ON WEIGHTED OVERPARTITIONS RELATED TO SOME q-SERIES IN RAMANUJAN'S LOST NOTEBOOK

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Dedicated to Bruce C. Berndt on his 80th birthday

Abstract. Motivated by certain q-series of Ramanujan, we examine two overpartition difference functions. We give both combinatorial and asymptotic formulas for the differences and show that they are always positive. We also briefly discuss similar differences for some other types of partitions. Our main tools are elementary q-series transformations and Ingham's Tauberian theorem.

1. Introduction

Recall the usual q-series notation

$$(a)_n = (a;q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}),$$

valid for $n \in \mathbb{N}_0 \cup \{\infty\}$. Let $g(q)$ be the q-series defined by

$$g(q) = \sum_{n \geq 0} \frac{q^{n(n+1)/2}}{(q)_n^2} = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + \cdots . \tag{1.1}$$

This series appears in Ramanujan's last letter to Hardy as an example of a function which does *not* have the mock theta property. (See [23, p. 58 equation (C)].) It also appears in Ramanujan's lost notebook, in the identity [4, Entry 1.4.9]

$$\sum_{n \geq 0} \frac{q^{n(n+1)/2}}{(q)_n^2} = \frac{(-q)_\infty}{(q)_\infty} \sum_{n \geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(q^2;q^2)_n}. \tag{1.2}$$

As was typically the case, Ramanujan did not discuss the combinatorial significance of his identity or of the coefficients of $g(q)$. Many years later, Andrews interpreted $g(q)$ as the generating function for the number of gradual stacks with summit of size n [1, equation (3.5)].

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Bringmann and Mahlburg [12] also studied these stacks, proving an asymptotic formula for the coefficients of $g(q)$. Taking a partition-theoretic approach, Berndt, Yee, and the first author examined the combinatorics of the identity (1.2) and gave a bijective proof using partition pairs [8, Theorem 5.2].

Ramanujan also recorded a "signed" version of $g(q)$,

$$g^s(q) = \sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n^2} = 1 - q + 2q^2 - 2q^3 + 2q^4 - 4q^5 + 5q^6 - 3q^7 + \cdots . \tag{1.3}$$

This series appears on the left-hand side of a notoriously difficult identity from the lost notebook [4, Entry (7.4.1)],

$$\sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n^2} = \sum_{n\geq 0} \frac{(-q)^{n(n+1)/2}}{(-q^2;q^2)_n} - 2\sum_{n\geq 1} \frac{(-1)^n q^{2n^2}}{(-q;q^2)_{2n}}. \tag{1.4}$$

Andrews proved Ramanujan's identity [1] and also observed empirically [2, p. 710] that the coefficients of $g^s(q)$ have "a lengthy sign change pattern which alters fairly infrequently." Specifically, the coefficients alternate in sign except for sporadic pairs of consecutive coefficients which have the same sign. For example, the alternating pattern is respected up to q^{500} except for the pairs $(14,15), (49,50), (102,103), (175,176), (268,269)$, and $(379,380)$.

In this paper we examine two signed q-series which are also closely related to $g(q)$ but whose sign patterns are much simpler – in fact, the coefficients will all be positive. Instead of the stacks and partition pairs mentioned above, our combinatorial framework is that of overpartitions [13]. Recall that an overpartition is a partition in which the first occurrence of a number may be overlined. For example, the 14 overpartitions of 4 are

$$\begin{gathered}(4), (\overline{4}), (3,1), (\overline{3},1), (3,\overline{1}), (\overline{3},\overline{1}), (2,2), (\overline{2},2), \\ (2,1,1), (\overline{2},1,1), (2,\overline{1},1), (\overline{2},\overline{1},1), (1,1,1,1), (\overline{1},1,1,1).\end{gathered} \tag{1.5}$$

Also recall that overpartitions may be represented by a Frobenius symbol, that is, a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{pmatrix},$$

where the top row is a partition into distinct non-negative parts, the bottom row is an overpartition into non-negative parts, and $n = m + \sum a_i + \sum b_i$. With this representation, the 14 overpartitions of 4 are

$$\begin{gathered}\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ \overline{0} \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ \overline{1} \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ \overline{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ \overline{3} \end{pmatrix}, \\ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ \overline{0} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \overline{1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & \overline{0} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \overline{1} & \overline{0} \end{pmatrix}.\end{gathered} \tag{1.6}$$

For more on the correspondence between overpartitions and their Frobenius representations, see [13].

