# Asymmetric Generalizations of Schur's theorem

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*For Krishna Alladi on his 60th birthday*

**Abstract** We extend a theorem of Alladi and Gordon asymmetrically to overpartitions. As special cases, we find asymmetric generalizations of Schur's theorem and partition identities closely related to Capparelli's identity and the Alladi-Andrews dual of Göllnitz' theorem.

**Key words:** partitions, overpartitions, Schur's theorem, Capparelli's theorem, weighted words

## 1 Introduction and Statement of Results

### *1.1 Introduction*

Recall that a partition $\lambda$ of $n$ is a non-increasing sequence of positive integers $(\lambda_1, \lambda_2, \dots, \lambda_k)$ whose sum is $n$. Like many papers on partition identities, this one begins with an influential theorem of Schur [15].

**Theorem 1 (Schur).** *Let $S(n)$ denote the number of partitions of n such that*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 6, & \text{if } \lambda_i \equiv \lambda_{i+1} \equiv 0 \pmod 3, \\ 3, & \text{otherwise}. \end{cases} \tag{1.1}$$

*Then $S(n)$ is equal to the number of partitions of n into parts congruent to* 1 *or* 5 *modulo* 6.

In terms of generating functions, Schur's theorem may be written

$$\sum_{n \geq 0} S(n) q^n = \frac{1}{(q;q^6)_\infty (q^5;q^6)_\infty}, \tag{1.2}$$

where we use the usual $q$-hypergeometric notation,

$$(a)_n = (a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \tag{1.3}$$

valid for $n \in \mathbb{N} \cup \infty$. Given the simple fact that

$$\frac{1}{(q;q^6)_\infty (q^5;q^6)_\infty} = (-q;q^3)_\infty (-q^2;q^3)_\infty, \tag{1.4}$$

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the $S(n)$ in Schur's theorem is also equal to the number of partitions of $n$ into distinct parts not divisible by 3. From this perspective, Alladi and Gordon [7, 8] gave a generalization and refinement of Schur's theorem, which we now describe.

Consider the positive integers in the three colors $a$, $b$, and $ab$, with the order

$$ab < a < b, \tag{1.5}$$

so that the integers are ordered

$$1_{ab} < 1_a < 1_b < 2_{ab} < 2_a < 2_b < \cdots . \tag{1.6}$$

Let $S(u,v,n)$ denote the number of three-colored partitions of $n$ with no part $1_{ab}$, $u$ parts colored $a$ or $ab$, $v$ parts colored $b$ or $ab$, and satisfying the difference conditions in the matrix

$$A = \begin{array}{c} \\ a \\ b \\ ab \end{array} \begin{array}{c} \begin{array}{ccc} a & b & ab \end{array} \\ \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \end{array}. \tag{1.7}$$

By this we mean that the entry $(x,y)$ gives the minimal difference between $\lambda_i$ of color $x$ and $\lambda_{i+1}$ of color $y$. Alladi and Gordon [7, 8] established the following elegant generating function for $S(u,v,n)$.

**Theorem 2 (Alladi-Gordon).** *We have*

$$\sum_{u,v,n\geq 0} S(u,v,n)a^u b^v q^n = (-aq;q)_\infty(-bq;q)_\infty. \tag{1.8}$$

Setting $q = q^3$, $a = aq^{-2}$ and $b = bq^{-1}$, the three-colored positive integers become the ordinary positive integers, with parts congruent to 0, 1, or 2 modulo 3 labelled $ab$, $a$, or $b$, respectively. The matrix of difference conditions in (1.7) becomes

$$\begin{array}{c} \\ a \\ b \\ ab \end{array} \begin{array}{c} \begin{array}{ccc} a & b & ab \end{array} \\ \begin{pmatrix} 3 & 5 & 4 \\ 4 & 3 & 5 \\ 5 & 4 & 6 \end{pmatrix} \end{array}, \tag{1.9}$$

which is equivalent to (1.1), and we recover Schur's theorem. In fact, we have a refinement of Schur's theorem, thanks to the extra parameters $a$ and $b$.

Alladi and Gordon's treatment of Schur's theorem marked the beginning of the so-called *method of weighted words*, which would subsequently be used to find refinements and generalizations of partition identities such as those of Göllnitz [5], Capparelli [6], and Siladić [12], as well as to discover a number of new identities. For more on this, see [1, 2, 4].

