HECKE-TYPE FORMULAS FOR FAMILIES OF UNIFIED WITTEN-RESHETIKHIN-TURAEV INVARIANTS

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Abstract. Every closed orientable 3-manifold can be constructed by surgery on a link in S^3. In the case of surgery along a torus knot, one obtains a Seifert fibered manifold. In this paper we consider three families of such manifolds and study their unified Witten-Reshetikhin-Turaev (WRT) invariants. Thanks to recent computation of the coefficients in the cyclotomic expansion of the colored Jones polynomial for $(2, 2t+1)$-torus knots, these WRT invariants can be neatly expressed as q-hypergeometric series which converge inside the unit disk. Using the Rosso-Jones formula and some rather non-standard techniques for Bailey pairs, we find Hecke-type formulas for these invariants. We also comment on their mock and quantum modularity.

1. Introduction

Recall that for a knot K, Habiro's cyclotomic expansion of the colored Jones polynomial is given by [8]

$$J_N(K;q) = \sum_{n=0}^{\infty} C_n(K;q)\,(q^{1+N})_n\,(q^{1-N})_n, \tag{1.1}$$

where $C_n(K;q) \in \mathbb{Z}[q,q^{-1}]$. Here we have used the usual q-hypergeometric notation,

$$(a)_n = (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \tag{1.2}$$

valid for $n \in \mathbb{N} \cup \{\infty\}$. Simple expressions for the cyclotomic coefficients $C_n(K;q)$ are known for select families of knots. For example, for the pth twist knot K_p with $p > 0$, Masbaum [22] found that

$$C_n(K_p;q) = q^n \sum_{n=s_p \geq s_{p-1} \cdots \geq s_1 \geq 0} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}, \tag{1.3}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ is the usual q-binomial coefficient (or Gaussian polynomial),

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q)_n}{(q)_{n-k}(q)_k}. \tag{1.4}$$

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For another example, in the case of the left-handed torus knots $T^*_{(2,2t+1)}$, the authors recently computed that [15]

$$C_n(T^*_{(2,2t+1)};q) = q^{n+1-t} \sum_{n+1=k_t \geq k_{t-1} \geq \cdots \geq k_1 \geq 1} \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1}+k_i-i+2\sum_{j=1}^{i-1} k_j \\ k_{i+1}-k_i \end{bmatrix}. \quad (1.5)$$

One important application of the cyclotomic coefficients $C_n(K;q)$ is to formulate the unified WRT invariants of 3-manifolds M constructed by surgery along a knot K. Recall that the unified WRT invariant $I(M;q)$ [8] is a certain q-series which gives the original WRT invariant $\tau_N(M)$ when evaluated at the N-th root of unity $\exp(2\pi i/N)$,

$$\tau_N(M) = \mathrm{ev}_{q=\exp(2\pi \mathrm{i}/N)}\, I(M;q).$$

For example, when M is obtained from $(-1/b)$-surgery (for b a positive integer) on the knot K, then its unified WRT invariant $I(M;q)$ is expressed in terms of the cyclotomic coefficients as [4, 5]

$$(1-q)\, I(M;q) = \sum_{s_b=0}^{\infty} C_{s_b}(K;q) \cdot (q^{s_b+1})_{s_b+1} \sum_{s_b \geq \cdots \geq s_2 \geq s_1 \geq 0} \prod_{i=1}^{b-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}. \quad (1.6)$$

In the case of (± 2)-surgery on K we have, respectively [4, 5, 6]

$$(1-q)\, I(M;q) = \sqrt{2}\, q^{\frac{1}{4}} \sum_{n=0}^{\infty} C_n(K;q) \cdot (-1)^n q^{-\frac{n}{2}} (q^{\frac{1}{2}}; -q^{\frac{1}{2}})_{2n+1}, \quad (1.7)$$

$$(1-q)\, I(M;q) = \sqrt{2}\, q^{\frac{1}{4}} \sum_{n=0}^{\infty} C_n(K;q) \cdot (q^{\frac{1}{2}}; -q^{\frac{1}{2}})_{2n+1}. \quad (1.8)$$

Before continuing, we mention that the 3-manifolds that we will encounter in this paper are the Brieskorn homology sphere $\Sigma_{(p_1,p_2,p_3)}$ and the Seifert manifolds $M(b;(a_1,b_1),(a_2,b_2),(a_3,b_3))$, which are obtained from the Dehn surgery on a 4-component link in S^3, where a single unknot has a linking number 1 with three mutually unlinking unknots (see, e.g., [28]).

If $C_n(K;q) \in \mathbb{Z}[q]$, then the series in (1.6) - (1.8) converge not only when q is a root of unity, but also for $|q|<1$. In light of the nearly modular behavior at roots of unity when M is a Seifert manifold (arising from relations between the WRT invariant and Eichler integrals [11, 12, 14, 19]), it is natural to seek information about these unified WRT invariants as functions of q inside the unit disk. We briefly review some previous work in this direction.

In [13], the first author studied the case when $b=1$ and K is the pth twist knot K_p with $p>0$. The manifold obtained by (-1)-surgery along K_p is the integral homology sphere $\Sigma_{(2,3,6p-1)}$, and from (1.3) and (1.6) we have the expression

$$(1-q) I(\Sigma_{(2,3,6p-1)};q) = \sum_{s_p \geq s_{p-1} \cdots \geq s_1 \geq 0} q^{s_p} (q^{s_p+1})_{s_p+1} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}. \quad (1.9)$$

When $p = 1$ we have the *false* theta function [19],

$$1 + q(1-q)I(\Sigma_{(2,3,5)}); q) = \sum_{n \geq 1} \chi(n) q^{\frac{n^2-1}{120}}, \tag{1.10}$$

where

$$\chi(n) = \begin{cases} 1, & \text{if } n \equiv 1, 11, 19, 29 \pmod{60} \\ -1, & \text{if } n \equiv 31, 41, 49, 59 \pmod{60} \\ 0, & \text{otherwise.} \end{cases} \tag{1.11}$$

The main result in [13] is the following Hecke-type expansion involving positive-definite quadratic forms:

$$(1-q)I(\Sigma_{(2,3,6b-1)}; q) = \frac{1}{(q)_\infty} \left(\sum_{r,s \geq 0} - \sum_{r,s<0} \right) (-1)^{r+s} q^{(b+\frac{1}{2})r^2 + 2rs + \frac{3}{2}s^2 + (b+\frac{1}{2})r + \frac{5}{2}s}. \tag{1.12}$$

Later Bringmann and the authors [6] considered the case of (± 2)-surgery on the trefoil knot K_1, which gives the Seifert manifolds $M_{(2,3,8)} = M(0; (2,1), (3,-2), (8,1))$ and $M_{(2,3,4)} = M(0; (2,1), (3,-2), (4,1))$, respectively. They found that

$$(1-q)I(M_{(2,3,8)}; q) = \sqrt{2} q^{\frac{1}{4}} \overline{\phi}_0(-q^{\frac{1}{2}}), \tag{1.13}$$

where

$$\overline{\phi}_0(q) = \frac{(-q)_\infty}{(q)_\infty} \left(\sum_{\substack{r,s \geq 0 \\ r \not\equiv s \pmod{2}}} - \sum_{\substack{r,s<0 \\ r \not\equiv s \pmod{2}}} \right) (-1)^{\frac{r-s-1}{2}} q^{\frac{(r+s+1)^2}{4} + (3r+2)s} \tag{1.14}$$

is a mock theta function, while

$$(1-q)I(M_{(2,3,4)}; q) = \sqrt{2} q^{\frac{1}{4}} g(-q^{\frac{1}{2}}), \tag{1.15}$$

where

$$2 + 2q^2 g(q) = (-q)_\infty + \sum_{n \geq 0} q^{n(3n+1)/2}(1 - q^{2n+1}) \tag{1.16}$$

is the sum of a modular form and a false theta function.

