ARITHMETIC PROPERTIES OF SCHUR-TYPE OVERPARTITIONS

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ABSTRACT. We investigate arithmetic properties of an overpartition counting function that first arose in connection with Schur's partition theorem and a universal mock theta function. Motivated by work of Basil Gordon on the Rogers-Ramanujan identities, we first give a complete characterization of the parity of this overpartition function in the progressions $2n+1$, $4n+2$, and $8n+4$ in terms of the factorization of $An+B$ for certain A and B. We then find similar characterizations of the residue modulo 4 in the progressions $8n+5$ and $8n+7$. Finally, we prove some Ramanujan-type congruences modulo 5. Our proofs use basic facts about modular forms and some elementary algebraic number theory.

1. Introduction and Statement of Results

A number of recent works have investigated the parity of the counting functions in some classical partition identities. For an elegant example, consider the Rogers-Ramanujan identities. These state that for $i = 1$ or 2,

$$\sum_{n\geq 0} H_i(n)q^n = \frac{1}{(q^i;q^5)_\infty (q^{5-i};q^5)_\infty}, \tag{1.1}$$

where $H_i(n)$ denotes the number of partitions of n where parts are at least i and differ by at least 2. Here we have used the usual q-series notation,

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n). \tag{1.2}$$

Using the classical theory of quadratic forms, Gordon [7] showed that if n is odd then $H_1(n)$ is odd if and only if

$$60n-1 = p^{4a+1}m^2 \tag{1.3}$$

for some prime p not dividing m, and if n is even then $H_2(n)$ is odd if and only if

$$60n+11 = p^{4a+1}m^2 \tag{1.4}$$

for some prime p not dividing m.

To give another nice example, let $A(n)$ denote the number of partitions of n where parts differ by at least 3 and multiples of 3 differ by at least 6. Schur's

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celebrated partition theorem then says that

$$\sum_{n\geq 0} A(n)q^n = \frac{1}{(q;q^6)_\infty (q^5;q^6)_\infty}. \tag{1.5}$$

Cao and Chen [4] showed that the generating function

$$\sum_{n\geq 0} A(2n+1)q^{24n+11} \tag{1.6}$$

is congruent modulo 2 to a weight 3/2 Hecke eigenform and used this to give linear congruences for $A(n)$ modulo 2. Specifically, if $p \geq 5$ is a prime such that

$$A\left(\frac{11p^2+1}{12}\right) \equiv 0 \pmod 2, \tag{1.7}$$

then

$$A\left(2p^2 n + \frac{11p^2+1}{12}\right) \equiv 0 \pmod 2 \tag{1.8}$$

whenever $n \not\equiv \frac{-11}{24} \pmod p$. Further studies of the parity of $H_i(n)$ and $A(n)$ can be found in [5, 6].

There are numerous results like Schur's theorem and the Rogers-Ramanujan identities in the theory of partitions, and it is natural to ask about the parity (or other arithmetic properties) of the counting functions associated with these identities. Here we consider a certain counting function $S(n)$, which first arose in an overpartition identity related to Schur's partition theorem and a universal mock theta function [2, 14]. We take a moment to give the combinatorial definition of $S(n)$, though all that is required in the sequel is the generating function in (1.10) below.

Recall that an overpartition is a partition in which the final occurrence of a given integer may be overlined. We define the matrix

$$\overline{A}_{3,1} = \begin{array}{c} \\ \overline{1} \\ \overline{2} \\ \overline{3} \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \overline{1} & \overline{2} & \overline{3} & 3 \end{array} \\ \begin{pmatrix} 3 & 2 & 4 & 1 \\ 4 & 3 & 5 & 2 \\ 5 & 4 & 6 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix} \end{array}. \tag{1.9}$$

Now let $S(n)$ denote the number of overpartitions $(\lambda_1, \lambda_2, \dots, \lambda_s)$ of n where only parts divisible by 3 may occur non-overlined, with the conditions

(1) The smallest part is $\overline{1}$, $\overline{2}$, $\overline{3}$, or 6 modulo 6;
(2) For $u, v \in \{\overline{1}, \overline{2}, \overline{3}, 3\}$, if $\lambda_i \equiv u \pmod 3$ and $\lambda_{i+1} \equiv v \pmod 3$, then $\lambda_i - \lambda_{i+1} \geq \overline{A}_{3,1}(u,v)$;
(3) For $u, v \in \{\overline{1}, \overline{2}, \overline{3}, 3\}$, if $\lambda_i \equiv u \pmod 3$ and $\lambda_{i+1} \equiv v \pmod 3$, then $\lambda_i - \lambda_{i+1} \equiv \overline{A}_{3,1}(u,v) \pmod 6$. In words, the *actual* difference between two parts must be congruent modulo 6 to the *smallest allowable* difference.

