n (1-xq^n)}. \tag{2.5}$$

Here we note the helpful relation

$$(a)_{-n} = \frac{(-1)^n q^{\binom{n+1}{2}}}{a^n (q/a)_n}.$$

Next we observe that

$$\sum_{r,s,n\geq 0} \eta_{2v}(r,s,n) d^r e^s q^n = \frac{1}{(2v)!} \left[\frac{\partial^{2v}}{\partial x^{2v}} x^{v-1} N(d,e,x;q)\right]_{x=1}.$$

Now computing the derivatives and simplifying as in [3, p.41-42], we arrive at $\mathcal{N}_{2v}(d,e;q)$. □

As with ordinary partitions, the symmetrized rank moments for overpartition pairs can be expressed in terms of the ordinary rank moments, and vice versa. In particular we note that we have

$$N_2(r,s,n) = 2\eta_2(r,s,n).$$

2.2. **A generalized k-marked Durfee symbol.** The second partition-theoretic object that Andrews associated to $\mathcal{N}_{2v}(0,0;q)$ is the k-marked Durfee symbol [3, Section 4]. Its definition is considerably more involved than that of the kth symmetrized rank moment. We start with k copies of the natural numbers $\{1_1, 2_1, 3_1, \dots\}$, $\{1_2, 2_2, 3_2, \dots\}, \dots, \{1_k, 2_k, 3_k, \dots\}$. We then form the *$k$-marked Durfee symbol* as a two-rowed array with a subscript S. Each row contains a partition using these k copies of the natural numbers where parts are at most S. The rows need not be of equal length. In addition we require that:

(1) The sequence of parts and the sequence of subscripts be non-increasing in each row,
(2) Each of the subscripts smaller than k occurs at least once in the top row,
(3) If $M_1, N_2, \dots, V_{k-2}, W_{k-1}$ are the largest parts with their respective subscripts in the top row, then all parts in the bottom row with subscript 1 lie in the interval $[1, M]$, with subscript 2 lie in $[M, N], \dots,$ with subscript $k-1$ lie in $[V, W]$, and with subscript k lie in $[W, S]$.

We let n be the sum of S^2 and all of the parts in the array and we say that the Durfee symbol is *related* to n. We denote by $\mathcal{D}_k(n)$ the number of k-marked Durfee symbols related to n. Andrews [3] has shown that $\mathcal{D}_{v+1}(n) = \eta_{2v}(n)$.

We now define a generalized *k-marked Durfee symbol* whose generating function will be $\mathcal{N}_{2v}(d,e;q)$. The only difference here is that the subscript S (contributing S^2) will be replaced by a triple (S, μ, ν), μ and ν being partitions into distinct parts between 0 and $S-1$. For such a partition, we say that a number $k \in [0, S-1]$ is *missing* if it does not occur. Let r denote the number of missing numbers in μ and s the number of missing numbers in ν. The number n to which such a Durfee symbol is related is the sum of S, all of the parts in the array, and all of the parts in μ and ν. When both μ and ν are "full" , i.e., $r = s = 0$, we get S^2, the case of the ordinary k-marked Durfee symbols. For example,

$$\begin{pmatrix} 4_3 & 3_2 & 3_1 & 2_1 & 1_1 & \\ 4_3 & 4_3 & 3_2 & 3_1 & 3_1 & 1_1 \end{pmatrix}_{4,(3,2,0),(2,1)} \tag{2.6}$$

is a 3-marked Durfee symbol related to $n = 43$, with $r = 1$ and $s = 2$.

Let $\mathcal{D}_k(r,s,n)$ be the number of generalized k-marked Durfee symbols described above. Following Andrews we define k ranks associated with k-marked Durfee symbols. For such a symbol δ and for each i we denote the number of entries in the top (resp. bottom) row with subscript i by $\tau_i(\delta)$ (resp. $\beta_i(\delta)$). Then the *ith rank* of δ is defined as

$$\rho_i(\delta) := \begin{cases} \tau_i(\delta) - \beta_i(\delta) - 1 & \text{for } 1 \leq i < k, \\ \tau_i(\delta) - \beta_i(\delta) & \text{for } i = k. \end{cases}$$

For example, the Durfee symbol in (2.6) has all three of its ranks equal to -1.

Let $\mathcal{D}_k(r,s,m_1,m_2,\dots,m_k,n)$ denote the number of generalized k-marked Durfee symbols counted by $\mathcal{D}_k(r,s,n)$ with ith rank equal to m_i.

Theorem 2.2. *For $k \geq 2$ we have*

$$\sum_{m_1,m_2,\dots,m_k \in \mathbb{Z}} \sum_{r,s,n \geq 0} \mathcal{D}_k(r,s,m_1,m_2,\dots,m_k,n) x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k} d^r e^s q^n$$
$$= \frac{(-dq,-eq)_\infty}{(q,deq)_\infty} \sum_{n \geq 1} \frac{(-1)^{n-1}(1+q^n)(1-q^n)^2(-1/d,-1/e)_n (de)^n q^{\binom{n}{2}+kn}}{(-dq,-eq)_n \prod_{j=1}^{k}(1-x_j q^n)(1-q^n/x_j)}. \tag{2.7}$$