It is readily seen that the q-series in (1.1) and (1.3) are generating functions for overpartitions where there are no non-overlined parts larger than the number of overlined parts. It is then natural to consider $\overline{p}(m,n)$, the number of overpartitions of n having m non-overlined parts larger than the number of overlined parts, and as a companion, $\overline{p}^u(m,n)$, the number of overpartitions of n having m overlined parts larger than the number of non-overlined parts. Using elementary combinatorial arguments we have the two-variable generating functions

$$F(z,q) := \sum_{n,m\geq 0} \overline{p}(m,n) z^m q^n = \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n^2 (zq^{n+1})_\infty}, \tag{1.7}$$

$$F^u(z,q) := \sum_{n,m\geq 0} \overline{p}^u(m,n) z^m q^n = \sum_{n\geq 0} \frac{q^n(-q)_n(-zq^{n+1})_\infty}{(q)_n}. \tag{1.8}$$

Note that $F(0,q) = g(q)$ is Ramanujan's q-series.

Our first pair of results concern the specializations

$$\begin{aligned} F(-1,q) &= \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n^2(-q^{n+1})_\infty} \\ &= 1 + 2q^2 + 2q^3 + 4q^4 + 4q^5 + 8q^6 + 10q^7 + 16q^8 + 20q^9 + 30q^{10} + \cdots \end{aligned}$$

and

$$\begin{aligned} F^u(-1,q) &= \sum_{n\geq 0} \frac{q^n(-q)_n(q^{n+1})_\infty}{(q)_n} \\ &= 1 + 2q^2 + 4q^3 + 6q^4 + 10q^5 + 16q^6 + 26q^7 + 40q^8 + 62q^9 + 92q^{10} + \cdots. \end{aligned}$$

Observe that $F(-1,q)$ is the generating function for $\overline{p}_e(n) - \overline{p}_o(n)$, where $\overline{p}_e(n)$ (resp. $\overline{p}_o(n)$) is the number of overpartitions of n such that there are an even (resp. odd) number of non-overlined parts larger than the number of overlined parts, while $F^u(-1,q)$ is the generating function for $\overline{p}^u_e(n) - \overline{p}^u_o(n)$, where $\overline{p}^u_e(n)$ (resp. $\overline{p}^u_o(n)$) is the number of overpartitions of n having an even (resp. odd) number of overlined parts larger than the number of non-overlined parts.

Despite the signs in the generating functions, these overpartition differences both turn out to be positive. We exhibit this positivity via two combinatorial identities involving the Frobenius symbol of an overpartition. For some other identities involving these Frobenius symbols, see [14, 18, 19].

Theorem 1.1. *Let $a(n)$ denote the number of overpartitions of n whose Frobenius symbols have only odd parts in the top row. Then for all positive integers n we have*

$$\overline{p}_e(n) - \overline{p}_o(n) = a(n). \tag{1.9}$$

In particular, for all positive integers $n > 1$,

$$\overline{p}_e(n) > \overline{p}_o(n). \tag{1.10}$$

Theorem 1.2. *Let $b(n)$ denote the number of overpartitions of n whose Frobenius symbols have only positive parts in the top row. Then for all positive integers n we have*

$$\overline{p}_e^u(n) - \overline{p}_o^u(n) = b(n). \tag{1.11}$$

In particular, for all positive integers $n > 1$,

$$\overline{p}_e^u(n) > \overline{p}_o^u(n). \tag{1.12}$$

The overpartition identities are proved using elementary q-series transformations, and the positivity will follow from the transformed generating function. As an illustration, take $n = 4$. Using (1.5) and (1.6) we find that $\overline{p}_e(4) = 9, \overline{p}_o(4) = 5$, and $a(4) = 4$, while $\overline{p}_e^u(4) = 10, \overline{p}_o^u(4) = 4$, and $b(4) = 6$.

It turns out that $F(-1, q)$ and $F^u(-1, q)$ both have elegant expressions in terms of false theta functions. We state these together as one result.

Theorem 1.3. *We have*

$$\sum_{n\geq 0} \left(\overline{p}_e(n) - \overline{p}_o(n)\right) q^n = \frac{1}{(q)_\infty} \sum_{n\geq 0} q^{n(3n+1)/2}(1 - q^{2n+1}), \tag{1.13}$$

$$\sum_{n\geq 0} \left(\overline{p}_e^u(n) - \overline{p}_o^u(n)\right) q^n = \frac{(-q)_\infty}{(q)_\infty} \left[1 - 2\sum_{n\geq 1} q^{n(3n-1)/2}\left(1 - q^n\right)\right]. \tag{1.14}$$

Theorem 1.3 allows us to apply Ingham's Tauberian theorem to deduce asymptotic formulas for $\overline{p}_e(n) - \overline{p}_o(n)$ and $\overline{p}_e^u(n) - \overline{p}_o^u(n)$. We use the usual notation $p(n)$ for the number of partitions of n and $\overline{p}(n)$ for the number of overpartitions of n.

Theorem 1.4. *As $n \to \infty$,*

$$\overline{p}_e(n) - \overline{p}_o(n) \sim \frac{1}{6\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \sim \frac{2}{3}p(n).$$

Theorem 1.5. *As $n \to \infty$,*

$$\overline{p}_e^u(n) - \overline{p}_o^u(n) \sim \frac{1}{3}\overline{p}(n).$$

These results are illustrated in Tables 1 and 2.