It turns out that Theorem 2 is a special case of an identity for overpartitions. Recall that an overpartition is a partition in which the first occurrence of a given integer may be overlined. We consider overpartitions with the same three colors and the same ordering as in (1.5) and (1.6), allowing the first occurrence of a given colored integer to be overlined. We append the label $d$ to the color of a non-overlined part. Let $\overline{S}(u,v,m,n)$ denote the number of overpartitions of $n$ having $m$ non-overlined parts, no part $1_{abd}$ or $\overline{1}_{ab}$, $u$ parts having $a$ in their color, $v$ parts having $b$ in their color, and satisfying the difference conditions in the matrix

$$\overline{A} = \begin{array}{c} \\ a \\ b \\ ab \\ ad \\ bd \\ abd \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & ab & ad & bd & abd \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 1 & 1 \end{pmatrix} \end{array}. \tag{1.10}$$

These overpartitions are also generated by a simple infinite product.

**Theorem 3 (See [16]).** *We have*

$$\sum_{u,v,m,n\geq 0} \overline{S}(u,v,m,n)a^u b^v d^m q^n = \frac{(-aq;q)_\infty(-bq;q)_\infty}{(adq;q)_\infty(bdq;q)_\infty}. \tag{1.11}$$

When $d=0$ we recover Theorem 2. With the same substitutions as before, $q=q^3$, $a=aq^{-2}$ and $b=bq^{-1}$, the matrix $\overline{A}$ becomes

$$\begin{array}{c} \\ a \\ b \\ ab \\ ad \\ bd \\ abd \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & ab & ad & bd & abd \end{array} \\ \begin{pmatrix} 3 & 5 & 4 & 0 & 2 & 1 \\ 4 & 3 & 5 & 1 & 0 & 2 \\ 5 & 4 & 6 & 2 & 1 & 3 \\ 3 & 5 & 4 & 0 & 2 & 1 \\ 4 & 3 & 5 & 1 & 0 & 2 \\ 5 & 4 & 6 & 2 & 1 & 3 \end{pmatrix} \end{array}, \tag{1.12}$$

and we have a result known as *Schur's theorem for overpartitions*.

**Theorem 4 (See [16]).** *Let $\overline{S}(m,n)$ denote the number of overpartitions of $n$ with $m$ non-overlined parts, such that*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 3, & \text{if } \lambda_{i+1} \text{ is overlined or if } \lambda_i \equiv \lambda_{i+1} \equiv 0 \pmod 3, \\ 6, & \text{if } \lambda_{i+1} \text{ is overlined and } \lambda_i \equiv \lambda_{i+1} \equiv 0 \pmod 3. \end{cases} \tag{1.13}$$

*Then $\overline{S}(m,n)$ is equal to the number of overpartitions of $n$ into parts not divisible by 3, $m$ of which are non-overlined.*

Note that when there are no non-overlined parts (i.e., $m=0$) we recover Schur's theorem. See [11] for a proof of Theorem 4 using $q$-difference equations and [14] for a bijective proof.

## 1.2 Statement of Results

In this paper we prove two asymmetric extensions of Theorem 2 to overpartitions. The word *asymmetric* refers to the fact that one of the terms in the denominator of (1.11) is missing from each of (1.14) and (1.15) below.

**Theorem 5.** *The following are true.*

*(i) Let $\overline{S}_1(u,v,m,n)$ denote the number of overpartitions counted by $\overline{S}(u,v,m,n)$ where, in addition, the $s$ smallest parts must be overlined, where $s$ is the number of parts of color $b$ or $bd$. Then*

$$\sum_{u,v,m,n\geq 0} \overline{S}_1(u,v,m,n)a^u b^v d^m q^n = \frac{(-aq;q)_\infty(-bq;q)_\infty}{(adq;q)_\infty}, \tag{1.14}$$

*(ii) Let $\overline{S}_2(u,v,m,n)$ denote the number of overpartitions counted by $\overline{S}(u,v,m,n)$ where, in addition, the $r$ smallest parts must be overlined, where $r$ is the number of parts of color $a$ or $ad$. Then*

$$\sum_{u,v,m,n\geq 0} \overline{S}_2(u,v,m,n)a^u b^v d^m q^n = \frac{(-aq;q)_\infty(-bq;q)_\infty}{(bdq;q)_\infty}. \tag{1.15}$$

Note that if $m=0$ in either (1.14) or (1.15), we recover the Alladi-Gordon result in Theorem 2. Also note that although we recover the overpartitions in Theorem 3 if either of the extra conditions is omitted, Theorem 5 is not a special case of Theorem 3.

With the usual substitutions $q=q^3$, $a=aq^{-2}$, and $b=bq^{-1}$, we obtain a pair of results which may be compared with Schur's theorem and Schur's theorem for overpartitions.