Proof. Arguing as in [3, Proof of Thm 10] and using the fact that $(-1/y)_n y^n$ is the generating function for partitions into distinct parts between 0 and $n-1$, with the exponent of y counting the number of missing numbers, we have that

$$\sum_{m_1,m_2,\ldots,m_k\in\mathbb{Z}}\sum_{r,s,n\geq 0}\mathcal{D}_k(r,s,m_1,m_2,\ldots,m_k,n)x_1^{m_1}x_2^{m_2}\cdots x_k^{m_k}d^re^sq^n =$$
$$\sum_{\substack{n_1>0\\ n_2,n_3,\ldots,n_k\geq 0}}\frac{(-1/d,-1/e)_{n_1+n_2+\cdots+n_k}(de)^{n_1+n_2+\cdots+n_k}q^{(n_1+n_2+\cdots+n_k)+(n_1+n_2+\cdots+n_{k-1})+\cdots+n_1}}{(x_1q,q/x_1)_{n_1}(x_2q^{n_1},q^{n_1}/x_2)_{n_2+1}}$$
$$\times\frac{1}{(x_3q^{n_1+n_2},q^{n_1+n_2}/x_3)_{n_3+1}\cdots(x_kq^{n_1+\cdots+n_{k-1}},q^{n_1+\cdots+n_{k-1}}/x_k)_{n_k+1}}.$$

Now in the k-fold generalization of Watson's q-analogue of Whipple's theorem [3, eq. (2.4)], replace k by $k+1$, set $b_j = 1/c_j = x_j$ for $1\leq j\leq k$, set $a=1$, set $b_{k+1} = -1/d$ and $c_{k+1} = -1/e$, and finally let $N\to\infty$. Then exactly as in [3, Proof of Thm 3] the right hand side above may be seen to be equal to the right hand side of (2.7). □

Setting each $x_j = 1$ in Theorem 2.2, we obtain the following:

Corollary 2.3. *For $v\geq 1$ we have $\mathcal{D}_{v+1}(r,s,n) = \eta_{2v}(r,s,n)$.*

2.3. **The full rank.** We now define a statistic on generalized k-marked Durfee symbols, called the *full rank*. This will not be required in the sequel, but it plays an important role in the study of ordinary k-marked Durfee symbols [3, 7, 8] and will certainly do so for the generalized symbols as well. The full rank of such a symbol δ is

$$FR(\delta) := \rho_1(\delta)+2\rho_2(\delta)+3\rho_3(\delta)+\cdots+k\rho_k(\delta).$$

Let $NF_k(r,s,m,n)$ denote the number of generalized k-marked Durfee symbols counted by $\mathcal{D}_k(r,s,n)$ whose full rank is equal to m. Evidently the two-variable generating function for $NF_k(r,s,m,n)$ is

$$\sum_{\substack{m\in\mathbb{Z}\\ r,s,n\geq 0}} NF_k(r,s,m,n)d^re^sx^mq^n = R_k\left(d,e,x,x^2,\ldots,x^k;q\right),$$

where for $k\geq 2$, $R_k(d,e,x_1,x_2,\ldots,x_k;q)$ denotes the left-hand side of (2.7). This function $R_k(d,e,x_1,x_2,\ldots,x_k;q)$ for $k\geq 2$ can in fact be expressed in terms of $N(d,e,x;q)$. Exactly as in Theorem 7 of [3], we can show:

Theorem 2.4. *If $x_i\neq x_j, x_j^{-1}$ for $i\neq j$ and $x_i^2\neq 1$, then we have:*

$$R_k(d,e,x_1,x_2,\ldots,x_k;q) = \sum_{i=1}^{k}\frac{N(d,e,x_i;q)}{\prod_{\substack{j=1\\ j\neq i}}^{k}(x_i-x_j)\left(1-\frac{1}{x_ix_j}\right)}. \tag{2.8}$$

Remark 2.5. *If $x_i\in\{x_j,x_j^{-1}\}$ or $x_i^2=1$, then a relation similar to (2.8) can be defined via analytic continuation.*

3. The case $(d, e, q) = (1, 1, q)$

Here we prove Theorem 1.1. We recall from [9] that the two-variable generating function for $\overline{NN}(m, n)$, the number of overpartition pairs of n with rank m, has the very special form given by

$$\sum_{\substack{n\geq 1\\ m\in\mathbb{Z}}} \overline{NN}(m,n)x^m q^n = \frac{-4x}{(1+x)^2} + \frac{4x(-q)_\infty^2}{(1+x)^2\,(xq, q/x)_\infty}. \tag{3.1}$$

Hence the ordinary kth rank moment generating function $M_k(1, 1; q)$ is simply

$$\left[\delta_x^k\left(\frac{-4x}{(1+x)^2} + \frac{4x(-q)_\infty^2}{(1+x)^2\,(xq, q/x)_\infty}\right)\right]_{x=1}, \tag{3.2}$$

where $\delta_x := x\frac{\partial}{\partial x}$. Now this gives that the $M_k(1, 1; q)$ are quasimodular forms (see [5, Section 4] for a discussion of the quasimodularity of terms like those in (3.2)). Since the $\mathcal{N}_{2v}(1, 1; q)$ can be written as linear combinations of the $M_k(1, 1; q)$, this completes the proof of Theorem 1.1.