In this paper we find Hecke-type formulas for the unified WRT invariants of the Seifert manifolds constructed from $(-1/b)$-, $(+2)$-, and (-2)-surgeries on the left-handed torus knot $T^*_{(2,2t+1)}$. In the first case we obtain the integral homology sphere $\Sigma_{(2,2t+1,4bt+2b-1)}$ (see, *e.g.*, [24]), and from (1.5) and (1.6) we have the q-hypergeometric series

$$q^{t-1}(1-q)\, I(\Sigma_{(2,2t+1,4bt+2b-1)}; q)$$
$$= \sum_{\substack{s_b \geq s_{b-1} \geq \cdots \geq s_1 \geq 0 \\ s_b+1 = k_t \geq k_{t-1} \geq \cdots \geq k_1 \geq 1}} q^{s_b}(q^{s_b+1})_{s_b+1} \prod_{i=1}^{b-1} \prod_{j=1}^{t-1} q^{s_i(s_i+1)+k_j^2} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix} \begin{bmatrix} k_{j+1} + k_j - j + 2\sum_{\ell=1}^{j-1} k_\ell \\ k_{j+1} - k_j \end{bmatrix}. \tag{1.17}$$

When $t = 1$ this reduces to (1.9). However, instead of Hecke-type expansions with positive-definite quadratic forms as in (1.12), we shall prove that these unified WRT invariants may be expressed in terms of indefinite ternary theta functions. In stating the formulas, we make use of the Jacobi theta function

$$j(x, q) := \sum_{n \in \mathbb{Z}} (-x)^n q^{\binom{n}{2}} = (x)_\infty (q/x)_\infty (q)_\infty. \tag{1.18}$$

Theorem 1.1. *For positive integers b, t, i, define the indefinite ternary theta function*

$$S(b,t,i;q) = \frac{1}{(q)_\infty^3}\left(\sum_{\substack{n,u,v\geq 0\\ u\not\equiv v \pmod 2}} + \sum_{\substack{n,u,v<0\\ u\not\equiv v \pmod 2}}\right)(-1)^{\frac{u-v-1}{2}} q^{Q(b,t,i,n,u,v)}, \tag{1.19}$$

where

$$\begin{aligned}Q(b,t,i,n,u,v) = b(2b+1)n^2 + ((b+\frac{1}{2})(u+v-1)+2bi)n + \frac{1}{8}u^2 + \frac{1}{8}v^2 \\ + \frac{4t+3}{4}uv + \frac{u}{2}(1+i+t) + \frac{v}{2}(-1+i+t).\end{aligned} \tag{1.20}$$

Then we have

$$(1-q)\, I(\Sigma_{(2,2t+1,4bt+2b-1)};q) = q^{\frac{3}{8}-\frac{t}{2}} \sum_{i=1}^{2b} (-1)^i j(q^i, q^{2b+1}) q^{\binom{i}{2}-\frac{i}{2}} S(b,t,i;q). \tag{1.21}$$

In the case of (-2)-surgery on $T^*_{(2,2t+1)}$, we have the Seifert fibered manifold $M_{(2,2t+1,4t)} = M(0;(2,1),(2t+1,-t-1),(4t,1))$ [24]. Equations (1.5) and (1.8) give the q-series

$$\begin{aligned}&(1-q)\, I(M_{(2,2t+1,4t)};q)\\ &= \sqrt{2}q^{\frac{1}{4}} \sum_{\substack{n\geq 0\\ n+1=k_t\geq k_{t-1}\geq\cdots\geq k_1\geq 1}} q^{n+1-t}(q^{\frac{1}{2}};-q^{\frac{1}{2}})_{2n+1} \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1}+k_i-i+2\sum_{j=1}^{i-1}k_j \\ k_{i+1}-k_i\end{bmatrix}.\end{aligned} \tag{1.22}$$

For general t we have the following formula, again in terms of indefinite ternary theta functions.

Theorem 1.2. *For t a positive integer, define the indefinite ternary theta function $S_1(t;q)$ by*

$$S_1(t;q) = \frac{-q^{-t}}{(1-q)(q)_\infty(q^2;q^2)_\infty}\left(\sum_{\substack{n,u,v\geq 0\\ u\not\equiv v \pmod 2}} + \sum_{\substack{n,u,v<0\\ u\not\equiv v \pmod 2}}\right)(-1)^{n+\frac{u-v-1}{2}} q^{Q_1(t,n,u,v)}(1-q^n), \tag{1.23}$$

where

$$Q_1(t,n,u,v) = \frac{1}{2}n^2 + \frac{3}{2}n + (n+t+3)u + (n+t+1)v + \frac{1}{4}u^2 + \frac{1}{4}v^2 + \frac{4t+3}{2}uv + \frac{7}{4}. \tag{1.24}$$

Then we have

$$(1-q)\, I(M_{(2,2t+1,4t)};q) = \sqrt{2}q^{\frac{1}{4}} S_1(t;-q^{\frac{1}{2}}). \tag{1.25}$$

Finally, in the case of $(+2)$-surgery on $T^*_{(2,2t+1)}$, we obtain the Seifert manifold $M_{(2,2t+1,4t+4)} = M(0;(2,1),(2t+1,-t-1),(4t+4,1))$ [24]. Equations (1.5) and (1.7) give

$$\begin{aligned}&(1-q)\, I(M_{(2,2t+1,4t+4)};q)\\ &= \sqrt{2}q^{\frac{1}{4}} \sum_{\substack{n\geq 0\\ n+1=k_t\geq k_{t-1}\geq\cdots\geq k_1\geq 1}} (-1)^n q^{\frac{n}{2}+1-t}(q^{\frac{1}{2}};-q^{\frac{1}{2}})_{2n+1} \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1}+k_i-i+2\sum_{j=1}^{i-1}k_j \\ k_{i+1}-k_i\end{bmatrix}.\end{aligned} \tag{1.26}$$

For general t we have the following formulas, this time in terms of indefinite *binary* theta functions.