**Corollary 1.** *For $j = 1$ or 2, let $\overline{S}_j(m,n)$ denote the number of overpartitions counted by $\overline{S}(m,n)$ in Theorem 4 with the extra condition that the smallest s parts are overlined, where s is the number of parts congruent to $3-j$ modulo 3. Then $\overline{S}_j(m,n)$ is equal to the number of overpartitions of n into overlined parts not divisible by 3 and m non-overlined parts congruent to j modulo 3.*

We highlight two other special cases of Theorem 5, where the overpartitions become ordinary partitions.

**Corollary 2.** *Let $C(n)$ denote the number of partitions of n satisfying the difference conditions*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 5, & \text{if } \lambda_{i+1} \text{ is even or if } \lambda_{i+1} \equiv 5 \pmod 6 \text{ and } \lambda_i \equiv 0,5 \pmod 6, \\ 11, & \text{if } \lambda_{i+1} \equiv 0 \pmod 6 \text{ and } \lambda_i \equiv 0,5 \pmod 6, \end{cases} \tag{1.16}$$

*and, in addition, having the s smallest parts even, where s is the number of parts congruent to 1 or 2 modulo 6. Then $C(n)$ is equal to the number of partitions of n into distinct parts not congruent to $\pm 1$ modulo 6.*

**Corollary 3.** *Let $G(n)$ denote the number of partitions of n satisfying the difference conditions in (1.16), and, in addition, having the r smallest parts even, where r is the number of parts congruent to 3 or 4 modulo 6. Then $G(n)$ is equal to the number of partitions of n into parts congruent $1,2,$ or 4 modulo 6, where only parts congruent to 1 modulo 6 may repeat.*

Note that the partitions into distinct parts not congruent to $\pm 1$ modulo 6 in Corollary 2 are precisely those in Capparelli's partition identity [6], while the partitions into parts $1,2$, or 4 modulo 6 in Corollary 3 are nearly those in the dual Göllnitz theorem due to Alladi and Andrews [3]. For other partitions related to Capparelli's identity, see [10].

The remainder of the paper is organized as follows. In the next section we prove Theorem 5 using a $q$-series identity, reviewing the work on Schur's theorem and Schur's theorem for overpartitions along the way. The colored partitions are similar in all three cases, but while Schur's theorem uses a staircase and the overpartition version uses a generalized staircase, the asymmetric version uses what we call a *partial staircase*. In Section 3 we give a bijective proof of Theorem 5. In Section 4 we deduce Corollaries 2 and 3 from Theorem 5. In Section 5 Corollaries 2 - 3 are illustrated with examples. We close in Section 6 with some final remarks.

## 2 Weighted words and the proof of Theorem 5

### *2.1 Schur's theorem*

Recall that we have been considering the positive integers in the three colors $a$, $b$, and $ab$, with the order $ab < a < b$. Take one ordinary partition with parts colored $a$, another ordinary partition with parts colored $b$, and one partition into distinct parts $\geq 2$ colored $ab$. If we then order the three-colored integers accordingly, we obtain a three-colored partition with no $1_{ab}$ and the matrix of difference conditions,

$$A' = \begin{array}{c} \\ a \\ b \\ ab \end{array} \begin{array}{c} \begin{array}{ccc} a & b & ab \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array}. \tag{2.17}$$

Let $S'(u,v,n)$ denote the number of such three-colored partitions of $n$, where $u$ is the number of parts with $a$ in their color and $v$ is the number of parts with $b$ in their color. Then it is quite clear that

$$\sum_{u,v,n \geq 0} S'(u,v,n) a^u b^v q^n = \sum_{r,s,t \geq 0} \frac{a^r q^r}{(q)_r} \frac{b^s q^s}{(q)_s} \frac{(ab)^t q^t q^{\binom{t+1}{2}}}{(q)_t}, \tag{2.18}$$

the first two terms corresponding to the ordinary partitions colored $a$ and $b$ and the third term to the partition into distinct parts $\geq 2$ colored $ab$.

Next let us add a "staircase" to the three-colored partition. That is, we add 0 to the smallest part, 1 to the next smallest part, and so on. This augments each minimal difference by one, giving us a partition with no part $1_{ab}$ and the difference conditions in (1.7). The quantities $u$ and $v$ do not change, and so we have the partitions counted by our $S(u,v,n)$ defined in the introduction.