4. The case $(d, e, q) = (1, 0, q)$

We begin by explicitly stating and proving the case $v = 1$. Let

$$E_2(z) := 1 - 24\sum_{n\geq 1}\frac{nq^n}{(1-q^n)} \tag{4.1}$$

be the usual weight 2 (quasimodular) Eisenstein series. Define the integral

$$\overline{NH}(z) := \frac{1}{2\sqrt{2}\pi i}\int_{-\bar{z}}^{i\infty}\frac{\eta^2(\tau)}{\eta(2\tau)(-i(\tau+z))^{\frac{3}{2}}}\,d\tau.$$

Theorem 4.1. *The function*

$$\mathcal{N}_2(1, 0; q) + \frac{(-q)_\infty}{(q)_\infty}\left(\frac{1}{12} + \frac{1}{6}E_2(2z)\right) - \overline{NH}(z)$$

is a weak Maass form of weight $\frac{3}{2}$ on $\Gamma_0(16)$.

Proof. First we recall an identity involving generalized Lambert series,

$$\sum_{n\in\mathbb{Z}}\frac{(-1)^{n-1}q^{n^2}(1-x)}{(1-xq^n)} + \sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{n^2}(1-x)}{(1+xq^n)} = \frac{-2\left(q^2;q^2\right)_\infty^2}{(1+1/x)\,(x^2q^2, q^2/x^2; q^2)_\infty}.$$

This is the case $y = -1/x$ of [16, eq. (4.3), corrected]. Differentiating twice with respect to x, setting $x = 1$, and multiplying by $(-q)_\infty/(q)_\infty$ yields

$$-4\mathcal{N}_2(1, 0; q) + 4\frac{(-q)_\infty}{(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{n^2+n}}{(1+q^n)^2} = \frac{(-q)_\infty}{(q)_\infty} - 8\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{2nq^{2n}}{(1-q^{2n})}. \tag{4.2}$$

Next we recall from [10] that the function

$$\overline{\mathcal{M}}(z) := 4\frac{(-q)_\infty}{(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{n^2+n}}{(1+q^n)^2} - 4\overline{NH}(z) \tag{4.3}$$

is a weak Maass form of weight $\frac{3}{2}$ on $\Gamma_0(16)$. The theorem follows by substituting for the sum above using (4.2). $\square$

Now to prove the case $v > 1$, we use a partial differential equation which is analogous to the "rank-crank PDE" of Atkin and Garvan [5]. We define $C(x;q)$ and $C^*(x;q)$ by

$$C(x;q) := \frac{(q)_\infty}{(xq, q/x)_\infty}, \qquad C^*(x;q) := \frac{C(x;q)}{(1-x)}.$$

For the function $N(d,e,x;q)$ in (2.1), we define

$$N^*(d,e,x;q) := \frac{N(d,e,x;q)}{(1-x)}. \tag{4.4}$$

We use the differential operator

$$\delta_q := q\frac{\partial}{\partial q}.$$

Furthermore let

$$J(x;q) := (x, q/x)_\infty .$$

We will prove the following partial differential equations:

Theorem 4.2. *We have*

$$x\frac{(q)_\infty^2}{(-q)_\infty}[C^*(x;q)]^3 J(-x;q) = \Big(2(1+x)\delta_q + \frac{1}{2}x + x\delta_x + \frac{1}{2}(1+x)\delta_x^2\Big)N^*(1,0,x;q), \tag{4.5}$$

and

$$\begin{aligned} &x\frac{(q)_\infty^2}{(-q)_\infty}[C(x;q)]^3 J(-x;q) \\ &\quad= \Big(2(1-x)^2(1+x)\delta_q + x(1+x) + 2x(1-x)\delta_x + \frac{1}{2}(1+x)(1-x)^2\delta_x^2\Big)N(1,0,x;q). \end{aligned} \tag{4.6}$$

Proof. Define

$$S_1(x,\zeta;q) := \sum_{n\in\mathbb{Z}} \frac{(-1)^n\zeta^n q^{n^2+n}}{1-xq^n}.$$

By Lemma 4.1 in [27], we have

$$\begin{aligned} S_1\left(x\zeta^{-1},\zeta^{-2};q\right) + \zeta^2 S_1\left(x\zeta,\zeta^2;q\right) &-\zeta\frac{J\left(\zeta^2;q\right)J\left(-q;q\right)}{J\left(\zeta;q\right)J\left(-\zeta;q\right)}S_1\left(x,1;q\right) \\ &= \frac{J\left(\zeta;q\right)J\left(\zeta^2;q\right)J\left(-x;q\right)(q)_\infty^2}{J\left(-\zeta;q\right)J\left(x\zeta;q\right)J\left(x\zeta^{-1};q\right)J(x;q)}. \end{aligned} \tag{4.7}$$

Equation (4.7) was one of the key results used to prove identities for rank differences for overpartitions in [27]. Let $g(\zeta)$ denote the right side of (4.7). Note that $g(\zeta)$ has a double zero at 1 and that

$$g''(1) = \frac{2(q)_\infty^3}{(-q)_\infty^2}\left[C^*(x;q)\right]^3 J\left(-x;q\right).$$

We let $h(\zeta)$ be the sum of the first two terms on the left side of (4.7). We find that $h''(1)$ equals

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{n(n+1)}\left(\frac{8n^2+8n+2}{1-xq^n}+\frac{2x(3+4n)q^n}{(1-xq^n)^2}+\frac{4x^2q^{2n}}{(1-xq^n)^3}\right)=\left(2+8\delta_q+4\delta_x+2\delta_x^2\right)S_1\left(x,1;q\right).$$