Theorem 1.3. *For t a positive integer define the indefinite binary theta function $S_2(t;q)$ by*

$$S_2(t;q) = \frac{-q^{-t}(-q)_\infty}{(q)_\infty}\left(\sum_{\substack{r,s\geq 0\\ r\not\equiv s \pmod 2}} - \sum_{\substack{r,s<0\\ r\not\equiv s \pmod 2}}\right)(-1)^{\frac{r-s-1}{2}}q^{Q_2(t,r,s)}, \tag{1.27}$$

where

$$Q_2(t,r,s) = \frac{1}{4}r^2 + \frac{1}{4}s^2 + \frac{4t+3}{2}rs + \frac{2t-1}{2}r + \frac{2t+3}{2}s + \frac{1}{4}. \tag{1.28}$$

Then we have

$$(1-q)\,I(M_{(2,2t+1,4t+4)};q) = \sqrt{2}q^{\frac{1}{4}}S_2(t;-q^{\frac{1}{2}}). \tag{1.29}$$

The proofs of Theorems 1.1 - 1.3 depend on the Rosso-Jones formula for the colored Jones polynomial of the torus knots knots $T^*_{(2,2t+1)}$ along with a number of facts about Bailey pairs. The relevant Bailey machinery is reviewed in the next section and the proofs are presented in Section 3.

In Section 4 we touch on the modular properties of the unified WRT invariants. Given that the base cases in (1.10) and (1.15) involve false theta functions, we do not necessarily expect any nice modular properties coming from Theorems 1.1 or 1.2. Indeed, we note that while these unified WRT invariants can be written in terms of indefinite theta series described in [1], a sign condition prevents us from finding a modular completion. On the other hand, we shall see that the q-series in (1.26) are mock theta functions.

We close in Section 5 by exhibiting the quantum modular properties of all of the unified WRT invariants in Theorems 1.1 - 1.3. See Proposition 5.1.

2. Bailey pairs

We require the notion of a Bailey pair [3, 30]. Recall that a Bailey pair relative to (a,q) is a pair of sequences $\alpha_n(a,q)$ and $\beta_n(a,q)$ related by

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}. \tag{2.1}$$

We shall often drop the q and simply refer to a Bailey pair relative to a. The classical Bailey lemma [2] is the following.

Lemma 2.1. *If (α_n,β_n) is a Bailey pair relative to a, then so is (α'_n,β'_n), where*

$$\alpha'_n = \frac{(b)_n(c)_n(aq/bc)^n}{(aq/b)_n(aq/c)_n}\alpha_n, \tag{2.2}$$

and

$$\beta'_n = \sum_{k=0}^{n} \frac{(b)_k(c)_k(aq/bc)_{n-k}(aq/bc)^k}{(aq/b)_n(aq/c)_n(q)_{n-k}}\beta_k. \tag{2.3}$$

Inserting (2.2) and (2.3) back in the definition (2.1) and letting $n \to \infty$, we have the following corollary.

Corollary 2.2. *If (α_n, β_n) is a Bailey pair relative to a, then*

$$\sum_{n\geq 0} (b)_n (c)_n (aq/bc)^n \beta_n = \frac{(aq/b)_\infty (aq/c)_\infty}{(aq)_\infty (aq/bc)_\infty} \sum_{n\geq 0} \frac{(b)_n (c)_n (aq/bc)^n}{(aq/b)_n (aq/c)_n} \alpha_n. \tag{2.4}$$

While the Bailey lemma is sufficient for many of the uses of Bailey pairs, for our purposes we will require several non-standard facts. First, we have two results of the second author and Osburn [21].

Lemma 2.3 (see Theorem 2.1 of [21])**.** *If (α_n, β_n) is a Bailey pair relative to 1 with $\alpha_0 = \beta_0 = 0$, then (α'_n, β'_n) is also a Bailey pair relative to 1, where*

$$\alpha'_n = \frac{-1}{1-q^{2n+2}} \alpha_{n+1} + \frac{q^{2n-2}}{1-q^{2n-2}} \alpha_{n-1}, \tag{2.5}$$

and

$$\beta'_n = -(1-q^{2n+1}) \beta_{n+1}. \tag{2.6}$$

Lemma 2.4 (see Theorem 1.2 of [21])**.** *If (α_n, β_n) is a Bailey pair relative to 1 with $\alpha_0 = \beta_0 = 0$, then (α^*_n, β^*_n) is a Bailey pair relative to q, where*

$$\alpha^*_n = \frac{1}{1-q} \left(\frac{-\alpha_{n+1}}{1-q^{2n+2}} + \frac{q^{2n}\alpha_n}{1-q^{2n}} \right), \tag{2.7}$$

and

$$\beta^*_n = -\beta_{n+1}. \tag{2.8}$$

Next we have a result of the second author [20].

Lemma 2.5 (see equation (1.21) of [20])**.** *If (α_n, β_n) is a Bailey pair relative to (a^2, q^2), then*

$$\sum_{n\geq 0} (-aq)_n q^{2n} \beta_n = \frac{1}{(aq)_\infty (q^2;q^2)_\infty (1-q)} \sum_{n,r\geq 0} (-a)^n q^{\binom{n+1}{2}+2nr+2r} (1-q^{n+1}) \alpha_r. \tag{2.9}$$

Finally, we need a new result.

Lemma 2.6. *If (α_n, β_n) is a Bailey pair relative to 1, then for any $k \geq 1$ we have*

$$\sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} (q^{n_k+1})_{n_k} q^{n_k} \beta_{n_k} q^{n_{k-1}^2 + n_{k-1} + \cdots + n_1^2 + n_1} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix} \begin{bmatrix} n_{k-1} \\ n_{k-2} \end{bmatrix} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}$$
$$= \frac{1}{(q)_\infty^3} \sum_{i=1}^{2k} (-1)^{i-1} q^{\binom{i}{2}} j(q^i, q^{2k+1}) \sum_{r,n\geq 0} q^{kn((2k+1)n+2i)+(2k+1)rn+ri} \alpha_r. \tag{2.10}$$

Proof. For $k=1$ this is the case $a=1$ of [20, Equation (1.10)]. For $k\geq 2$ Warnaar [31, Proposition 6.1] has shown that

$$\begin{aligned}\sum_{n\geq 0}\frac{(a^2q)_{2n}q^n}{(a^2q)_n(q)_n}&\sum_{n_{k-1},\ldots,n_1\geq 0}\frac{q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}}{(aq)_{n-n_{k-1}}(q)_{n_{k-1}-n_{k-2}}\cdots(q)_{n_2-n_1}(q)_{n_1}}\\&=\frac{1}{(q)_\infty^2(aq)_\infty}\sum_{i=1}^{2k}(-a)^{i-1}q^{\binom{i}{2}}j(q^i,q^{2k+1})\sum_{n\geq 0}a^{(2k+1)n}q^{kn((2k+1)n+2i)}.\end{aligned}\tag{2.11}$$