Note that the asymptotic behavior is quite different for the two difference functions. From work of Zuckerman [25], we have

$$\overline{p}(n) \sim \frac{1}{8n} \exp\left(\pi\sqrt{n}\right),$$

which implies that $\overline{p}_e(n) - \overline{p}_o(n)$ is small compared to $\overline{p}(n)$. On the other hand, $\overline{p}_e^u - \overline{p}_o^u(n)$ is approximately $\overline{p}(n)/3$.

The rest of the paper is organized as follows. In the next section we prove the main results. In Section 3 we examine other classes of partitions using a similar weight on specific parts. By doing so, we generate several interesting new partition functions and find relations

Table 1. Comparison of $p(n)$ and $\overline{p}_e(n)-\overline{p}_o(n)$ for $n \leq 10000$ (values rounded to four decimal places)

n	$p(n)$	$\overline{p}_e(n)-\overline{p}_o(n)$	$(\overline{p}_e(n)-\overline{p}_o(n))/p(n)$
100	1.906×10^{8}	1.288×10^{8}	0.6756
500	2.300×10^{21}	1.543×10^{21}	0.6708
1000	2.406×10^{31}	1.611×10^{31}	0.6696
5000	1.698×10^{74}	1.134×10^{74}	0.6680
10000	3.617×10^{106}	2.415×10^{106}	0.6676

Table 2. Comparison of $\overline{p}(n)$ and $\overline{p}^u_e(n)-\overline{p}^u_o(n)$ for $n \leq 10000$ (values rounded to four decimal places)

n	$\overline{p}(n)$	$\overline{p}^u_e(n)-\overline{p}^u_o(n)$	$(\overline{p}^u_e(n)-\overline{p}^u_o(n))/\overline{p}(n)$
100	5.329×10^{10}	1.895×10^{10}	0.3557
500	7.945×10^{26}	2.730×10^{26}	0.3436
1000	1.729×10^{39}	5.890×10^{38}	0.3406
5000	7.447×10^{91}	2.507×10^{91}	0.3366
10000	3.413×10^{131}	1.146×10^{131}	0.3357

to Ramanujan's mock theta functions or false theta functions. In one case we prove an unexpected Ramanujan-type congruence modulo 5. We close in Section 4 with some remarks.

2. Proofs of Theorems 1.1 – 1.5

In this section we prove the main results. We begin with Theorem 1.1.

Proof of Theorem 1.1. We require a transformation of Jackson [16, Appendix III, eq. (III.4)],

$$\sum_{n\geq 0} \frac{(a)_n(b)_n}{(q)_n(c)_n} z^n = \frac{(az)_\infty}{(z)_\infty} \sum_{n\geq 0} \frac{(a)_n(c/b)_n(-bz)^n q^{\binom{n}{2}}}{(q)_n(c)_n(az)_n}. \tag{2.1}$$

Using this with $z=-q/a$, $c=-b=q$, and $a\to\infty$ we have

$$\begin{aligned} F(-1,q) &= \frac{1}{(-q)_\infty} \sum_{n\geq 0} \frac{(-q)_n q^{n(n+1)/2}}{(q)_n^2} \\ &= \sum_{n\geq 0} \frac{q^{n^2+n}(-1)_n}{(q)_n(q^2;q^2)_n}. \end{aligned} \tag{2.2}$$

The nth summand of the last q-series generates a Frobenius symbol as follows. First, q^n counts the number of columns. Second, the term $q^{n^2}/(q^2;q^2)_n$ generates a partition into n distinct odd parts for the top row. Finally, the term $(-1)_n/(q)_n$ contributes an overpartition into n non-negative parts for the bottom row (as explained in [13]). Therefore

$$F(-1,q) = \sum_{n\geq 0} a(n)q^n,$$

and (1.9) follows. The positivity in (1.10) is deduced from (2.2). $\square$

We now turn to Theorem 1.2. The proof is similar, with Heine's transformation in place of Jackson's transformation.

Proof of Theorem 1.2. We require the third Heine transformation [16, Appendix III, eq. (III.3)],

$$\sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n(q)_n} z^n = \frac{(abz/c)_\infty}{(z)_\infty} \sum_{n\geq 0} \frac{(c/a)_n(c/b)_n}{(c)_n(q)_n} (abz/c)^n \qquad \text{for } |z|, |abz/c| < 1. \tag{2.3}$$

Using this with $c = z = -b = q$ and $a = 0$ we have

$$F^u(-1,q) = (q)_\infty \sum_{n\geq 0} \frac{q^n(-q)_n}{(q)_n^2} \tag{2.4}$$

$$= \sum_{n\geq 0} \frac{(-1)_n q^{n(n+3)/2}}{(q)_n^2}. \tag{2.5}$$

Here the nth summand generates a Frobenius symbol much like before. The term q^n again counts the number of columns, while $(-1)_n/(q)_n$ generates an overpartition into non-negative parts in the bottom row. The remaining part, $q^{n(n+1)/2}/(q)_n$, contributes a partition into distinct positive parts in the top row. Therefore

$$F^u(-1,q) = \sum_{n\geq 0} b(n)q^n,$$

and (1.11) follows. The positivity in (1.12) is deduced from (2.5). $\square$

Next we treat Theorem 1.3. The proof uses the Heine transformation and an identity from the lost notebook in the first case and an identity of Warnaar in the second case.