Now, to compute the generating function for $S(u,v,n)$, we observe that adding a staircase simply corresponds to multiplying the summand on the right-hand side of (2.18) by $q^{\binom{r+s+t}{2}}$, and we have the generating function

$$\sum S(u,v,n)a^u b^v q^n = \sum_{r,s,t\geq 0} \frac{a^r q^r}{(q)_r} \frac{b^s q^s}{(q)_s} \frac{(ab)^t q^t q^{\binom{t+1}{2}}}{(q)_t} q^{\binom{r+s+t}{2}}. \tag{2.19}$$

To simplify this sum (and some later ones), we recall several basic $q$-series facts (see [13]). First, we have

$$(a)_{n-k} = \frac{(a)_n}{(q^{1-n}/a)_k}(-q/a)^k q^{\binom{k}{2}-nk}, \tag{2.20}$$

so that

$$(q)_{n-k} = \frac{(q)_n}{(q^{-n})_k}(-1)^k q^{\binom{k}{2}-nk} \tag{2.21}$$

and

$$\frac{(aq^{-n})_n}{(bq^{-n})_n} = \frac{(q/a)_n}{(q/b)_n}(a/b)^n. \tag{2.22}$$

We also recall the $q$-Chu-Vandermonde summation,

$$\sum_{k=0}^{n} \frac{(a)_k (q^{-n})_k q^k}{(q)_k (c)_k} = \frac{(c/a)_n a^n}{(c)_n}, \tag{2.23}$$

and the $q$-binomial identity,

$$\sum_{n\geq 0} \frac{z^n (-a)_n}{(q)_n} = \frac{(-az)_\infty}{(z)_\infty}, \tag{2.24}$$

noting the special cases

$$\sum_{k=0}^{n} \frac{(q^{-n})_k (q^{-m})_k q^k}{(q)_k} = q^{-mn} \tag{2.25}$$

and

$$\sum_{n\geq 0} \frac{q^{\binom{n+1}{2}} z^n}{(q)_n} = (-zq)_\infty. \tag{2.26}$$

We now evaluate (2.19) as follows:

$$
\begin{aligned}
\sum S(u,v,n)a^u b^v q^n &= \sum_{r,s,t\geq 0} \frac{q^{\binom{r+s+t}{2}+r+s+t+\binom{t+1}{2}} a^{r+t} b^{s+t}}{(q)_r(q)_s(q)_t} \\
&= \sum_{\substack{r,s,t\geq 0 \\ t\leq \min\{r,s\}}} \frac{q^{\binom{r+s-t}{2}+r+s+\binom{t}{2}} a^r b^s}{(q)_{r-t}(q)_{s-t}(q)_t} \qquad ((r,s)=(r-t,s-t)) \\
&= \sum_{\substack{r,s,t\geq 0 \\ t\leq \min\{r,s\}}} \frac{q^{\binom{r+1}{2}+\binom{s+1}{2}+rs} a^r b^s (q^{-r})_t (q^{-s})_t q^t}{(q)_r(q)_s(q)_t} \qquad \text{(from (2.21))} \\
&= \sum_{r,s\geq 0} \frac{q^{\binom{r+1}{2}+\binom{s+1}{2}} a^r b^s}{(q)_r(q)_s} \qquad \text{(by (2.25))} \\
&= (-aq)_\infty(-bq)_\infty \qquad \text{(by (2.26))}.
\end{aligned}
$$

This is Theorem 2.

## 2.2 Schur's theorem for overpartitions

Now let us go back to the three-colored partitions counted by $S'(u,v,n)$. Instead of a adding a staircase to such a partition, we will add a *generalized staircase*. This corresponds the term $d^{r+s+t}(-1/d)_{r+s+t}$, as follows. For each part $k$ between 0 and $r+s+t-1$ in the partition into distinct parts generated by $(-1/d)_{r+s+t}$, we add 1 to each of the $k$ largest parts and then overline the $k+1$st part. Notice that the exponent of $d$ counts the number of non-overlined parts, and when $d=0$ we just obtain the staircase $q^{\binom{r+s+t}{2}}$.

Thus we obtain a three-colored overpartition $\lambda$ where the minimal difference between $\lambda_i$ and $\lambda_{i+1}$ is as in (1.7) if $\lambda_{i+1}$ is overlined, but as in (2.17) if $\lambda_{i+1}$ is non-overlined; that is, as in (1.10). There is no part $1_{ab}$ or $1_{abd}$, and $u$ and $v$ count the same quantities as before. With $m$ counting the number of non-overlined parts, then, we have the overpartitions counted by $\overline{S}(u,v,m,n)$. So,

$$
\sum \overline{S}(u,v,m,n)a^u b^v d^m q^n = \sum_{r,s,t\geq 0} \frac{a^r q^r}{(q)_r} \frac{b^s q^s}{(q)_s} \frac{(ab)^t q^t q^{\binom{t+1}{2}}}{(q)_t} (-1/d)_{r+s+t} d^{r+s+t}. \tag{2.27}
$$

We emphasize that the only difference with the generating function for $S(u,v,n)$ in (2.19) is that the generalized staircase $(-1/d)_{r+s+t}d^{r+s+t}$ replaces the staircase $q^{\binom{r+s+t}{2}}$.