Here we have extended the notation in (1.2) to all integers n using

$$(a)_n:=\frac{(a)_\infty}{(aq^n)_\infty},\tag{2.12}$$

so that, in particular, $1/(q)_n=0$ if $n<0$. We set $a=q^r$ in Warnaar's identity and shift the summation on the left-hand side by r to obtain

$$\begin{aligned}\sum_{n\geq r}\frac{(q^{2r+1})_{2n-2r}q^{n-r}}{(q^{2r+1})_{n-r}(q)_{n-r}}&\sum_{n_{k-1},\ldots,n_1\geq 0}\frac{q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}}{(q^{r+1})_{n-n_{k-1}-r}(q)_{n_{k-1}-n_{k-2}}\cdots(q)_{n_2-n_1}(q)_{n_1}}\\&=\frac{1}{(q)_\infty^2(q^{r+1})_\infty}\sum_{i=1}^{2k}(-1)^{i-1}q^{\binom{i}{2}+r(i-1)}j(q^i,q^{2k+1})\sum_{n\geq 0}q^{(2k+1)nr+kn((2k+1)n+2i)}.\end{aligned}\tag{2.13}$$

Simplifying the q-factorials on the left-hand side, we have

$$\begin{aligned}\sum_{n\geq r}\frac{(q)_{2n}q^n}{(q)_{n+r}(q)_{n-r}}&\sum_{n\geq n_{k-1}\geq n_1\geq 0}\frac{q^{n_{k-1}^2+n_{k-1}+\cdots+n_1^2+n_1}}{(q)_{n-n_{k-1}}(q)_{n_{k-1}-n_{k-2}}\cdots(q)_{n_2-n_1}(q)_{n_1}}\\&=\frac{1}{(q)_\infty^3}\sum_{i=1}^{2k}(-1)^{i-1}q^{\binom{i}{2}+ri}j(q^i,q^{2k+1})\sum_{n\geq 0}q^{(2k+1)nr+kn((2k+1)n+2i)}.\end{aligned}\tag{2.14}$$

Next on both sides we multiply by α_r and sum over the non-negative integers r. Interchanging the summations over n and r on the left-hand side, recalling the definition of the Bailey pair in (2.1), and rewriting the multisum in terms of q-binomial coefficients gives the statement of the lemma. □

3. Proofs of Theorems 1.1 – 1.3

Recall the polynomials $C_n(T^*_{(2,2t+1)};q)$ defined in (1.5). A key role in all of the proofs in this section is played by the following Bailey pair, which comes from the Rosso–Jones formula [27] for the colored Jones polynomial and an inversion relation between the colored Jones polynomial and its cyclotomic coefficients (see [15]).

Proposition 3.1. *The sequences (α_n,β_n) form a Bailey pair relative to 1, where*

$$\alpha_n=q^{(t+1)n^2-n}(1-q^{2n})\sum_{k=-n}^{n-1}(-1)^kq^{-(t+\frac{1}{2})k^2-(t-\frac{1}{2})k},\tag{3.1}$$

$$\beta_n=-q^{t-n}C_{n-1}(T^*_{(2,2t+1)};q).\tag{3.2}$$

Note that we have $\alpha_0=\beta_0=0$. We prove Theorems 1.1 – 1.3 in reverse order, beginning with $(+2)$-surgery.

Proof of Theorem 1.3. We require the case $a = 1$, $q = q^2$, $b = -1$ and $c = -q$ of Corollary 2.2. If $\alpha_0 = \beta_0 = 0$ (which is the case for us), then we have

$$\sum_{n\geq 1}(-q)_{2n-1}q^n\beta_n(1,q^2) = \frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{q^n}{1+q^{2n}}\alpha_n(1,q^2). \tag{3.3}$$

Now, we compute

$$\begin{aligned}
&\sum_{n\geq 0}q^{-n}(-q)_{2n+1}C_n(T^*_{(2,2t+1)};q^2)\\
&\quad= \sum_{n\geq 1}q^{-n+1}(-q)_{2n-1}C_{n-1}(T^*_{(2,2t+1)};q^2)\\
&\quad= -q^{1-2t}\sum_{n\geq 1}q^n(-q)_{2n-1}\beta_n(1,q^2) \qquad\qquad \text{(by (3.2))}\\
&\quad= \frac{-q^{1-2t}(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{q^n}{1+q^{2n}}\alpha_n(1,q^2) \qquad\qquad \text{(by (3.3))}\\
&\quad= \frac{-q^{1-2t}(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}q^{2(t+1)n^2-n}(1-q^{2n})\sum_{j=-n}^{n-1}(-1)^j q^{-(2t+1)j^2-(2t-1)j}\\
&\quad= \frac{-q^{1-2t}(-q)_\infty}{(q)_\infty}\left(\sum_{\substack{n\geq 1\\ -n\leq j\leq n-1}} - \sum_{\substack{n\leq 1\\ n\leq j\leq -n-1}}\right)(-1)^j q^{2(t+1)n^2-n-(2t+1)j^2-(2t-1)j}.
\end{aligned}$$

Letting $n = (r+s+1)/2$ and $j = (r-s-1)/2$ in these last two sums and simplifying gives

$$\sum_{n\geq 0}q^{-n}(-q)_{2n+1}C_n(T^*_{(2,2t+1)};q^2) = S_2(t;q). \tag{3.4}$$

Setting $q = -q^{1/2}$ and comparing with (1.7) gives (1.29) and completes the proof. $\square$

Next we turn to the (-2)-surgery.

Proof of Theorem 1.2. We begin by applying Lemma 2.4 to the Bailey pair in Proposition 3.1 with $q = q^2$. This gives a Bailey pair (α^*_n, β^*_n) relative to (q^2, q^2), where

$$\begin{aligned}
\alpha^*_n = \frac{1}{1-q^2}\Bigg(&-q^{2(t+1)(n+1)^2-2(n+1)}\sum_{k=-n-1}^{n}(-1)^k q^{-(2t+1)k^2-(2t-1)k}\\
&+q^{2(t+1)n^2+2n}\sum_{k=-n}^{n-1}(-1)^k q^{-(2t+1)k^2-(2t-1)k}\Bigg)
\end{aligned} \tag{3.5}$$

and

$$\beta^*_n = q^{2t-2n-2}C_n(T^*_{(2,2t+1)};q^2). \tag{3.6}$$

Next we note the case $a = q$ of Lemma 2.5,

$$\sum_{n\geq 0}(-q)_{2n+1}q^{2n}\beta_n = \frac{1+q}{(q)_\infty(q^2;q^2)_\infty}\sum_{n,r\geq 0}(-1)^n q^{\binom{n+1}{2}+2nr+2r+n}(1-q^{n+1})\alpha_r. \tag{3.7}$$