A special case of the main result in [2] may be written as the identity

$$\sum_{n\geq 0} S(n)q^n = \frac{1}{(q;q^6)_\infty (q^5;q^6)_\infty (q^6;q^6)_\infty}. \tag{1.10}$$

In light of the similarity to (1.5), along with the fact that Schur's partition identity also corresponds to a universal mock theta function [3], we say that the overpartitions counted by $S(n)$ are of *Schur-type*.[1]

In the first part of this paper we study the parity of $S(n)$ in the progressions $2n+1$, $4n+2$, and $8n+4$. In the first case we find that the generating function is congruent modulo 2 to a unary theta function, while in the second two cases it is congruent to a binary theta function. In these cases we then obtain characterizations of the parity of $S(n)$ in terms of the factorization of $An+B$ for certain A and B, reminiscent of Gordon's results for the Rogers-Ramanujan functions $H_i(n)$.

Theorem 1.1. *For all $n \geq 0$, we have*

(1) *$S(2n+1)$ is odd if and only if $3n+1 = m^2$,*

(2) *$S(4n+2)$ is odd if and only if $12n+5 = p^{4a+1}m^2$ for some prime p not dividing m,*

(3) *$S(8n+4)$ is odd if and only if $24n+11 = p^{4a+1}m^2$ for some prime p not dividing m.*

From Theorem 1.1 one may easily deduce many families of linear congruences. We limit ourselves to just two examples.

Corollary 1.2. *If $p \equiv 1 \pmod{12}$ is prime, then*

$$S\left(4pn + \frac{5(p-1)}{3} + 2\right) \equiv 0 \pmod 2 \tag{1.11}$$

whenever $n \not\equiv \frac{5(p-1)}{12} \pmod p$. In particular, if $x \not\equiv \frac{5(p-1)}{12} \pmod p$ we have

$$S\left(4p^2n + 4px + \frac{5(p-1)}{3} + 2\right) \equiv 0 \pmod 2. \tag{1.12}$$

Corollary 1.3. *If $p \equiv 1 \pmod{24}$ is prime, then*

$$S\left(8pn + \frac{11(p-1)}{3} + 4\right) \equiv 0 \pmod 2 \tag{1.13}$$

whenever $n \not\equiv \frac{11(p-1)}{24} \pmod p$. In particular, if $x \not\equiv \frac{11(p-1)}{24} \pmod p$ we have

$$S\left(8p^2n + 8px + \frac{11(p-1)}{3} + 4\right) \equiv 0 \pmod 2. \tag{1.14}$$

Next we examine $S(n)$ modulo 4 in the subprogressions $8n+5$ and $8n+7$. In these cases we find results of the same nature as Theorem 1.1.

Theorem 1.4. *For $n \geq 0$, we have*

(1) *$S(8n+5)$ is not divisible by 4 (in which case it is 2 modulo 4) if and only if n is even and $12n+7 = p^{4a+1}m^2$ for some prime p not dividing m,*

(2) *$S(8n+7)$ is not divisible by 4 (in which case it is 2 modulo 4) if and only if $6n+5 = p^{4a+1}m^2$ for some prime $p \equiv 1, 3 \pmod 8$ not dividing m.*

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[1] These should not be confused with the overpartitions in the generalizations of Schur's theorem described in [8, 9]

Again, these results may be used to give congruences in arithmetic progressions, one example of which is the following.

Corollary 1.5. *If $p \equiv 1 \pmod{24}$ is prime, then*

$$S\left(16pn + \frac{14(p-1)}{3} + 5\right) \equiv 0 \pmod{4} \tag{1.15}$$

whenever $n \not\equiv \frac{7(p-1)}{24} \pmod{p}$. In particular, if $x \not\equiv \frac{7(p-1)}{24} \pmod{p}$ we have

$$S\left(16p^2n + 16px + \frac{14(p-1)}{3} + 5\right) \equiv 0 \pmod{4}. \tag{1.16}$$

In the last part of the paper we prove some congruences for $S(n)$ modulo 5. A key result is the following congruence relating $S(5n+2)$ and $S(20n+7)$.