Proof of Theorem 1.3. We begin with the first Heine transformation [16, Appendix III, eq. (III.1)],

$$\sum_{n\geq 0} \frac{(a)_n(b)_n}{(q)_n(c)_n} z^n = \frac{(b)_\infty(az)_\infty}{(c)_\infty(z)_\infty} \sum_{n\geq 0} \frac{(c/b)_n(z)_n}{(q)_n(az)_n} b^n \qquad \text{for } |q|, |z|, |b| < 1. \tag{2.6}$$

By setting $a = 1/z$, $c = -q$, $z = zq$ and $b \to 0$ in (2.6) and (2.3), we find the following two-variable version of Ramanujan's identity (1.2),

$$\begin{aligned} \frac{(q)_\infty}{(-q)_\infty (zq)_\infty} \sum_{n \geq 0} \frac{(zq)_n q^{n(n+1)/2}}{(q)_n^2} &= \sum_{n \geq 0} \frac{(1/z)_n (zq)^n}{(q^2;q^2)_n} \\ &= \frac{1}{(zq)_\infty} \sum_{n \geq 0} \frac{(-zq)_n (-1)^n q^{n(n+1)/2}}{(q^2;q^2)_n}. \end{aligned} \tag{2.7}$$

This implies that

$$F(z,q) = \frac{(-q)_\infty}{(q)_\infty (zq)_\infty} \sum_{n \geq 0} \frac{(-zq)_n (-1)^n q^{n(n+1)/2}}{(q^2;q^2)_n}.$$

From the lost notebook [3, Entry 9.4.2] we have the false theta identity

$$\sum_{n \geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(-q)_n} = \sum_{n \geq 0} q^{n(3n+1)/2} (1 - q^{2n+1}), \tag{2.8}$$

which gives

$$(q)_\infty F(-1,q) = \sum_{n \geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(-q)_n} = \sum_{n \geq 0} q^{n(3n+1)/2} (1 - q^{2n+1}), \tag{2.9}$$

and this is (1.13).

For (1.14) we simply use the case $a = 1$ of Warnaar's identity [24, p. 390],

$$\sum_{n \geq 0} \frac{(-aq)_n q^n}{(q)_n (a^2 q)_n} = \frac{(-aq)_\infty}{(q)_\infty (a^2 q)_\infty} \left[1 - (1+a) \sum_{n \geq 1} a^{3n-2} q^{n(3n-1)/2} (1 - aq^n) \right] \tag{2.10}$$

applied to (2.4). $\square$

Before continuing, we give two corollaries of Theorem 1.3. First, using

$$\frac{1}{(q)_\infty} = \sum_{n \geq 0} p(n) q^n$$

and

$$\frac{(-q)_\infty}{(q)_\infty} = \sum_{n \geq 0} \overline{p}(n) q^n$$

together with (1.13) and (1.14), we obtain formulas for $\overline{p}_e(n) - \overline{p}_o(n)$ and $\overline{p}_e^u(n) - \overline{p}_o^u(n)$ in terms of $p(n)$ and $\overline{p}(n)$, respectively.

Corollary 2.1. *For all positive integers n we have*

$$\overline{p}_e(n) - \overline{p}_o(n) = p(n) - p(n-1) + p(n-2) - p(n-5) + \cdots, \tag{2.11}$$

and

$$\overline{p}_e^u(n) - \overline{p}_o^u(n) = \overline{p}(n) - 2\overline{p}(n-1) + 2\overline{p}(n-2) - 2\overline{p}(n-5) + \cdots \tag{2.12}$$

Next we give a characterization of $\overline{p}_e^u(n) - \overline{p}_o^u(n)$ modulo 4 which follows from (1.14).