Now using (3.5), (3.6), and (3.7) we compute

$$
\begin{aligned}
&\sum_{n\geq 0} C_n(T^*_{(2,2t+1)};q^2)(-q)_{2n+1} \\
&\quad = q^{2-2t}\sum_{n\geq 0}\beta^*_n q^{2n}(-q)_{2n+1} \\
&\quad = \frac{q^{2-2t}}{(1-q)(q^2;q^2)_\infty (q)_\infty} \\
&\qquad \times \Bigg(\sum_{\substack{n\geq 0\\ r\geq 0}} \sum_{k=-r}^{r-1} (-1)^{n+k}(1-q^{n+1}) q^{\binom{n+1}{2}+n+2nr+2r+2(t+1)r^2+2r-(2t+1)k^2-(2t-1)k} \\
&\qquad\quad - \sum_{\substack{n\geq 0\\ r\geq 0}} \sum_{k=-r-1}^{r} (-1)^{n+k}(1-q^{n+1}) q^{\binom{n+1}{2}+n+2nr+2r+2(t+1)(r+1)^2-2(r+1)-(2t+1)k^2-(2t-1)k} \Bigg).
\end{aligned}
$$

Replacing n by $n-1$ and letting $r=(u+v+1)/2$ and $k=(u-v-1)/2$ in the first sum, replacing n by $-n-1$ and letting $r=(-u-v-3)/2$ and $k=(u-v-1)/2$ in the second sum, and then simplifying we obtain

$$
\sum_{n\geq 0} C_n(T^*_{(2,2t+1)};q^2)(-q)_{2n+1} = S_1(t;q). \tag{3.8}
$$

Setting $q=-q^{1/2}$ and comparing with (1.7) gives (1.25) and completes the proof. $\square$

Finally, we treat $(-1/b)$-surgery.

Proof of Theorem 1.1. Recall our key Bailey pair from Proposition 3.1. Inserting this into Lemma 2.3 gives a Bailey pair (α'_n,β'_n) relative to 1, where

$$
\begin{aligned}
\alpha'_n = &-q^{(t+1)(r+1)^2-(r+1)} \sum_{j=-r-1}^{r} (-1)^j q^{-(t+\frac{1}{2})j^2-(t-\frac{1}{2})j} \\
&+ q^{(t+1)(r-1)^2-(r-1)+(2r-2)} \sum_{j=-r+1}^{r-2} (-1)^j q^{-(t+\frac{1}{2})j^2-(t-\frac{1}{2})j},
\end{aligned} \tag{3.9}
$$

and

$$
\beta'_n = (1-q^{2n+1})q^{t-n-1}C_n(T^*_{(2,2t+1)};q). \tag{3.10}
$$

Using this together with Lemma 2.6 we compute as follows:

$$
\begin{aligned}
&\sum_{s_b \geq 0} (q^{s_b+1})_{s_b+1} C_{s_b}(T^*_{(2,2t+1)};q) \sum_{s_b \geq s_{b-1} \geq \cdots \geq s_1 \geq 0} \prod_{i=1}^{b-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix} \\
&= \sum_{s_b \geq 0} (q^{s_b+1})_{s_b} (1-q^{2s_b+1})(-q^{s_b-t+1}) \beta_{s_b} \sum_{s_b \geq s_{b-1} \geq \cdots \geq s_1 \geq 0} \prod_{i=1}^{b-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix} \\
&= q^{1-t} \sum_{s_b \geq 0} (q^{s_b+1})_{s_b} \beta'_{s_b} \sum_{s_b \geq s_{b-1} \geq \cdots \geq s_1 \geq 0} \prod_{i=1}^{b-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix} \\
&= \frac{q^{1-t}}{(q)_\infty^3} \sum_{i=1}^{2k} (-1)^{i-1} q^{\binom{i}{2}} j(q^i, q^{2k+1}) \sum_{r,n \geq 0} q^{kn((2k+1)n+2i)+(2k+1)rn+ri} \alpha'_r \\
&= -\frac{q^{1-t}}{(q)_\infty^3} \sum_{i=1}^{2k} (-1)^{i-1} q^{\binom{i}{2}} j(q^i, q^{2k+1}) \\
&\quad \times \sum_{r,n \geq 0} q^{kn((2k+1)n+2i)+(2k+1)rn+ri} \cdot q^{(t+1)(r+1)^2-(r+1)} \sum_{j=-r-1}^{r} (-1)^j q^{-(t+\frac{1}{2})j^2-(t-\frac{1}{2})j} \\
&+ \frac{1}{(q)_\infty^3} \sum_{i=1}^{2k} (-1)^{i-1} q^{\binom{i}{2}} j(q^i, q^{2k+1}) \\
&\quad \times \sum_{r,n \geq 0} q^{kn((2k+1)n+2i)+(2k+1)rn+ri} \cdot q^{(t+1)(r-1)^2-(r-1)+(2r-2)} \sum_{j=-r+1}^{r-2} (-1)^j q^{-(t+\frac{1}{2})j^2-(t-\frac{1}{2})j}.
\end{aligned}
$$

Substituting $(r,j,n,i) = ((u+v-1)/2, (u-v-1)/2, n, i)$ in the first sum on the right-hand side and then $(r,j,n,i) = ((-u-v+1)/2, (u-v-1)/2, -n-1, 2k+1-i)$ in the second sum and simplifying gives

$$
\begin{aligned}
&\sum_{s_b \geq 0} (q^{s_b+1})_{s_b+1} C_{s_b}(T^*_{(2,2t+1)};q) \sum_{s_b \geq s_{b-1} \geq \cdots \geq s_1 \geq 0} \prod_{i=1}^{b-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix} \\
&\qquad = \frac{-q^{\frac{3}{8}-\frac{1}{2}t}}{(q)_\infty^3} \sum_{i=1}^{2k} (-1)^{i-1} q^{\binom{i}{2}-\frac{i}{2}} j(q^i, q^{2k+1}) \\
&\qquad\quad \times \Bigg(\sum_{\substack{n,u,v \geq 0 \\ u \not\equiv v \pmod 2}} + \sum_{\substack{n,u,v < 0 \\ u \not\equiv v \pmod 2}} \Bigg) (-1)^{(u-v-1)/2} Q(k,t,i,n,u,v),
\end{aligned}
\tag{3.11}
$$

with $Q(k,t,i,n,u,v)$ defined in (1.20). This completes the proof of Theorem 1.1. $\square$

4. Modularity

In this section we examine the modularity of our unified WRT invariants as functions of q for $|q| < 1$, or equivalently via $q := e^{2\pi i z}$, as functions of z for $z \in \mathbb{C}$ with $\Im(z) > 0$. The quantum modularity as functions at rational numbers z (or roots of unity q) is treated in Section 5.