Letting $j(\zeta)$ be the third term on the left side of (4.7), one can show that

$$\begin{aligned} j''(1)&=-4\left(-\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2}-3\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\right)S_1(x,1;q)\\ &=-4\left(-\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2}-3\Phi_1(q)\right)S_1\left(x,1;q\right), \end{aligned}$$

where

$$\Phi_1(q):=\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

One can check that

$$\delta_q(q)_\infty=-\Phi_1(q)(q)_\infty. \tag{4.8}$$

We next need the following identity, which follows from (2.5) and the fact that

$$\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{n^2+n}}{(1+q^n)}=\frac{1}{2}\frac{(q)_\infty}{(-q)_\infty}: \tag{4.9}$$

$$xS_1\left(x,1;q\right)=\frac{(q)_\infty}{2(-q)_\infty}\left(-1+(1+x)N^*\left(1,0,x;q\right)\right). \tag{4.10}$$

Applying δ_q to both sides of (4.10) and using (4.8), we get

$$x\delta_qS_1\left(x,1;q\right)=\left(-\Phi_1(q)-\sum_{k=1}^{\infty}\frac{kq^k}{1+q^k}\right)xS_1\left(x,1;q\right)+\frac{(q)_\infty}{2(-q)_\infty}\delta_q\left(1+x\right)N^*\left(1,0,x;q\right).$$

Similarly, we find that

$$x\delta_xS_1\left(x,1;q\right)=\frac{(q)_\infty}{2(-q)_\infty}\delta_x\left(1+x\right)N^*\left(1,0,x;q\right)-xS_1\left(x,1;q\right)$$

and

$$x\delta_x^2S_1\left(x,1;q\right)=xS_1\left(x,1;q\right)+\frac{(q)_\infty}{2(-q)_\infty}\left(\delta_x^2-2\delta_x\right)N^*\left(1,0,x;q\right)\left(1+x\right).$$

Combining the above now easily yields

$$\begin{aligned} x\frac{(q)_\infty^2}{(-q)_\infty}\left[C^*(x;q)\right]^3J(-x;q)&=\left(2\delta_q+\frac{1}{2}\delta_x^2\right)N^*\left(1,0,x;q\right)\left(1+x\right)\\ &+2x\frac{(-q)_\infty}{(q)_\infty}S_1\left(x,1;q\right)\left[\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-2\sum_{n=1}^{\infty}\frac{nq^n}{1+q^n}+\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2}\right]. \end{aligned} \tag{4.11}$$

Note that the terms in brackets in (4.11) sum to 0. This may be seen, for example, by writing these sums in terms of divisor functions.

Next an application of the product rule yields (4.5). From (4.4) we find that

$$\delta_x N^*(1,0,x;q) = \frac{\delta_x N(1,0,x;q) + xN^*(1,0,x;q)}{1-x}, \tag{4.12}$$

$$\delta_x^2 N^*(1,0,x;q) = \frac{\delta_x^2 N(1,0,x;q) + 2x\delta_x N^*(1,0,x;q) + xN^*(1,0,x;q)}{1-x}, \tag{4.13}$$

$$\delta_q N^*(1,0,x;q) = \frac{\delta_q N(1,0,x;q)}{1-x}. \tag{4.14}$$

This easily yields (4.6). □

We may now prove the case $v > 1$ inductively using Theorem 4.2. Actually we shall argue using the ordinary rank moment generating functions $M_{2v}(1,0;q)$, but as we have already mentioned the functions $\mathcal{N}_{2v}(1,0;q)$ may be written in terms of the $M_{2v}(1,0;q)$. First, since $M_2(1,0;q) = 2\mathcal{N}_2(1,0;q)$, Theorem 4.1 implies that the former is a quasimock theta function. Next, appealing to (2.1), we have that $M_v(1,0;q)$ is $[\delta_x^v N(1,0,x;q)]_{x=1}$. Now apply δ_x to (4.6) $2v$ times and then set $x=1$. We first consider the left-hand side. First, from [5, Section 4] we have that $[\delta_x^r C(x;q)]_{x=1}$ is a quasimodular form. Moreover we have

$$\begin{aligned}\delta_x\left(J(-x;q)\right) &= \left(\frac{x}{1+x} + x\sum_{m=1}^{\infty}\frac{q^m}{1+xq^m} - x^{-1}\sum_{m=1}^{\infty}\frac{q^m}{1+x^{-1}q^m}\right)J(-x;q)\\ &= \left(\frac{x}{x+1} - \sum_{m=1}^{\infty}\frac{(-1)^m q^m}{1-q^m}\left(x^m - x^{-m}\right)\right)J(-x;q).\end{aligned}$$

Thus $\left[\delta_x^{2v} J(-x;q)\right]_{x=1}$ is a linear combination of terms of the form

$$\left(c_r - (1-(-1)^r)\sum_{m=1}^{\infty}\frac{(-1)^m\, m^r\, q^m}{1-q^m}\right)^l J(-1;q)$$

for integers r and l with some constant c_r. The theory of Eisenstein series on congruence subgroups (see Section III.3 in [22]) yields that up to a constant term the above sum is a modular form for odd $r \geq 3$. For $r=1$, observe that

$$\delta_q \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{-\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty}\sum_{n\geq 1}\frac{(-1)^n n\, q^n}{1-q^n},$$

and hence we have a quasimodular form in this case. So, applying δ_x $2v$ times and then setting $x=1$ yields a quasimodular form on the left hand side. Now the claim follows by induction, since the term $\delta_x^{2v} N(1,0,x;q)$ occurs with multiplicity $(2v-2)(2v-1) \neq 0$ for $v > 1$ and the other terms on the right are derivatives of quasimock theta functions.