Corollary 2.2. *For all positive integers $n \geq 1$,*

$$\overline{p}_e^u(n) - \overline{p}_o^u(n) \equiv \begin{cases} 2 \pmod 4, & \text{if } n \text{ is a square or generalized pentagonal number} \\ & \text{(but not both)}, \\ 0 \pmod 4, & \text{otherwise.} \end{cases}$$

Proof. Using the identity

$$\frac{(q)_\infty}{(-q)_\infty} = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2},$$

the generating function for overpartitions satisfies

$$\frac{(-q)_\infty}{(q)_\infty} = \frac{1}{1 + 2\sum_{n\geq 1}(-1)^n q^{n^2}} \equiv 1 + 2\sum_{n\geq 1} q^{n^2} \pmod 4.$$

Together with (1.14) we then have

$$\sum_{n\geq 0} (\overline{p}_e^u(n) - \overline{p}_o^u(n))q^n \equiv \left(1 + 2\sum_{n\geq 1} q^{n^2}\right)\left(1 + 2\sum_{n\geq 1} q^{n(3n-1)/2}(1+q^n)\right) \pmod 4,$$

and the result follows. □

Note that there exist numbers which are both squares and pentagonal numbers (see $A036353$ in [22]).

We are now ready to prove Theorems 1.4 and 1.5. For these we use Ingham's Tauberian theorem [17, Theorem 1.1], as presented in [10].

Theorem 2.3 (Theorem 1.1 in [10])**.** *Let $f(q) = \sum_{n\geq 0} a(n)q^n$ be a power series whose radius of convergence is equal to 1 and whose coefficients $a(n)$ are non-negative and weakly increasing. Suppose that for $A > 0, \lambda, \alpha \in \mathbb{R}$,*

$$f(e^{-t}) \sim \lambda t^\alpha e^{A/t} \quad \text{as } t \to 0^+, \qquad f(e^{-z}) \ll |z|^\alpha e^{A/|z|} \quad \text{as } z \to 0,$$

with $z = x + iy$ ($x > 0, y \in \mathbb{R}$) in each region of the form $|y| \leq cx$ for $c > 0$. Then

$$a(n) \sim \frac{\lambda}{2\sqrt{\pi}} \frac{A^{\frac{\alpha}{2}+\frac{1}{4}}}{n^{\frac{\alpha}{2}+\frac{3}{4}}} e^{2\sqrt{An}}$$

as $n \to \infty$.

Proof of Theorem 1.4. Let $q = e^{-z}$ and $f(z) = e^{-\frac{3}{2}z^2}$. By Euler-Maclaurin summation [10, Theorem 1.2],

$$\begin{aligned}&\sum_{n\geq 0} q^{n(3n-1)/2}(1 - q^{2n+1}) \\ &= e^{-z/24} \sum_{n\geq 0} \left[e^{-\frac{3}{2}\left(n+\frac{1}{6}\right)^2 z} - e^{-\frac{3}{2}\left(n+\frac{5}{6}\right)^2 z}\right]\end{aligned}$$

$$= e^{-z/24}\sum_{n\geq 0}\left[f\left(\left(n+\frac{1}{6}\right)\sqrt{z}\right)-f\left(\left(n+\frac{5}{6}\right)\sqrt{z}\right)\right]$$
$$= e^{-z/24}\left[\frac{1}{\sqrt{z}}\int_0^\infty f(t)dt - B_1\left(\frac{1}{6}\right)-\frac{1}{\sqrt{z}}\int_0^\infty f(t)dt + B_1\left(\frac{5}{6}\right)+\mathcal{O}(z)\right]$$
$$= e^{-z/24}\left(\frac{2}{3}+\mathcal{O}(z)\right)$$

uniformly, as $z \to 0$ in $\{x+iy : |y| \leq cx\}$ for every fixed $c > 0$, where $B_n(t)$ is the nth Bernoulli polynomial. Hence we obtain, as $z \to 0$ in $\{x+iy : |y| \leq cx\}$ for every fixed $c > 0$,

$$F(-1, e^{-z}) \sim \sqrt{\frac{2z}{9\pi}}\exp\left(\frac{\pi^2}{6z}\right)$$

with the asymptotic

$$\log(q)_\infty \sim -\frac{\pi^2}{6z}+\frac{1}{2}\log\left(\frac{2\pi}{z}\right)+\frac{z}{24}.$$

To apply Ingham's Tauberian theorem, we now need to prove that $\overline{p}_e(n)-\overline{p}_o(n)$ is weakly increasing. To see this, note that (2.2) gives

$$\begin{aligned}(1-q)\sum_{n\geq 0}\left(\overline{p}_e(n)-\overline{p}_o(n)\right)q^n &= (1-q)\sum_{n\geq 0}\frac{(-1)_n q^{n^2+n}}{(q)_n(q^2;q^2)_n}\\ &= (1-q)+\sum_{n\geq 1}\frac{(-1)_n q^{n^2+n}}{(q^2)_{n-1}(q^2;q^2)_n},\end{aligned}$$

which has non-negative coefficients for $n \geq 2$. Therefore $\overline{p}_e(n)-\overline{p}_o(n)$ is weakly increasing for $n \geq 2$. By Theorem 2.3, we may now conclude that

$$\overline{p}_e(n)-\overline{p}_o(n) \sim \frac{1}{6\sqrt{3n}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