We now evaluate (2.27) as follows:

$$\begin{aligned}
\sum \overline{S}(u,v,m,n)a^u b^v d^m q^n &= \sum_{r,s,t\geq 0}(-1/d)_{r+s+t}d^{r+s+t}\frac{a^r q^r}{(q)_r}\frac{b^s q^s}{(q)_s}\frac{(ab)^t q^t q^{\binom{t+1}{2}}}{(q)_t} \\
&= \sum_{\substack{r,s,t\geq 0\\ t\leq \min\{r,s\}}}\frac{(-1/d)_{r+s-t}d^{r+s-t}q^{r+s+\binom{t}{2}}a^r b^s}{(q)_{r-t}(q)_{s-t}(q)_t} \qquad ((r,s)=(r-t,s-t)) \\
&= \sum_{\substack{r,s,t\geq 0\\ t\leq \min\{r,s\}}}\frac{(-1/d)_{r+s}d^{r+s}a^r b^s q^{r+s}(q^{-r})_t(q^{-s})_t q^t}{(q)_r(q)_s(q)_t(-dq^{1-r-s})_t} \qquad \text{(by (2.21) and (2.20))} \\
&= \sum_{r,s\geq 0}\frac{(-1/d)_{r+s}d^{r+s}a^r b^s q^{r+s}}{(q)_r(q)_s}\frac{(-dq^{1-s})_s}{(-dq^{1-r-s})_s}q^{-rs} \qquad \text{(by (2.23))} \\
&= \sum_{r,s\geq 0}\frac{(-1/d)_{r+s}d^{r+s}a^r b^s q^{r+s}}{(q)_r(q)_s}\frac{(-1/d)_s}{(-q^r/d)_s} \qquad \text{(by (2.22))} \\
&= \sum_{r,s\geq 0}\frac{(-1/d)_{r+s}d^{r+s}a^r b^s q^{r+s}}{(q)_r(q)_s}\frac{(-1/d)_s(-1/d)_r}{(-1/d)_{r+s}} \\
&= \sum_{r,s\geq 0}\frac{(-1/d)_r(-1/d)_s d^{r+s}a^r b^s q^{r+s}}{(q)_r(q)_s} \\
&= \frac{(-aq)_\infty(-bq)_\infty}{(adq)_\infty(bdq)_\infty} \qquad \text{(by (2.24))}.
\end{aligned}$$

This is Theorem 3.

## *2.3 The asymmetric Schur's theorem for overpartitions*

Finally we turn to the asymmetric case. Instead of a staircase or generalized staircase, we use a *partial staircase*, which is a kind of generalized staircase which is an actual staircase at the top. If we have $r+s+t$ parts, we require that the $s$ largest steps in the staircase occur, namely $r+s+t-1, r+s+t-2, \ldots, r+t$. Then we allow a generalized staircase from $r+t-1$ down to 0. The result is the partial staircase corresponding to the term

$$q^{\binom{r+s+t}{2}-\binom{r+t}{2}}(-1/d)_{r+t}d^{r+t} = q^{\binom{s}{2}+rs+st}(-1/d)_{r+t}d^{r+t}. \tag{2.28}$$

Adding such a partial staircase to a three-colored partition counted by $S'(u,v,n)$ gives an overpartition counted by $\overline{S}_1(u,v,m,n)$, where as usual $m$ denotes the number of non-overlined parts.

In terms of generating functions, we have

$$\sum \overline{S}_1(u,v,m,n)r^u s^v d^m q^n = \sum_{r,s,t\geq 0}\frac{a^r q^r}{(q)_r}\frac{b^s q^s}{(q)_s}\frac{(ab)^t q^t q^{\binom{t+1}{2}}}{(q)_t}q^{\binom{s}{2}+rs+st}(-1/d)_{r+t}d^{r+t}, \tag{2.29}$$

which may compared with (2.27) and (2.19). This triple sum may be evaluated as follows:

$$
\begin{aligned}
&\sum \overline{S}_1(u,v,m,n)a^u b^v d^m q^n \\
&\quad = \sum_{r,s,t\geq 0} \frac{q^{\binom{s}{2}+rs+st}(-1/d)_{r+t}d^{r+t}q^{r+s+t+\binom{t+1}{2}}a^{r+t}b^{s+t}}{(q)_r(q)_s(q)_t} \\
&\quad = \sum_{\substack{r,s,t\geq 0 \\ t\leq \min\{r,s\}}} \frac{q^{\binom{s-t}{2}+(r-t)(s-t)+(s-t)t}(-1/d)_r d^r q^{r+s+\binom{t}{2}}a^r b^s}{(q)_{r-t}(q)_{s-t}(q)_t} \qquad ((r,s)=(r-t,s-t)) \\
&\quad = \sum_{\substack{r,s,t\geq 0 \\ t\leq \min\{r,s\}}} \frac{q^{\binom{s+1}{2}+rs+r+t}(-1/d)_r d^r a^r b^s (q^{-r})_t(q^{-s})_t}{(q)_r(q)_s(q)_t} \quad \text{(by (2.21))} \\
&\quad = \sum_{r,s\geq 0} \frac{q^{\binom{s+1}{2}+r}(-1/d)_r d^r a^r b^s}{(q)_r(q)_s} \quad \text{(by (2.25))} \\
&\quad = \frac{(-aq)_\infty(-bq)_\infty}{(adq)_\infty} \quad \text{(by (2.26) and (2.24))}.
\end{aligned}
$$

This is the first part of Theorem 5. Note that by symmetry we can exchange the roles of $r$ and $s$ in the partial staircase (2.28) and the same argument would give the product

$$\frac{(-aq)_\infty(-bq)_\infty}{(bdq)_\infty}, \tag{2.30}$$

corresponding to the overpartitions counted by $\overline{S}_2(u,v,m,n)$. This completes the proof. $\square$

## 3 A bijective proof

Here we give a bijective proof of Theorem 5. We give details only for the first part. We start with the product side, namely a partition $\lambda$ corresponding to $(-bq)_\infty$ and an overpartition $\mu$ corresponding to $(-aq)_\infty/(adq)_\infty$. To illustrate the steps in the bijection, we follow the example

$$\lambda = (23_b, 22_b, 19_b, 15_b, 14_b, 11_b, 7_b, 4_b, 3_b, 1_b)$$

and

$$\mu = (15_a, \overline{13}_a, 13_a, \overline{10}_a, \overline{9}_a, 8_a, 8_a, 8_a, \overline{5}_a, 5_a, 5_a, 4_a, 3_a, \overline{1}_a).$$

(We omit the label $d$ from the colors of the non-overlined parts.) Let $r$ be the number of parts in $\mu$. Then, for each part $x_b$ of $\lambda$ which is $\leq r$, we add 1 to the $x$ largest parts of $\mu$ and change the color of the $x$th part to $ab$. This gives us $\lambda'$ and $\mu'$. In our example, we have

$$\lambda' = (23_b, 22_b, 19_b, 15_b)$$

and

$$\mu' = (21_{ab}, \overline{18}_a, 18_{ab}, \overline{14}_{ab}, \overline{12}_a, 11_a, 11_{ab}, 10_a, \overline{7}_a, 7_a, 7_{ab}, 5_a, 4_a, \overline{2}_{ab}).$$

Next, we remove a generalized staircase from $\mu'$ and then remove $r$ from the smallest part of $\lambda'$, $r+1$ from the next smallest part, and so on. The result is $\lambda''$, $\mu''$, and the removed parts in $\nu$. In our example, we have

$$\lambda'' = (6_b, 6_b, 4_b, 1_b),$$

$$\mu'' = (16_{ab}, 14_a, 14_{ab}, 11_{ab}, 10_a, 9_a, 9_{ab}, 8_a, 6_a, 6_a, 6_{ab}, 4_a, 3_a, 2_{ab}),$$

and the partial staircase

$$\nu = (17, 16, 15, 14, 13, 8, 4, 3, 1).$$

Since there were 4 parts in $\lambda'$, the 4 largest parts of $\nu$ form a staircase. Now we recall the order $ab < a < b$ and put the parts of $\lambda''$ into $\mu''$ in the proper place. Continuing our example, we have a partition

$$\mu''' = (16_{ab}, 14_a, 14_{ab}, 11_{ab}, 10_a, 9_a, 9_{ab}, 8_a, 6_b, 6_b, 6_a, 6_a, 6_{ab}, 4_b, 4_a, 3_a, 2_{ab}, 1_b).$$

Finally, we add the partial staircase $\nu$ back on to $\mu'''$. In our case, we have

$$(25_{ab}, \overline{22}_a, 22_{ab}, \overline{18}_{ab}, \overline{16}_a, 15_a, 15_{ab}, 14_a, \overline{11}_b, 11_b, 11_a, 11_a, 11_{ab}, \overline{8}_b, \overline{7}_a, \overline{5}_a, \overline{3}_{ab}, \overline{1}_b).$$

Notice that because we are adding the partial staircase in the manner described in Section 2.2, and since the $s$ largest possible parts of $\nu$ occur (where $s$ is the number of $b$-parts), the $s$ smallest parts of the final overpartition will be overlined. (In our example, $s = 4$.) A little thought reveals that the difference conditions between parts match what is claimed in (1.10) and that the operation is reversible.