4.1 Mock Modularity

Let $g(z)$ be a modular form of weight $2-k$. Following Zagier [33], a mock modular form of weight k with shadow $g(z)$ is a series

$$F(q) = \sum_{n \geq n_0} a(n) q^n, \tag{4.1}$$

such that for some rational number λ a completion

$$q^\lambda \widehat{F}(q) := q^\lambda F(q) + \int_{-\overline{z}}^{\infty} \frac{g(w)}{(z+w)^k} dw \tag{4.2}$$

transforms like a modular form of weight k for some congruence subgroup of $SL_2(\mathbb{Z})$. A mock theta function is a mock modular form of weight 1/2 whose shadow is a unary theta series. A fundamental building block of mock theta functions is the Appell-Lerch series, which we write following Hickerson and Mortenson [9] as

$$m(x,q,z) = \frac{1}{j(z,q)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n z^n q^{\binom{n}{2}}}{1 - zxq^{n-1}}. \tag{4.3}$$

Unlike the indefinite ternary theta functions in Theorems 1.1 and 1.2, the Hecke-type series in Theorem 1.3 are indefinite binary theta functions. This leads to the following.

Corollary 4.1. *The functions $S_2(t;q)$ are mock theta functions.*

Proof. The series in (1.27) are written in terms of the fundamental Hecke-type series

$$f_{a,b,c}(x,y,q) = \left(\sum_{r,s \geq 0} - \sum_{r,s<0} \right) (-1)^{r+s} x^r y^s q^{a\binom{r}{2} + brs + c\binom{s}{2}}$$

as

$$S_2(t;q) = \frac{(-q)_\infty}{(q)_\infty} \left(-f_{1,4t+3,1}(q^{2t+1}, q^{6t+7}, q^2) + q^2 f_{1,4t+3,1}(q^{2t+5}, q^{6t+3}, q^2) \right). \tag{4.4}$$

From work of Zwegers [34], products of modular forms and $f_{a,b,c}(x,y,q)$ like those above are known to be mixed mock modular forms, by which we mean finite sums $\sum_i f_i g_i$, where f_i is a modular form and g_i is a mock theta function. To help see whether these are pure mock theta functions, one may use a result of Hickerson and Mortenson [9, Theorem 1.3] to express them in terms of Appell-Lerch series. Their result is too involved to quote here in full, but applying it to (4.4) we find that up to the addition of an explicit modular form, $S_2(t;q)$ is

$$\begin{aligned} &(-1)^{t+1} q^{-t^2} m(-q^{8t^2+24t+14}, q^{32t^2+48t+16}, -1) \\ &\quad + (-1)^{t+1} q^{-t^2-2t+1} m(-q^{8t^2+16t+10}, q^{32t^2+48t+16}, -1) \\ &\quad\quad + (-1)^t q^{-t^2-4t-2} m(-q^{8t^2+8t-2}, q^{32t^2+48t+16}, -1) \\ &\quad\quad\quad + (-1)^t q^{-t^2-6t-5} m(-q^{8t^2-6}, q^{32t^2+48t+16}, -1). \end{aligned} \tag{4.5}$$

Since the Appell–Lerch functions are mock theta functions, this completes the proof. $\square$

Note that the case $t=1$ of (4.4) appears in [23] (where $S_2(1;q)$ is called $\overline{\phi_0}(q)$, following [6]).

A modular completion of $S_2(t;q)$ can also be deduced from work of Zwegers [34] as follows. Letting $(r,s)=(2r,2s+1)$ or $(2r+1,2s)$ in Theorem 1.3 we have

$$\frac{(q)_\infty}{(-q)_\infty}S_2(t;q)=q^2\left(\sum_{r,s\geq 0}-\sum_{r,s<0}\right)(-1)^{r+s}q^{r^2+s^2+(8t+6)rs+(2t+4)s+(6t+2)r}$$
$$-\left(\sum_{r,s\geq 0}-\sum_{r,s<0}\right)(-1)^{r+s}q^{r^2+s^2+(8t+6)rs+(6t+6)s+2tr}$$

By use of $B(x,y)=x^\top\begin{pmatrix}2 & 2(4t+3)\\ 2(4t+3) & 2\end{pmatrix}y$ and $Q(x)=\frac{1}{2}B(x,x)$, we get

$$\begin{aligned}2\frac{(q)_\infty}{(-q)_\infty}\zeta_{8(t+1)}^{4t+3}q^{\frac{24t^3+32t^2-9}{8(2t+1)(t+1)}}S_2(t;q)\\ =\sum_{\nu\in\mathbb{Z}^2+a_1}\left(\operatorname{sgn}B(\nu,c_1)-\operatorname{sgn}B(\nu,c_2)\right)q^{Q(\nu)}e^{2\pi iB(\nu,b)}\\ -\sum_{\nu\in\mathbb{Z}^2+a_2}\left(\operatorname{sgn}B(\nu,c_1)-\operatorname{sgn}B(\nu,c_2)\right)q^{Q(\nu)}e^{2\pi iB(\nu,b)},\end{aligned}\tag{4.6}$$

where

$$c_1=\begin{pmatrix}-1\\ 4t+3\end{pmatrix},\qquad c_2=\begin{pmatrix}-4t-3\\ 1\end{pmatrix},$$
$$a_1=\begin{pmatrix}\frac{4t^2+8t+5}{8(2t+1)(t+1)}\\ \frac{12t^2+12t+1}{8(2t+1)(t+1)}\end{pmatrix},\qquad a_2=\begin{pmatrix}\frac{12t^2+20t+9}{8(2t+1)(t+1)}\\ \frac{4t^2-3}{8(2t+1)(t+1)}\end{pmatrix},\qquad b=\begin{pmatrix}\frac{1}{16(t+1)}\\ \frac{1}{16(t+1)}\end{pmatrix}.$$

A modular completion of the right hand side of (4.6) is then given by [34, Chapter 2]

$$\begin{aligned}\left(\sum_{\nu=\binom{r}{s}\in\mathbb{Z}^2+a_1}-\sum_{\nu=\binom{r}{s}\in\mathbb{Z}^2+a_2}\right)\left(E(2r\sqrt{8(2t+1)(t+1)y})+E(2s\sqrt{8(2t+1)(t+1)y})\right)\\ \times q^{Q(\nu)}e^{2\pi iB(\nu,b)}.\end{aligned}\tag{4.7}$$

where $y=\Im(\tau)$ with $q=e^{2\pi i\tau}$ and where the error function is $E(x)=2\int_0^x e^{-\pi w^2}dw$.

4.2 "False" indefinite ternary theta functions

In a recent paper [1] (see also [17]), Alexandrov et al proposed an explicit modular completion of indefinite theta series of signature $(n-2,2)$ (see [7, 25] for signature $(n-r,r)$). Therein a non-holomorphic theta function was constructed by use of a generalized error function as a solution of Vignéras' differential equation [29].