as $n \to \infty$. □

Proof of Theorem 1.5. As in the proof of Theorem 1.4, we can obtain the asymptotic behavior of the generating function for $\overline{p}_e^u(n)-\overline{p}_o^u(n)$. Setting $q = e^{-z}$ and $f(z) = \exp(-\frac{3}{2}z^2)$, Euler-Maclaurin summation gives

$$\begin{aligned}\sum_{n\geq 1} q^{n(3n-1)/2}(1-q^{2n}) &= e^{-z/24}\sum_{n\geq 0}\left[f\left(\left(n+\frac{5}{6}\right)\sqrt{z}\right)-f\left(\left(n+\frac{7}{6}\right)\sqrt{z}\right)\right]\\ &= e^{-z/24}\left(\frac{1}{3}+\mathcal{O}(z)\right)\end{aligned}$$

uniformly, as $z \to 0$ in $\{x+iy : |y| \leq cx\}$ for every fixed $c > 0$.

Hence we have as $z \to 0$ in $\{x+iy : |y| \le cx\}$ for every fixed $c>0$,

$$\sum_{n\ge 0} \left(\overline{p}_e^u(n) - \overline{p}_o^u(n)\right) q^n \sim \frac{1}{6}\sqrt{\frac{z}{\pi}} \exp\left(\frac{\pi^2}{4z}\right).$$

Since we can see that $\overline{p}_e^u(n) - \overline{p}_o^u(n)$ is weakly increasing for $n \ge 2$ from

$$(1-q)\sum_{n\ge 0} \left(\overline{p}_e^u(n) - \overline{p}_o^u(n)\right) q^n = (1-q)\sum_{n\ge 0} \frac{(-1)_n q^{n(n+3)/2}}{(q)_n^2},$$

by applying Theorem 2.3, we have the asymptotic formula

$$\overline{p}_e^u(n) - \overline{p}_o^u(n) \sim \frac{1}{24n} \exp\left(\pi\sqrt{n}\right)$$

as $n \to \infty$. $\square$

3. Other weighted partition functions

In this section we replace overpartitions by other types of partitions but use a similar weight on specific types of parts. We touch on three different cases, the first of which is closely related to $F^u(-1, q)$.

3.1. **Bipartitions.** A bipartition π of n is a pair of partitions λ_r and λ_b where λ_r and λ_b are ordinary partitions and the sum of their parts is n. Let us say that the parts in λ_r are colored red and the parts in λ_b are colored blue. Define $b_e(n)$ (resp. $b_o(n)$) to be the number of bipartitions of n which have an even (resp. odd) number of red parts larger than the number of blue parts. Let $B(q)$ denote the generating function for $b_e(n) - b_o(n)$,

$$\begin{aligned} B(q) &:= \sum_{n\ge 0} (b_e(n) - b_o(n)) q^n \\ &= 1 + 3q^2 + 4q^3 + 10q^4 + 14q^5 + 29q^6 + 44q^7 + 79q^8 + 120q^9 + 199q^{10} + \cdots \end{aligned}$$

Elementary combinatorial arguments followed by an appeal to (2.4) give that

$$\begin{aligned} B(q) &= \sum_{n\ge 0} \frac{q^n}{(q)_n^2 (-q^{n+1})_\infty} \\ &= \frac{1}{(q^2;q^2)_\infty} F^u(-1, q), \end{aligned}$$

from which we conclude the following.

Theorem 3.1. *For all positive integers $n > 1$,*

$$b_e(n) > b_o(n).$$

Moreover, we have the expression

$$\sum_{n\geq 0}(b_e(n)-b_o(n))q^n = \frac{1}{(q)_\infty^2}\left[1-2\sum_{n\geq 1}q^{n(3n-1)/2}\left(1-q^n\right)\right]. \tag{3.1}$$

Using "Ramanujan's method" [6, 7], the false theta identity (3.1) can be used to deduce the following, perhaps unexpected, Ramanujan-type congruence.

Theorem 3.2. *For all non-negative integers n, we have*

$$b_e(5n+4)-b_o(5n+4)\equiv 0 \pmod 5.$$

Proof. Recall Jacobi's identity

$$(q)_\infty^3 = \sum_{k\geq 0}(-1)^k(2k+1)q^{k(k+1)/2}.$$

Combined with (3.1), we have

$$\begin{aligned} B(q) \equiv \frac{1}{(q^5;q^5)_\infty}\Bigg(&\sum_{k\geq 0}(-1)^k(2k+1)q^{k(k+1)/2} \\ &-2\sum_{\substack{k\geq 0\\ n\geq 1}}(2k+1)q^{k(k+1)/2+n(3n-1)/2}(1-q^n)\Bigg) \pmod 5. \end{aligned}$$