5. The case $(d,e,q) = (1,1/q,q^2)$

We begin again with the case $v = 1$.

Theorem 5.1. *The function*

$$\mathcal{N}_2\left(1,1/q;q^2\right) + \frac{(-q)_\infty}{(q)_\infty}\left(\frac{1}{12} + \frac{1}{24}E_2\left(2z\right)\right) - \frac{\overline{NH}(z)}{2}$$

is a weak Maass form of weight $3/2$ *on* $\Gamma_0(16)$*.*

Proof. We begin with the following identity, which is obvious:

$$\sum_{n\geq 1}\frac{(-1)^n q^{n^2+n}\left(1+q^{2n}\right)}{\left(1-q^{2n}\right)^2} + 2\sum_{n\geq 1}\frac{(-1)^n q^{n^2+2n}}{\left(1-q^{2n}\right)^2} = \sum_{n\geq 1}\frac{(-1)^n q^{n^2+n}}{\left(1-q^{n}\right)^2}.$$

In terms of symmetrized rank moments, this says that

$$\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^n q^{n^2+n}\left(1+q^{2n}\right)}{\left(1-q^{2n}\right)^2} - \mathcal{N}_2\left(1,1/q;q^2\right) = -\frac{\mathcal{N}_2\left(1,0;q\right)}{2}. \tag{5.1}$$

Now, taking $(a,b,c,d,q) = (1,b,1/b,\infty,q^2)$ in the ${}_6\phi_5$ summation [20],

$$\sum_{n\geq 0}\frac{(1-aq^{2n})(a,b,c,d)_n\,(aq/bcd)^n}{(1-a)\,(q,aq/b,aq/c,aq/d)_n} = \frac{(aq,aq/bc,aq/bd,aq/cd)_\infty}{(aq/b,aq/c,aq/d,aq/bcd)_\infty},$$

we obtain

$$1 + \sum_{n\geq 1}\frac{\left(1+q^{2n}\right)\left(b,1/b;q^2\right)_n(-1)^n q^{n^2+n}}{\left(bq^2,q^2/b;q^2\right)_n} = \frac{\left(q^2;q^2\right)_\infty^2}{\left(bq^2,q^2/b;q^2\right)_\infty}.$$

Taking $\partial^2/\partial b^2$, setting $b = 1$, and substituting into (5.1) gives

$$\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{nq^{2n}}{\left(1-q^{2n}\right)} + \mathcal{N}_2\left(1,1/q;q^2\right) = \frac{\mathcal{N}_2\left(1,0;q\right)}{2}.$$

Applying Theorem 4.1 completes the proof. □

Next we deduce a PDE for $N(1,1/q,x;q^2)$.

Theorem 5.2. *We have*

$$2x\left(q^2;q^2\right)_\infty^2\left[C^*(x;q^2)\right]^3 J\left(-x;q\right) = \Big((1+x)\delta_q + x + 2x\delta_x + (1+x)\delta_x^2\Big)N^*\left(1,1/q,x;q^2\right), \tag{5.2}$$

and

$$\begin{aligned} &2x\left(q^2;q^2\right)_\infty^2\left[C(x;q^2)\right]^3 J(-x;q) \\ &= \Big((1+x)(1-x)^2\delta_q + 2x(1+x) + 4x(1-x)\delta_x + (1+x)(1-x)^2\delta_x^2\Big)N\left(1,1/q,x;q^2\right). \end{aligned} \tag{5.3}$$

Proof. The proof is similar to that of Theorem 4.2. We define

$$S_2(x,\zeta;q) := \sum_{n\in\mathbb{Z}} \frac{(-1)^n\zeta^n q^{n^2+2n}}{1-xq^{2n}}.$$

Taking $(a_1, a_2, b_1, b_2, b_3, q) = (-x, -xq, x\zeta, x/\zeta, x, q^2)$ in the case $(r,s) = (2,3)$ of Theorem 2.1 of [16], we obtain

$$\begin{aligned} S_2\left(x\zeta^{-1},\zeta^{-1};q\right) + \zeta^2 S_2(x\zeta,\zeta;q) + 2\frac{J\left(\zeta^2;q^2\right)(-q;q)_\infty^2}{J(-\zeta;q)\,J\left(\zeta^{-1};q^2\right)} S_2(x,1;q) \\ = \frac{J(-x;q)J\left(\zeta^2;q^2\right)J\left(\zeta;q^2\right)\left(q^2;q^2\right)_\infty^2}{J\left(x\zeta;q^2\right)J\left(x\zeta^{-1};q^2\right)J(-\zeta;q)\,J\left(x;q^2\right)}. \end{aligned} \tag{5.4}$$

Let $g(\zeta)$ denote the right side of (5.4). Note that $g(\zeta)$ has a double zero at 1 and that

$$g''(1) = \frac{2\left(q^2;q^2\right)_\infty^3}{(-q;q)_\infty^2}\left[C^*(x;q^2)\right]^3 J(-x;q).$$

We let $h(\zeta)$ be the sum of the first two terms on the left side of (5.4). We find that $h''(1)$ equals

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{n(n+2)}\left(\frac{2n^2+4n+2}{1-xq^{2n}} + \frac{2x(3+2n)q^{2n}}{(1-xq^{2n})^2} + \frac{4x^2q^{4n}}{(1-xq^{2n})^3}\right) = \left(2+2\delta_q+4\delta_x+2\delta_x^2\right) S_2(x,1;q).$$