Theorem 1.6. *For all $n \geq 0$, we have*

$$S(20n+7) \equiv (-1)^{n+1}S(5n+2) \pmod{5}. \tag{1.17}$$

Therefore, given any *seed* congruence $S(An+B) \equiv 0 \pmod{5}$, where $An+B$ is a subprogression of $5n+2$, we immediately obtain a family of congruences. One such seed is

$$S(40n+12) \equiv 0 \pmod{5}, \tag{1.18}$$

leading to the family below.

Theorem 1.7. *For $n \geq 0$ and $\alpha \geq 0$ we have*

$$S\left(5 \cdot 2^{2\alpha+3}n + \frac{35 \cdot 4^{\alpha} + 1}{3}\right) \equiv 0 \pmod{5}. \tag{1.19}$$

The rest of the paper is organized as follows. In the following section, we collect some background on modular forms, eta-quotients, and representations of integers by binary quadratic forms. In Sections 3 and 4, we prove the main results. We close in Section 5 with some observations and expectations regarding congruences for $S(n)$.

2. Background

We first recall some classical results on the number of representations of a natural number by the quadratic forms $x^2 + ky^2$ for $k = 1, 2$, or 3. These can be found in many places, such as [10, 11, 12].

Proposition 2.1. *Let m have the prime factorization $m = 2^c p_1^{a_1} \cdots p_k^{a_k} q_1^{b_1} \cdots q_\ell^{b_\ell}$, with the $p_i \equiv 1 \pmod{4}$ and $q_j \equiv 3 \pmod{4}$. If $r_1(m)$ denotes the number of solutions to $x^2 + y^2 = m$ with $x, y \in \mathbb{Z}$, then*

$$r_1(m) = \begin{cases} 0, & \text{if any } b_j \text{ is odd,} \\ 4(a_1+1)\cdots(a_k+1), & \text{otherwise.} \end{cases} \tag{2.1}$$

Proposition 2.2. *Let m have the prime factorization $m = 2^c p_1^{a_1} \cdots p_k^{a_k} q_1^{b_1} \cdots q_\ell^{b_\ell}$, with the $p_i \equiv 1, 3 \pmod{8}$ and $q_j \equiv 5, 7 \pmod{8}$. If $r_2(m)$ denotes the number of solutions to $x^2 + 2y^2 = m$ with $x, y \in \mathbb{Z}$, then*

$$r_2(m) = \begin{cases} 0, & \text{if any } b_j \text{ is odd,} \\ 2(a_1+1)\cdots(a_k+1), & \text{otherwise.} \end{cases} \tag{2.2}$$

Proposition 2.3. *Let m have the prime factorization $m = 3^c p_1^{a_1} \cdots p_k^{a_k} q_1^{b_1} \cdots q_\ell^{b_\ell}$, with the $p_i \equiv 1 \pmod 3$ and $q_j \equiv 2 \pmod 3$. If $r_3(m)$ denotes the number of solutions to $x^2 + 3y^2 = m$ with $x, y \in \mathbb{Z}$, then*

$$r_3(m) = \begin{cases} 0, & \text{if any } b_j \text{ is odd,} \\ 2(a_1+1)\cdots(a_k+1), & \text{if all } b_j \text{ are even and } n \text{ is odd,} \\ 6(a_1+1)\cdots(a_k+1), & \text{if all } b_j \text{ are even and } n \text{ is even.} \end{cases} \tag{2.3}$$

Next we recall some basic facts about modular forms, especially those constructed from Dedekind's eta function,

$$\eta(z) = q^{1/24}(q;q)_\infty, \tag{2.4}$$

where $q = e^{2\pi i z}$. All of these facts can be found in [13]. In what follows, let $M_k(\Gamma_0(N), \chi)$ be the complex vector space of holomorphic, weight k, level N modular forms with character χ. When the character is trivial, we omit it and write $M_k(\Gamma_0(N))$.