## 4 Proofs of Corollaries 2 and 3

We begin by treating Corollary 2. For this, we use (1.15) with substitutions $q = q^6$, $a = q^{-4}$, $b = q^{-2}$, and $d = q^{-1}$. The product side is then

$$\frac{(-q^2;q^6)_\infty(-q^4;q^6)_\infty}{(q^3;q^6)_\infty} = (-q^2;q^6)_\infty(-q^4;q^6)_\infty(-q^3;q^3)_\infty, \tag{4.31}$$

which is the generating function for partitions into distinct parts not congruent to $\pm 1$ modulo 6. On the other hand, in the colored partitions counted by $\overline{S}_2(u,v,m,n)$, a part $x$ of color $a, b, ab, ad, bd$, or $abd$ becomes the integer $6x-4, 6x-2, 6x-6, 6x-5, 6x-3$, or $6x-7$, respectively. (Recall that the label $d$ corresponds to a non-overlined part.) Since there was no part $1_{ab}$ or $1_{abd}$, this is the full set of postive integers. The matrix of difference conditions in (1.10) becomes

$$\begin{array}{c} \\ a \\ b \\ ab \\ ad \\ bd \\ abd \end{array}
\begin{array}{c} \begin{array}{cccccc} a & b & ab & ad & bd & abd \end{array} \\
\begin{pmatrix} 6 & 10 & 8 & 1 & 5 & 3 \\ 8 & 6 & 10 & 3 & 1 & 5 \\ 10 & 8 & 12 & 5 & 3 & 7 \\ 5 & 9 & 7 & 0 & 4 & 2 \\ 7 & 5 & 9 & 2 & 0 & 4 \\ 9 & 7 & 11 & 4 & 2 & 6 \end{pmatrix} \end{array}, \tag{4.32}$$

which is succintly summarized by the difference conditions in (1.16) To finish, we note that the parts colored $a$ or $ad$ in overpartitions counted by $\overline{S}_2(u,v,m,n)$ become parts of the form $6x-4$ or $6x-5$. This gives Corollary 2

Corollary 3 is similar. We use the same substitutions $q = q^6$, $a = q^{-4}$, $b = q^{-2}$, and $d = q^{-1}$, but this time in (1.14). On the product side we have

$$\frac{(-q^2;q^6)_\infty(-q^4;q^6)_\infty}{(q;q^6)_\infty}, \tag{4.33}$$

which is the generating function for the number of partitions into parts $1, 2$, or 4 modulo 6, where only parts congruent to 1 modulo 6 may be repeated. From $\overline{S}_1(u,v,m,n)$ we have the same difference conditions as in (4.32) (and hence (1.16)). Finally, the parts colored $b$ or $bd$ correspond to parts of the form $6x-3$ and $6x-2$.

# 5 Examples

## 5.1 Generalizations of Schur's theorem

Here we illustrate Theorem 4 and Corollary 1 for $n = 6$. To begin, there are 24 overpartitions of 6 satisfying the difference conditions in (1.13),

$$\begin{gathered}(\overline{6}),(6),(\overline{5},\overline{1}),(\overline{5},1),(5,\overline{1}),(5,1),(\overline{4},2),(4,2),(\overline{4},\overline{1},1),(\overline{4},1,1),(4,\overline{1},1),(4,1,1),\\(\overline{3},2,1),(3,2,1),(\overline{3},1,1,1),(3,1,1,1),(\overline{2},2,2),(2,2,2),(\overline{2},2,1,1,),(2,2,1,1),\\(\overline{2},1,1,1,1),(2,1,1,1,1),(\overline{1},1,1,1,1,1),(1,1,1,1,1,1),\end{gathered} \tag{5.34}$$

as well as 24 overpartitions of 6 into parts not divisible by 3,

$$\begin{gathered}(\overline{5},\overline{1}),(\overline{5},1),(5,\overline{1}),(5,1),(\overline{4},\overline{2}),(\overline{4},2),(4,\overline{2}),(4,2),(\overline{4},\overline{1},1),(\overline{4},1,1),(4,\overline{1},1),(4,1,1),\\(\overline{2},2,2),(2,2,2),(\overline{2},2,\overline{1},1),(\overline{2},2,1,1),(2,2,\overline{1},1),(2,2,1,1),\\(\overline{2},\overline{1},1,1,1),(\overline{2},1,1,1,1),(2,\overline{1},1,1,1),(2,1,1,1,1),(\overline{1},1,1,1,1,1),(1,1,1,1,1,1),\end{gathered} \tag{5.35}$$

confirming Theorem 4 for $n = 6$ (and $0 \leq m \leq 6$).