Letting $j(\zeta)$ be the third term on the left side of (5.4), one can show that

$$j''(1) = 4\left(\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2} + 3\Phi_1\left(q^2\right)\right) S_2(x,1;q).$$

We next need the following identity, which is again a consequence of (2.5) and (4.9):

$$xS_2(x,1;q) = \frac{(q)_\infty}{2(-q)_\infty}\left(-1+(1+x)N^*\left(1,1/q,x;q^2\right)\right). \tag{5.5}$$

Applying δ_q to both sides of (5.5) and using (4.8), we get

$$x\delta_q S_2(x,1;q) = \left(-\Phi_1(q) - \sum_{k=1}^{\infty}\frac{kq^k}{1+q^k}\right) xS_2(x,1;q) + \frac{(q)_\infty}{2(-q)_\infty}\delta_q(1+x)N^*\left(1,1/q,x;q^2\right).$$

Similarly, we find that

$$\begin{aligned} x\delta_x S_2(x,1;q) &= \frac{(q)_\infty}{2(-q)_\infty}\delta_x(1+x)N^*\left(1,1/q,x;q^2\right) - xS_2(x,1;q), \\ x\delta_x^2 S_2(x,1;q) &= xS_2(x,1;q) + \frac{(q)_\infty}{2(-q)_\infty}\left(\delta_x^2 - 2\delta_x\right)N^*\left(1,1/q,x;q^2\right)(1+x). \end{aligned}$$

Combining the above and simplifying now yields

$$2x\left(q^2;q^2\right)_\infty^2\left[C^*(x;q^2)\right]^3 J(-x;q) = \Big((1+x)\delta_q + x + 2x\delta_x + (1+x)\delta_x^2\Big)N^*\left(1,1/q,x;q^2\right)$$
$$+ 2x\frac{(-q)_\infty}{(q)_\infty}S_2\left(x,1;q\right)\left[-\Phi_1(q) - \sum_{n=1}^{\infty}\frac{nq^n}{1+q^n} + 2\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2} + 6\Phi_1\left(q^2\right)\right]. \tag{5.6}$$

Note that the terms in brackets in (5.6) sum to 0 and thus we have (5.2). Using the analogues of equations (4.12), (4.13), and (4.14), we may obtain (5.3). □

Now the general case $v > 1$ follows just as for $\mathcal{N}_{2v}(1,0;q)$ in the previous section. We omit the details.

6. The case $(d,e,q) = (0,1/q,q^2)$

Let us begin again with the case $v = 1$. Before stating it, we need a lemma. Define the function $g(z)$ by

$$g(z) := \frac{\left(q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\sum_{n\in\mathbb{Z}}\frac{q^{2n^2+3n+1}}{\left(1-q^{2n+1}\right)^2}$$

and the integral $NH_2(z)$ by

$$NH_2(z) := \frac{1}{4\sqrt{2}i\pi}\int_{-\bar{z}}^{i\infty}\frac{\eta^2(16\tau)}{\eta(8\tau)\left(-i\left(\tau+z\right)\right)^{\frac{3}{2}}}d\tau.$$

Moreover let

$$\overline{\mathcal{M}}_2(z) := q^{-1}g\left(8z\right) - NH_2(z).$$

Lemma 6.1. *The function $\overline{\mathcal{M}}_2(z)$ is a weak Maass form of weight $\frac{3}{2}$ on $\Gamma_0(16)$.*

Proof. This will follow from the work in [10, Section 4]. First, recall that the function $\overline{\mathcal{M}}(z)$ defined in (4.3) is a weak Maass form of weight $\frac{3}{2}$ on $\Gamma_0(16)$. As in the case of classical modular forms one can show that the function

$$\overline{\mathcal{N}}(z) := \frac{1}{2\sqrt{2}}\left(-i16z\right)^{-\frac{3}{2}}\overline{\mathcal{M}}\left(-\frac{1}{16z}\right)$$

is also a weak Maass form of weight $\frac{3}{2}$ on $\Gamma_0(16)$. It turns out that $\overline{\mathcal{N}}(z) = \overline{\mathcal{M}}_2(z)$. To see this, observe that the transformation law for $\overline{\mathcal{M}}(z)$ (see Corollary 4.4 and Lemma 4.5 of [10]) implies that

$$\mathcal{M}\left(-\frac{1}{z}\right) = 2\sqrt{2}\left(-iz\right)^{\frac{3}{2}}\mathcal{U}\left(\frac{z}{2}\right) - \frac{2}{\pi i}(-iz)^{\frac{3}{2}}\int_{-\bar{z}}^{i\infty}\frac{\eta^2(\tau)}{\eta\left(\frac{\tau}{2}\right)\left(-i\left(\tau+z\right)\right)^{\frac{3}{2}}}d\tau,$$

where

$$\mathcal{U}(z) := \frac{\eta(z)}{\eta^2(2z)}\sum_{\substack{n\in\mathbb{Z}\\ n\text{ odd}}}\frac{q^{\frac{n}{2}(n+1)}}{\left(1-q^n\right)^2}.$$

To finish the proof we make the change of variables $z \mapsto 16z$ and observe that $\mathcal{U}(z) = q^{-\frac{1}{8}}g(z)$. □

Remark 6.2. *We note that there is a typo in the definition of the function $\mathcal{U}(z)$ just above Corollary 4.2 in* [10].