For a signature $(2,1)$ matrix $\mathbf{A}$, let $F_{\mathbf{A}}(z_1,z_2,z_3;\tau)$ be the indefinite theta series

$$F_{\mathbf{A}}(z_1,z_2,z_3;\tau)=\left(\sum_{\ell,m,n\geq 0}+\sum_{\ell,m,n<0}\right)q^{\frac{1}{2}(\ell,m,n)\mathbf{A}\begin{pmatrix}\ell\\ m\\ n\end{pmatrix}}\zeta_1^\ell\zeta_2^m\zeta_3^n,\tag{4.8}$$

where $q=e^{2\pi i\tau}$ and $\zeta_j=e^{2\pi iz_j}$. The signed sum has a kernel $\left(\operatorname{sgn}(\ell+\frac{1}{2})+\operatorname{sgn}(m+\frac{1}{2})\right)\left(\operatorname{sgn}(\ell+\frac{1}{2})+\operatorname{sgn}(n+\frac{1}{2})\right)$, which can be rewritten as

$\Phi_{\mathbf{A}}(\left(\begin{smallmatrix} \ell+\frac{1}{2} \\ m+\frac{1}{2} \\ n+\frac{1}{2}\end{smallmatrix}\right))$, where

$$\Phi_{\mathbf{A}}(x) = \frac{1}{4}\left(\operatorname{sgn} B_{\mathbf{A}}(c_0, x) - \operatorname{sgn} B_{\mathbf{A}}(c_1, x)\right)\left(\operatorname{sgn} B_{\mathbf{A}}(c_0, x) - \operatorname{sgn} B_{\mathbf{A}}(c_2, x)\right),$$

with suitable c_i. In [1], $\Phi_{\mathbf{A}}(x)$ was completed by using a generalization of the error function $E(x)$, and a modular completion $\widehat{F}_{\mathbf{A}}(z_1, z_2, z_3; \tau)$ was constructed explicitly under certain conditions on the c_i. These conditions were refined in [17], but in any case we require that $B_{\mathbf{A}}(c_i, c_i) < 0$.

Our indefinite q-series take the form of (4.8); Theorem 1.1 proves

$$\begin{aligned} S(b,t,i;q) = \frac{q^{\frac{t+i}{2}-\frac{3}{8}}}{(q)_\infty^3}\Big(& q\, F_{\mathbf{A}_0}\left(2bi\tau, \tfrac{2i+2t+3}{2}\tau + \tfrac{1}{2}, \tfrac{6t+2i+1}{2}\tau+\tfrac{1}{2}; \tau\right) \\ & - F_{\mathbf{A}_0}\left(2bi\tau, \tfrac{2i+6t+5}{2}\tau + \tfrac{1}{2}, \tfrac{2t+2i-1}{2}\tau+\tfrac{1}{2}; \tau\right)\Big), \end{aligned} \tag{4.9}$$

where

$$\mathbf{A}_0 = \begin{pmatrix} 2b(2b+1) & 2b+1 & 2b+1 \\ 2b+1 & 1 & 4t+3 \\ 2b+1 & 4t+3 & 1 \end{pmatrix}.$$

Theorem 1.2 shows

$$\begin{aligned} S_1(t;q) = \frac{-q^3}{(1-q)(q)_\infty(q^2;q^2)_\infty}\Big(& q^2 F_{\mathbf{A}_1}\left(\tfrac{5}{2}\tau+\tfrac{1}{2}, (2t+7)\tau+\tfrac{1}{2}, (6t+5)\tau+\tfrac{1}{2}; \tau\right) \\ & - q^2 F_{\mathbf{A}_1}\left(\tfrac{7}{2}\tau+\tfrac{1}{2}, (2t+7)\tau+\tfrac{1}{2}, (6t+5)\tau+\tfrac{1}{2}; \tau\right) \\ & + F_{\mathbf{A}_1}\left(\tfrac{5}{2}\tau+\tfrac{1}{2}, (6t+9)\tau+\tfrac{1}{2}, (2t+3)\tau+\tfrac{1}{2}; \tau\right) \\ & - F_{\mathbf{A}_1}\left(\tfrac{7}{2}\tau+\tfrac{1}{2}, (6t+9)\tau+\tfrac{1}{2}, (2t+3)\tau+\tfrac{1}{2}; \tau\right)\Big), \end{aligned} \tag{4.10}$$

where

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 2(4t+3) \\ 2 & 2(4t+3) & 2 \end{pmatrix}.$$

Both $\mathbf{A}_0$ and $\mathbf{A}_1$ are of signature $(2,1)$, but the kernels $\Phi_{\mathbf{A}}(x)$ are given with positive c_i, $B(c_i, c_i) > 0$; for (4.9), $c_0 = \left(\begin{smallmatrix} -4(t+1) \\ 2b+1 \\ 2b+1 \end{smallmatrix}\right)$, $c_1 = \left(\begin{smallmatrix} -2(2t+1) \\ 1 \\ 8bt+4b-1 \end{smallmatrix}\right)$, $c_2 = \left(\begin{smallmatrix} -2(2t+1) \\ 8bt+4b-1 \\ 1 \end{smallmatrix}\right)$, and for (4.10), $c_0 = \left(\begin{smallmatrix} 4(t+1) \\ -1 \\ -1 \end{smallmatrix}\right)$, $c_1 = \left(\begin{smallmatrix} 4(2t+1) \\ -1 \\ -4t-1 \end{smallmatrix}\right)$, $c_2 = \left(\begin{smallmatrix} 4(2t+1) \\ -4t-1 \\ -1 \end{smallmatrix}\right)$. This suggests that these q-series are *false* theta functions, not mock theta functions.

5. Quantum Modularity

Following Zagier [32], a *quantum modular form* is a function $g(z)$ on the rational numbers (or a suitable subset) such that for a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, the function

$$g_\gamma(z) = g(z) - (cz+d)^{-k} g(\gamma z)$$

has "nice" properties such as continuity or analyticity. As with classical modular forms, we allow the automorphy factor $(cz+d)^{-k}$ to be multiplied by a character or replaced by a multiplier system. In this section we show that the unified WRT invariants studied in the first part of this paper are quantum modular forms.

The WRT invariant $\tau_N(M)$ for the Seifert manifold $M = M(b; (p_1, q_1), (p_2, q_2), (p_3, q_3))$ is given by use of the colored Jones polynomial for the 4-component link as [16, 18, 26]

$$\begin{aligned} e^{\frac{2\pi \mathrm{i}}{N}\left(\frac{\phi}{4}-\frac{1}{2}\right)} \left(e^{\frac{2\pi\mathrm{i}}{N}}-1\right) \cdot \tau_N(M) = \frac{e^{\frac{\pi\mathrm{i}}{4}}}{\sqrt{2Np_1p_2p_3}} \sum_{k=1}^{N-1} \frac{e^{-b\frac{k^2}{2N}\pi\mathrm{i}}}{e^{\frac{k}{N}\pi\mathrm{i}}-e^{-\frac{k}{N}\pi\mathrm{i}}} \\ \times \prod_{i=1}^{3} \sum_{n_i \mod p_i} e^{-\frac{q_i}{2Np_i}(k+2Nn_i)^2} \left(e^{\frac{1}{Mp_i}(k+2Nn_i)\pi\mathrm{i}} - e^{-\frac{1}{Mp_i}(k+2Nn_i)\pi\mathrm{i}}\right). \end{aligned} \tag{5.1}$$