Proposition 2.4. *Let*

$$f(z) = \prod_{1 \le \delta | N} \eta^{r_\delta}(\delta z), \tag{2.5}$$

and suppose that $f(z)$ satisifes the conditions

(1)
$$\sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24},$$

(2)
$$\sum_{\delta | N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

(3)
$$\sum_{\delta | N} \frac{(d,\delta)^2 r_\delta}{\delta} \ge 0.$$

Then $f(z) \in M_k(\Gamma_0(N), \chi)$, where $k = \sum_{\delta|N} \frac{1}{2} r_\delta$ and $\chi(d) = \left(\frac{(-1)^k s}{d}\right)$, with $s = \prod_{\delta|N} \delta^{r_\delta}$.

The first two parts of Proposition 2.4 ensure that $f(z)$ transforms like a modular form and the third condition guarantees that $f(z)$ is holomorphic at the cusps.

The next two results apply to generic modular forms. The first proposition allows one to reduce the proof of a congruence to a "reasonable" finite computation.

Proposition 2.5. *(Sturm's Criterion) Given $f(z) = \sum_{n=0}^{\infty} a(n) q^n$ and $f(z) \in M_k(\Gamma_0(N), \chi)$, if $f(z)$ satisfies*

(1) *$a(n) \in \mathbb{Z}$ for all n,*

(2) *$a(n) \equiv 0 \mod M$ for all $n \le \dfrac{kN}{12} \displaystyle\prod_{p|N} \left(1 + \frac{1}{p}\right)$,*

then $a(n) \equiv 0 \mod M$.

Note that this proposition applies equally well to identities, since $a(n) = 0$ if and only if $a(n) \equiv 0 \pmod{M}$ for every natural number M.

The second proposition determines how the U-operator preserves modularity, where the U-operator is defined via its action on q-expansions by

$$\sum_{n\geq 0} a(n)q^n \mid U(m) = \sum_{n\geq 0} a(mn)q^n. \tag{2.6}$$

Proposition 2.6. *Suppose that $f(z) \in M_k(\Gamma_0(N), \chi)$. If $m \mid N$, then*

$$f(z) \mid U(m) \in M_k(\Gamma_0(N), \chi). \tag{2.7}$$

Finally, we require three well-known theta function identities. The first is Jacobi's triple product identity, and the other two are corollaries of it.

Proposition 2.7 (Jacobi's Triple Product)**.** *If $z \neq 0$ is a complex number,*

$$(-zq; q^2)_\infty(-\frac{q}{z}; q^2)_\infty(q^2; q^2)_\infty = \sum_{n\in\mathbb{Z}} z^n q^{n^2}. \tag{2.8}$$

Proposition 2.8 (Euler's Pentagonal Number Theorem)**.**

$$(q; q)_\infty = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(3n+1)/2}. \tag{2.9}$$

Proposition 2.9 (Jacobi's identity)**.**

$$(q; q)_\infty^3 = \sum_{n\geq 0} (2n + 1)(-1)^n q^{n(n+1)/2}. \tag{2.10}$$

3. $S(n)$ modulo 2 and 4

Before proceeding to the proofs of the main results, we establish some dissection identities. First, we have 2-dissection identities for two eta quotients. These are certainly known, but we give proofs in order to emphasize the modular approach, which is systematic and can be easily applied to any identity. Of course, this requires knowing the identity in advance. To guess the infinite product corresponding to each arithmetic progression, we used the algorithm described in [1, Exercise 6.2]. Though we did not need it here, it should be noted that in the case where the generating function in an arithmetic progression is a linear combination of infinite products, one can use Smoot's implementation of Radu's algorithm [15] to guess (and even prove) the expression.

For the rest of the paper we use the notation

$$f_k = (q^k; q^k)_\infty. \tag{3.1}$$

Lemma 3.1. *We have the 2-dissections*

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{3.2}$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}. \tag{3.3}$$

Proof. Multiplying both sides by $f_1^7 f_3^7$ and converting to η notation using (2.4), the identity (3.2) is equivalent to

$$\eta^6(z)\eta^6(3z) = \frac{\eta^7(z)\eta^7(3z)\eta^2(8z)\eta^5(12z)}{\eta^2(2z)\eta(4z)\eta^4(6z)\eta^2(24z)} + \frac{\eta^7(z)\eta^7(3z)\eta^5(4z)\eta^2(24z)}{\eta^4(2z)\eta^2(6z)\eta^2(8z)\eta(12z)}. \tag{3.4}$$

By Proposition 2.4, this is an identity between modular forms in $M_6(\Gamma_0(24))$, and hence is confirmed by Proposition 2.5 after checking the first 24 coefficients in the q-expansion.