Since $k(k+1)/2 \not\equiv 4 \pmod 5$ the first term has no contribution modulo 5 to the coefficient of q^{5n+4} in $B(q)$. As for the second term, it is easy to check that if $k(k+1)/2+n(3n-1)/2$ or $k(k+1)/2+n(3n+1)/2$ is congruent to 4 modulo 5, then $k\equiv 2 \pmod 5$. In this case, the factor $(2k+1)$ ensures that there is again no contribution modulo 5 to the coefficient of q^{5n+4} in $B(q)$. □

3.2. **Partitions without repeated even parts I.** Let $ped(n)$ denote the number of partitions of n without repeated even parts. We define $ped_e(n)$ (resp. $ped_o(n)$) to be the number of partitions counted by $ped(n)$ having an even (resp. odd) number of odd parts larger than *twice* the number of even parts. Then we have

$$\begin{aligned} H(q) :&= \sum_{n\geq 0}(ped_e(n)-ped_o(n))q^n \\ &= \sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q^2)_n(q^2;q^2)_n(-q^{2n+1};q^2)_\infty} \\ &= 1-q+2q^2-q^3+4q^4-2q^5+7q^6-4q^7+10q^8-6q^9+17q^{10}-\cdots. \end{aligned} \tag{3.2}$$

While Ramanujan's series $g^s(q)$ is a reminder that looks can be deceiving, it turns out that the coefficients of $H(q)$ do indeed alternate as suggested by (3.2).

Theorem 3.3. *For all positive integers n we have*

$$(-1)^n \left(ped_e(n) - ped_o(n)\right) > 0.$$

Proof. We require the second Heine transformation [16, Appendix III, eq. (III.2)],

$$\sum_{n\geq 0} \frac{(a)_n (b)_n z^n}{(c)_n (q)_n} = \frac{(c/b)_\infty (bz)_\infty}{(c)_\infty (z)_\infty} \sum_{n\geq 0} \frac{(abz/c)_n (b)_n (c/b)^n}{(bz)_n (q)_n}. \tag{3.3}$$

With $q = q^2$, $a = -c = q$, $z = -q^2/b$, and $b \to \infty$ we find that

$$H(-q) = \frac{(-q^2;q^2)_\infty}{(q^2;q^4)_\infty} \sum_{n\geq 0} \frac{q^{n^2}}{(-q^2;q^2)_n}. \tag{3.4}$$

This clearly has positive coefficients. □

Recall Ramanujan's third order mock theta function

$$\phi(q) = \sum_{n\geq 0} \frac{q^{n^2}}{(-q^2;q^2)_n}.$$

By (3.4) we have

$$H(-q) = \frac{(-q^2;q^2)_\infty}{(q^2;q^4)_\infty} \phi(q).$$

Such products of modular forms and mock theta functions are called *mixed* mock modular forms [20]. It is possible to obtain asymptotic formulas for the coefficients of such forms (see [9, 11], for example), though we shall not pursue this here.

We close this subsection with a combinatorial identity for the coefficients of $H(-q)$. Following the lead of Fine [15, Section 26] in his treatment of Ramanujan's mock theta function

$$\psi(q) = \sum_{n\geq 1} \frac{q^{n^2}}{(q;q^2)_n},$$

we say that a partition into odd parts is *without gaps* if all odd parts less than the largest occur. If λ is a partition and $k \geq 0$, we refer to the k smallest parts of λ as an *initial partition.*

Theorem 3.4. *Let $c(n)$ denote the number of partitions counted by $ped(n)$ such that odd parts occur an even number of times, except possibly for an initial partition into odd parts without gaps $\{1, 3, \ldots, 2k-1\}$, wherein all parts occur an odd number of times, and in which case even parts are all $> 2k$. Then*

$$c(n) = (-1)^n (ped_e(n) - ped_o(n)).$$

Proof. We have

$$H(-q) = \frac{(-q^2;q^2)_\infty}{(q^2;q^4)_\infty} \sum_{n\geq 0} \frac{q^{n^2}}{(-q^2;q^2)_n}$$

$$= \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^4)_n} \times \frac{(-q^{2n+2};q^2)_\infty}{(q^{4n+2};q^4)_\infty},$$

and this is easily seen to be the generating function for $c(n)$. Specifically, the nth summand generates an initial partition into odd parts $\leq 2n-1$ without gaps (which is empty if $n=0$), each part occurring an odd number of times, while the product in the numerator contributes the distinct even parts $\geq 2n+2$ and the product in the denominator contributes odd parts $\geq 2n+1$ occurring an even number of times. $\square$

3.3. **Partitions without repeated even parts II.** In this section we again consider partitions without repeated parts, but this time we reverse the roles played by the even and odd parts. Namely, let $ped_e^e(n)$ (resp. $ped_o^e(n)$) be the number of partitions counted by $ped(n)$ having an even (resp. odd) number of even parts larger than twice the number of odd parts. Then we have

$$\begin{aligned} H^e(q) &:= \sum_{n\geq 0} \left(ped_e^e(n) - ped_o^e(n)\right) q^n \\ &= \sum_{n\geq 0} \frac{q^n(-q^2;q^2)_n(q^{2n+2};q^2)_\infty}{(q^2;q^2)_n} \\ &= 1 + q + 3q^3 + 2q^4 + 4q^5 + 7q^6 + 6q^7 + 12q^8 + 12q^9 + 21q^{10} + \cdots, \end{aligned} \tag{3.5}$$

We show that the coefficients are positive, as suggested by (3.5).