Here we use

$$\phi = \sum_{i=1}^{3} \left(12\, s(q_i, p_i) - \frac{q_i}{p_i} + 1\right),$$

where $s(b, a)$ is the Dedekind sum

$$s(b, a) = \frac{1}{4a} \sum_{k=1}^{a-1} \cot\left(\frac{k}{a}\pi\right) \cot\left(\frac{k}{a} b\pi\right).$$

A relationship between (5.1) and modular forms has been studied since the work of Lawrence–Zagier [19]. A building block in these studies is the Eichler integral $\widetilde{\Psi}_P^{(a)}(\tau)$ of the vector-valued modular form $\Psi_P^{(a)}(\tau)$,

$$\begin{aligned} \widetilde{\Psi}_P^{(a)}(\tau) &= \sum_{n=0}^{\infty} \psi_{2P}^{(a)}(n)\, q^{\frac{n^2}{4P}}, \\ \Psi_P^{(a)}(\tau) &= \frac{1}{2} \sum_{n\in\mathbb{Z}} n\, \psi_{2P}^{(a)}(n)\, q^{\frac{n^2}{4P}}, \end{aligned}$$

where

$$\psi_{2P}^{(a)}(n) = \begin{cases} \pm 1, & \text{for } n = \pm a \pmod{2P}, \\ 0, & \text{otherwise.} \end{cases}$$

As $\tau \to \frac{1}{N}$ ($N \in \mathbb{Z}_{>0}$), the Eichler integral reduces to

$$\widetilde{\Psi}_P^{(a)}(\tfrac{1}{N}) = -\sum_{k=0}^{2PN} \psi_{2P}^{(a)}(k)\, e^{\frac{k^2}{2PN}\pi\mathrm{i}}\, B_1\left(\frac{k}{2PN}\right), \tag{5.2}$$

where the Bernoulli polynomial is $B_1(x) = x - \frac{1}{2}$. A crucial property is the near modularity [10],

$$\widetilde{\Psi}_P^{(a)}(\tfrac{1}{N}) + \sqrt{\frac{N}{\mathrm{i}}} \sum_{b=1}^{P-1} \sqrt{\frac{2}{P}} \sin\left(\frac{ab}{P}\pi\right) \widetilde{\Psi}_P^{(b)}(-N) \simeq \sum_{k=0}^{\infty} \frac{L(-2k, \psi_{2P}^{(a)})}{k!} \left(\frac{\pi\mathrm{i}}{2PN}\right)^k. \tag{5.3}$$

Here a generating function of the Dirichlet L-function is

$$\frac{\sinh((P-a)z)}{\sinh(Pz)} = \sum_{k=0}^{\infty} \frac{L(-2k, \psi_{2P}^{(a)})}{(2k)!}\, z^{2k},$$

and the Eichler integral at $N \in \mathbb{Z}$ is

$$\widetilde{\Psi}_P^{(a)}(N) = \left(1 - \frac{a}{P}\right) e^{\frac{a^2}{2P}\pi\mathrm{i}N}.$$

A result of [19] on the Poincaré homology sphere $\Sigma_{(2,3,5)}$, which is obtained by a (-1)-surgery on $T^*_{(2,3)}$ (the case $t = b = 1$ in our notation), can be read as

$$e^{\frac{2\pi \mathrm{i}}{N}} (e^{\frac{2\pi \mathrm{i}}{N}} - 1)\, \tau_N(\Sigma_{(2,3,5)}) = 1 + \frac{1}{2} e^{-\frac{\pi \mathrm{i}}{60N}}\, \widetilde{\Psi}_{30}^{(1)+(11)+(19)+(29)}(\tfrac{1}{N}). \tag{5.4}$$

Here and thereafter we use $\Psi_P^{k_a(a)+k_b(b)+\cdots}(z) = k_a \Psi_P^{(a)}(z) + k_b \Psi_P^{(b)}(z) + \cdots$ for brevity. For other cases, an expression of the WRT invariant for the Brieskorn homology sphere [11] can be read as

$$\begin{aligned} & e^{\frac{2\pi \mathrm{i}}{N}\left(\frac{6t+2b-1}{8} - \frac{1}{2(2t+1)} + \frac{2t+1}{2(4bt+2b-1)}\right)} (e^{\frac{2\pi \mathrm{i}}{N}} - 1)\, \tau_N(\Sigma_{(2,2t+1,4bt+2b-1)}) \\ & \qquad = \frac{1}{2} \widetilde{\Psi}_{2(2t+1)(4bt+2b-1)}^{(8bt^2-6t-2b-1)-(8bt^2+2t-2b+3)-(8bt^2+16bt-6t+6b-5)+(8bt^2+16bt+2t+6b-1)}(\tfrac{1}{N}). \end{aligned} \tag{5.5}$$

In the case of (± 2)-surgery on the torus knot $T^*_{(2,2t+1)}$, both Seifert manifolds $M_{(2,2t+1,4t+4)}$ and $M_{(2,2t+1,4t)}$ have $H_1(M;\mathbb{Z}) = \mathbb{Z}_2$. As can be checked from (5.1), the WRT invariants vanish at odd N. At even N, the unified WRT invariants, (1.22) and (1.26), coincide with the WRT invariants (5.1). In [12, 14], the case $t = 1$ was studied. For general $t > 1$, we have explicitly

$$(e^{\frac{2\pi \mathrm{i}}{N}} - 1)\, \tau_N(M_{(2,2t+1,4t)}) = \frac{e^{-\frac{2\pi \mathrm{i}}{N} \frac{24t^3+16t^2-4t+1}{16t(2t+1)}}}{\sqrt{2}} \widetilde{\Psi}_{8t^2+4t}^{(4t^2-4t-1)-(4t^2+1)-(4t^2+4t-1)+(4t^2+8t+1)}(\tfrac{1}{N}), \tag{5.6}$$

$$(e^{\frac{2\pi \mathrm{i}}{N}} - 1)\, \tau_N(M_{(2,2t+1,4t+4)}) = \frac{e^{-\frac{2\pi \mathrm{i}}{N} \frac{24t^3+24t^2-12t-13}{16(2t+1)(t+1)}}}{\sqrt{2}} \widetilde{\Psi}_{4(t+1)(2t+1)}^{(4t^2-3)-(4t^2+4t-1)-(4t^2+8t+5)+(4t^2+12t+7)}(\tfrac{1}{N}). \tag{5.7}$$

An asymptotic expansion (5.3) of the Eichler integrals proves the quantum modularity of our unified WRT invariants, (1.17), (1.22), and (1.26) at the N-th root of unity $\exp(2\pi i/N)$.

Proposition 5.1. *The q-series* (1.21)*,* (1.25)*, and* (1.29) *are quantum modular forms.*

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