Theorem 6.3. *The function*

$$q^{-1}\mathcal{N}_2\left(0,-1/q^8;q^{16}\right)-\frac{\eta(8z)}{24\eta^2(16z)}\left(1-E_2\left(8z\right)\right)+NH_2(z)$$

is a weak Maass form of weight $3/2$ *on* $\Gamma_0(16)$.

Proof. Here we use the identity

$$\sum_{n\in\mathbb{Z}}\frac{(-1)^{n+1}q^{2n^2-n}}{(1-xq^{2n})}+\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{2n^2+n}}{(1+xq^{2n+1})}=\frac{\left(-q,q^2;q^2\right)_\infty^2}{(1/x,xq^2,-xq,-q/x;q^2)_\infty},$$

which is the case $q=q^2$ and $y=-1/xq$ of [16, eq. (4.3), corrected]. Differentiating twice with respect to x, setting $x=1$, and multiplying by $(q;q^2)_\infty/(q^2;q^2)_\infty$ gives

$$\frac{-\left(q;q^2\right)_\infty}{(q^2;q^2)_\infty}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{q^{2n^2+n}}{(1-q^{2n})^2}+\frac{\left(q;q^2\right)_\infty}{(q^2;q^2)_\infty}\sum_{n\in\mathbb{Z}}\frac{q^{2n^2+3n+1}}{(1-q^{2n+1})^2}=\frac{\left(q;q^2\right)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq1}\frac{nq^n}{(1-q^n)}.$$

Notice that the first term on the left hand side above is $\mathcal{N}_2(0,-1/q;q^2)$ and the second is $g(z)$. The theorem now follows from Lemma 6.1. □

Remark 6.4. *Notice that by replacing* z *by* $z+\frac{1}{16}$ *(i.e.* q *by* $e^{\pi i/8}q$*), we have that* $q^{-1}\mathcal{N}_2(0,1/q^8;q^{16})$ *is also a quasimock theta function, although the corresponding weak Maass form is on a much smaller group (* $\Gamma_1(256)$*, for example).*

We now prove a PDE for $N(0,1/q,x;q^2)$.

Theorem 6.5. *We have*

$$2x\frac{(q^2;q^2)_\infty^2}{(-q;q^2)_\infty}\left[C^*(x;q^2)\right]^3J\left(-xq;q^2\right)=\left(2\delta_q+\delta_x+\delta_x^2\right)N^*\left(0,1/q,x;q^2\right),\tag{6.1}$$

and

$$\begin{aligned}&2x\frac{\left(q^2;q^2\right)_\infty^2}{(-q;q^2)_\infty}\left[C\left(x;q^2\right)\right]^3J\left(-xq;q^2\right)\\&\quad=\left(2(1-x)^2\delta_q+(1+x)(1-x)\delta_x+2x+(1-x)^2\delta_x^2\right)N\left(0,1/q,x;q^2\right).\end{aligned}\tag{6.2}$$

Proof. To prove (6.1), we first define

$$S_3\left(x,\zeta;q\right):=\sum_{n\in\mathbb{Z}}\frac{(-1)^n\zeta^nq^{2n^2+3n}}{1-xq^{2n}}.$$

By Lemma 4.1 in [28], we have

$$\begin{aligned}&S_3\left(x\zeta^{-1},\zeta^{-2};q\right)+\zeta^3S_3\left(x\zeta,\zeta^2;q\right)-\zeta\frac{J\left(\zeta^2;q^2\right)\left(-q;q^2\right)_\infty^2}{J\left(\zeta;q^2\right)J\left(-q\zeta;q^2\right)}S_3\left(x,1;q\right)\\&\quad=\frac{J\left(-xq;q^2\right)J\left(\zeta^2;q^2\right)J\left(\zeta;q^2\right)\left(q^2;q^2\right)_\infty^2}{J\left(x\zeta^{-1};q^2\right)J\left(x\zeta;q^2\right)J\left(-q\zeta;q^2\right)J\left(x;q^2\right)}.\end{aligned}\tag{6.3}$$

Equation (6.3) was one of the key results used to prove identities for M_2-rank differences for partitions without repeated odd parts in [28]. Let $g(\zeta)$ denote the right side of (6.3). Note that $g(\zeta)$ has a double zero at 1 and that

$$g''(1) = \frac{4\left(q^2;q^2\right)_\infty^3}{\left(-q;q^2\right)_\infty^2}\left[C^*\left(x;q^2\right)\right]^3 J\left(-xq;q^2\right).$$

We let $h(\zeta)$ be the sum of the first two terms on the left side of (6.3). We find that $h''(1)$ equals

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{n(2n+3)}\left(\frac{8n^2+12n+6}{1-xq^{2n}}+\frac{8x(1+n)q^{2n}}{(1-xq^{2n})^2}+\frac{4x^2q^{4n}}{(1-xq^{2n})^3}\right)=\left(6+4\delta_q+6\delta_x+2\delta_x^2\right)S_3\left(x,1;q\right).$$

Letting $j(\zeta)$ be the third term on the left side of (6.3), one can show that

$$j''(1) = -2\left(1-2\sum_{n=1}^\infty\frac{q^{2n-1}}{(1+q^{2n-1})^2}-6\Phi_1(q^2)\right)S_3\left(x,1;q\right).$$