Theorem 3.5. *For all positive integers $n \neq 2$, we have*

$$ped_e^e(n) > ped_o^e(n).$$

Proof. By Jackson's transformation (2.1) with $q = q^2, z = q, c = -a = q^2$, and $b \to 0$ we have

$$\begin{aligned} H^e(q) &= (q^2;q^2)_\infty \sum_{n\geq 0} \frac{(-q^2;q^2)_n q^n}{(q^2;q^2)_n^2} \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n\geq 0} \frac{q^{2n^2+n}(-q^2;q^2)_n(-q^{2n+3};q^2)_\infty}{(q^2;q^2)_n^2}. \end{aligned} \tag{3.6}$$

The positivity then follows after recalling Gauss' identity,

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \sum_{n\geq 0} q^{n(n+1)/2}.$$

$\square$

4. Concluding Remarks

The weighting procedure described in this paper can be applied to many other types of partitions. To give just one more example, let $p_e(n)$ (resp. $p_o(n)$) be the number of partitions of n having an even (resp. odd) number of even parts larger than twice of the number of odd parts. Then, we find that

$$\begin{aligned}\sum_{n\geq 0}(p_e(n)-p_o(n))q^n &= \sum_{n\geq 0}\frac{q^n}{(q^2;q^2)_n^2(-q^{2n+2};q^2)_\infty}\\ &= 1+q+3q^3+3q^4+5q^5+7q^6+9q^7+16q^8+18q^9+28q^{10}+\cdots.\end{aligned}$$

Comparing this with (3.6), we see that

$$\sum_{n\geq 0}(p_e(n)-p_o(n))q^n = (-q;q^2)_\infty\sum_{n\geq 0}\frac{q^{2n^2+n}(-q^2;q^2)_n(-q^{2n+3};q^2)_\infty}{(q^2;q^2)_n^2},$$

and we conclude that

$$p_e(n) > p_o(n)$$

for all positive integers $n \neq 2$.

Our positivity proofs have used elementary q-series transformations, but there are some situations which might be more subtle. For example, reversing the roles of even and odd parts in the previous paragraph, let $p'_e(n)$ (resp. $p'_o(n)$) be the number of partitions of n having an even (resp. odd) number of odd parts larger than twice of the number of even parts. Then

$$\begin{aligned}\sum_{n\geq 0}(p'_e(n)-p'_o(n))q^n &= \frac{1}{(-q;q^2)_\infty}\sum_{n\geq 0}\frac{q^{2n}(-q;q^2)_n}{(q)_{2n}}\\ &= 1-q+2q^2-q^3+5q^4-q^5+9q^6-q^7+16q^8\\ &\quad +28q^{10}+4q^{11}+47q^{12}+11q^{12}+77q^{13}+26q^{15}+\cdots,\end{aligned}$$

leading one to suspect that $p'_e(n)-p'_o(n)>0$ for all $n\neq 1,3,5,7$. This does not appear to follow easily from transformations used in this paper.

Instead of weighting according to the number of parts of some type that are larger than the number of parts of another type, one could also use the number of parts of some type that are *less than or equal to* the number of parts of another type. For instance, if $\overline{p}'_e(n)$ (resp. $\overline{p}'_o(n)$) denotes the number of overpartitions of n having an even (resp. odd) number of non-overlined parts less than or equal to the number of overlined parts, then

$$\begin{aligned}\sum_{n\geq 0}\left(\overline{p}'_e(n)-\overline{p}'_o(n)\right)q^n &= \sum_{n\geq 0}\frac{q^{n(n+1)/2}}{(q)_n(-q)_n(q^{n+1})_\infty}\\ &= \frac{1}{(q)_\infty}\sum_{n\geq 0}\frac{q^{n(n+1)/2}}{(-q)_n}\end{aligned}\tag{4.1}$$

$$= 1 + 2q + 2q^2 + 6q^3 + 6q^4 + 12q^5 + 16q^6 + 26q^7 + 32q^8 + \cdots . \tag{4.2}$$

The positivity of $\overline{p}_e'(n) - \overline{p}_o'(n)$ follows immediately. Note that the sum in the middle line above is a famous q-series from Ramanujan's lost notebook [5]. For a different combinatorial interpretation of (4.1) and an asymptotic formula for the coefficients, see [21, Theorems 1.1, 1.7].

The possibilities are many, and there are undoubtedly more interesting q-series waiting be discovered.

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