Of the overpartitions in (5.34), 12 of them have their $s$ smallest parts overlined, where $s$ is the number of parts congruent to 2 modulo 3. These are

$$\begin{gathered}(\overline{6}),(6),(\overline{5},\overline{1}),(5,\overline{1}),(\overline{4},\overline{1},1),(\overline{4},1,1),(4,\overline{1},1),(4,1,1),\\(\overline{3},1,1,1),(3,1,1,1),(\overline{1},1,1,1,1,1),(1,1,1,1,1,1).\end{gathered}$$

And, as predicted by Theorem 1 for $j = 1$, there are 12 overpartitions in (5.35) whose non-overlined parts are all congruent to 1 modulo 3,

$$\begin{gathered}(\overline{5},\overline{1}),(\overline{5},1),(\overline{4},\overline{2}),(4,\overline{2}),(\overline{4},\overline{1},1),(\overline{4},1,1),(4,\overline{1},1),(4,1,1),\\(\overline{2},\overline{1},1,1,1),(\overline{2},1,1,1,1),(\overline{1},1,1,1,1,1),(1,1,1,1,1,1).\end{gathered}$$

Similarly, of the overpartitions in (5.34), there are 6 which have their $s$ smallest parts overlined, where $s$ is the number of parts congruent to 1 modulo 3,

$$(\overline{6}),(6),(\overline{5},\overline{1}),(5,\overline{1}),(\overline{2},2,2),(2,2,2),$$

and there are 6 overparititons in (5.35) whose non-overlined parts are all congruent to 2 modulo 3,

$$(\overline{5},\overline{1}),(5,\overline{1}),(\overline{4},\overline{2}),(\overline{4},2),(\overline{2},2,2),(2,2,2).$$

## 5.2 Corollaries 2 and 3

Next we illustrate Corollaries 2 and 3 for $n = 10$. There are 19 partitions of 10 which satisfy the difference conditions in (1.16). They are

$$\begin{gathered}(10),(9,1),(8,2),(8,1,1),(7,3),(7,2,1),(7,1,1,1),(6,3,1),(6,1,1,1,1),(5,3,1,1),\\(5,1,1,1,1,1),(4,3,3),(4,3,1,1,1),(4,1,1,1,1,1,1),(3,3,3,1),(3,3,1,1,1,1),\\(3,1,1,1,1,1,1,1),(2,1,1,1,1,1,1,1,1),(1,1,1,1,1,1,1,1,1,1).\end{gathered} \tag{5.36}$$

Of these, the ones that have their $s$ smallest parts even, where $s$ is the number of parts congruent to 1 or 2 modulo 6, are

$$(10),(8,2),(4,3,3).$$

Thus $C(10)=3$, and as predicted, there are 3 partitions of 10 into distinct parts not congruent to $\pm 1$ modulo 6,

$$(10),(8,2),(6,4).$$

On the other hand, nine of the partitions in (5.36) have their $r$ smallest parts even, where $r$ is the number of parts congruent to 3 or 4 modulo 6. These are

$$(10),(8,2),(8,1,1),(7,2,1),(7,1,1,1),(6,1,1,1,1),(5,1,1,1,1,1),$$
$$(2,1,1,1,1,1,1,1,1),(1,1,1,1,1,1,1,1,1,1).$$

Thus $G(10)=9$, and the nine partitions of 10 into parts congruent $1,2$, or 4 modulo 6 with only parts congruent to 1 modulo 6 allowed to repeat are

$$(10),(8,2),(8,1,1),(7,2,1),(7,1,1,1),(4,2,1,1,1,1),(4,1,1,1,1,1,1),$$
$$(2,1,1,1,1,1,1,1,1),(1,1,1,1,1,1,1,1,1,1).$$

## 6 Conclusion

In establishing Theorem 5, we have shown that using a partial staircase in the context of weighted words can lead to elegant infinite product generating functions, just as with staircases and generalized staircases. We have limited ourselves to the framework of Schur's theorem, but partial staircases can be used to asymmetrically extend other partition identities, such as Göllnitz' theorem [5] or the Alladi-Andrews-Berkovich identity [4]. We leave the details to the motivated reader.

We have also seen that partial staircases work well with bijective arguments. It remains to be seen, however, whether proofs of Schur's theorem [9] and Schur's theorem for overpartitions [12] using $q$-difference equations can be adapted to the asymmetric case.

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