One can check that

$$\delta_q\left(q^2;q^2\right)_\infty = -2\Phi_1\left(q^2\right)\left(q^2;q^2\right)_\infty \tag{6.4}$$

and

$$\delta_q\left(-q;q^2\right)_\infty = \left(-q;q^2\right)_\infty\sum_{k=0}^\infty\frac{(2k+1)q^{2k+1}}{(1+q^{2k+1})}. \tag{6.5}$$

From (2.5) we have the following identity:

$$xS_3\left(x,1;q\right) = \frac{\left(q^2;q^2\right)_\infty}{\left(-q;q^2\right)_\infty}\Big(-1+N^*\left(0,1/q,x;q^2\right)\Big). \tag{6.6}$$

Applying δ_q to both sides of (6.6) and using (6.4) and (6.5), we get

$$x\delta_q S_3\left(x,1;q\right) = \left(-2\Phi_1\left(q^2\right)-\sum_{k=0}^\infty\frac{(2k+1)q^{2k+1}}{1+q^{2k+1}}\right)xS_3\left(x,1;q\right)+\frac{\left(q^2;q^2\right)_\infty}{\left(-q;q^2\right)_\infty}\delta_q N^*\left(0,1/q,x;q^2\right).$$

Similarly, we find that

$$\begin{aligned} x\delta_x S_3(x,1;q) &= \frac{\left(q^2;q^2\right)_\infty}{\left(-q;q^2\right)_\infty}\delta_x N^*\left(0,1/q,x;q^2\right) - xS_3\left(x,1;q\right), \\ x\delta_x^2 S_3\left(x,1;q\right) &= xS_3\left(x,1;q\right)+\frac{\left(q^2;q^2\right)_\infty}{\left(-q;q^2\right)_\infty}\left(\delta_x^2-2\delta_x\right)N^*\left(0,1/q,x;q^2\right). \end{aligned}$$

Combining the above now yields

$$
\begin{aligned}
2x\frac{\left(q^2;q^2\right)_\infty^2}{\left(-q;q^2\right)_\infty}\left[C^*\left(x;q^2\right)\right]^3 J\left(-xq;q^2\right) &= \left(2\delta_q+\delta_x+\delta_x^2\right)N^*\left(0,1/q,x;q^2\right) \\
&+4x\frac{\left(-q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}S_3\left(x,1;q\right)\left[\Phi_1\left(q^2\right)-\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1+q^{2n+1}}+\sum_{n=1}^{\infty}\frac{q^{2n-1}}{\left(1+q^{2n-1}\right)^2}\right].
\end{aligned}
\tag{6.7}
$$

Observe that the terms in brackets in (6.7) sum to 0 and thus (6.1) follows. Using the analogues of equations (4.12), (4.13) and (4.14), we obtain (6.2). □

The case $v > 1$ follows by induction as before.

7. The smallest parts function

We now define a *smallest parts function* $spt(r, s, n)$ in the context of overpartition pairs (λ, μ) of n. It is the total number of appearances of the smallest parts in all of the overpartition pairs (λ, μ) of n, where r is the number of overlined parts in λ plus the number of non-overlined parts in μ and s is the number of parts in μ, such that the smallest part in (λ, μ) only occurs non-overlined and only in λ. Thus overpartition pairs like $((\overline{6}, 6, 5, 4, 4, 4, \overline{3}, \overline{1}), (7, 7, \overline{5}, 2, 2, 2))$ contribute nothing to $spt(r, s, n)$. We have the following generating function:

Theorem 7.1. *Recalling the definition of $Spt(d, e; q)$ from the introduction, we have*

$$\sum_{r,s,n\geq 0} spt(r,s,n)d^r e^s q^n = Spt(d,e;q).$$

Proof. We proceed as in [4, Proof of Theorem 4]. Briefly, we have that

$$
\begin{aligned}
&\sum_{r,s,n\geq 0} spt(r,s,n)d^r e^s q^n = \frac{(-dq,-eq)_\infty}{(deq,q)_\infty}\sum_{n\geq 1}\frac{(q,deq)_n\, q^n}{(1-q^n)^2\,(-dq,-eq)_n} \\
&= \frac{-\,(-dq,-eq)_\infty}{2\,(deq,q)_\infty}\left[\frac{\partial^2}{\partial x^2}\sum_{n\geq 0}\frac{(deq)_n\,(x,1/x)_n\, q^n}{(q,-dq,-eq)_n}\right]_{x=1} \\
&= \frac{-\,(-dq,-eq)_\infty}{2\,(deq,q)_\infty}\left[\frac{\partial^2}{\partial x^2}\frac{(xq,q/x)_\infty}{(q)_\infty^2}\left(1+\sum_{n\geq 1}\frac{(-de)^n q^{n(n+3)/2}(1+q^n)\,(-1/d,-1/e,x,1/x)_n}{(q/x,xq,-dq-eq)_n}\right)\right]_{x=1} \\
&= Spt(d,e;q),
\end{aligned}
$$

where the penultimate line follows from the Watson-Whipple transformation. □

As noted in the introduction, Corollary 1.3 follows immediately from Theorem 1.2, or more explicitly, from Theorems 4.1, 5.1, and 6.3. Regarding Remark 1.4, it is not hard to see that when $d = e = 1$, the symmetry in the overpartition pairs means that the smallest parts function for $n > 0$ is just counting $\overline{pp}(n)/4$, where $\overline{pp}(n)$ is the number of overpartitions of n. From the case $x = 1$ of (3.1), the generating function for $\overline{pp}(n)$ is $(-q)_\infty^2/(q)_\infty^2$.

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