Equation (3.3) can be proved in a similar manner. Multiplying both sides by $f_2 f_4 f_8^2$ and converting to η notation, the identity is equivalent to

$$\eta^2(z)\eta(2z)\eta(4z)\eta^2(8z) = \frac{\eta^2(2z)\eta^7(8z)}{\eta(4z)\eta^2(16z)} - 2\eta^2(2z)\eta(4z)\eta(8z)\eta^2(16z). \tag{3.5}$$

This is an identity between modular forms in $M_3(\Gamma_0(16), \chi)$, where $\chi(d) := \left(\frac{-2}{d}\right)$. Its validity is confirmed by checking the first 6 coefficients in the q-expansion. $\square$

Our next two results contain $2-$ and $4-$dissection identities for the generating function for $S(n)$, which by (1.10) can be expressed as

$$\sum_{n\geq 0} S(n)q^n = \frac{f_2 f_3}{f_1 f_6^2}. \tag{3.6}$$

Lemma 3.2. *We have*

$$\sum_{n\geq 0} S(n)q^n = \frac{f_4 f_{16} f_{24}^2}{f_2 f_6 f_8 f_{12} f_{48}} + q\frac{f_8^2 f_{48}}{f_2 f_6 f_{16} f_{24}}. \tag{3.7}$$

Proof. Using (3.6) and (2.4), we have that (3.7) is equivalent to the identity

$$\begin{aligned}\eta^2(z)\eta^2(2z)\eta^2(3z)\eta^2(6z) = &\frac{\eta^3(z)\eta(3z)\eta^3(6z)\eta(4z)\eta(16z)\eta^2(24z)}{\eta(8z)\eta(12z)\eta(48z)}\\ &+\frac{\eta^3(z)\eta(3z)\eta^3(6z)\eta^2(8z)\eta(48z)}{\eta(16z)\eta(24z)}.\end{aligned} \tag{3.8}$$

By Proposition 2.4, all three terms in the above identity are holomorphic modular forms in $M_4(\Gamma_0(48))$. Hence, by Proposition 2.5 we only need to check that the q-expansions agree up to $n = 32$. This proves the result. $\square$

Lemma 3.3. *We have*

$$\sum_{n=0}^{\infty} S(n)q^n = \frac{f_{16}^3 f_{24}^7}{f_4 f_{48}^3 f_{12}^5 f_8^2} + q\frac{f_{16} f_8 f_{24}^4}{f_4^2 f_{48} f_{12}^4} + q^2\frac{f_8^4 f_{48} f_{24}}{f_4^3 f_{12}^3 f_{16}} + q^3\frac{f_8^7 f_{48}^3}{f_4^4 f_{12}^2 f_{16}^3 f_{24}^2} \tag{3.9}$$

Proof. This is equivalent to the identity

$$\begin{aligned}\eta^2(z)\eta^2(2z)\eta^2(3z)\eta^2(6z) = &\frac{\eta^3(z)\eta(2z)\eta(3z)\eta^4(6z)\eta^3(16z)\eta^7(24z)}{\eta(4z)\eta^2(8z)\eta^5(12z)\eta^3(48z)}\\ &+\frac{\eta^3(z)\eta(2z)\eta(3z)\eta^4(6z)\eta(8z)\eta(16z)\eta^4(24z)}{\eta^2(4z)\eta^4(12z)\eta(48z)}\\ &+\frac{\eta^3(z)\eta(2z)\eta(3z)\eta^4(6z)\eta^4(8z)\eta(24z)\eta(48z)}{\eta^3(4z)\eta^3(12z)\eta(16z)}\\ &+\frac{\eta^3(z)\eta(2z)\eta(3z)\eta^4(6z)\eta^7(8z)\eta^3(48z)}{\eta^4(4z)\eta^2(12z)\eta^3(16z)\eta^2(24z)}.\end{aligned} \tag{3.10}$$

Again by Proposition 2.4, all of the terms in this identity are holomorphic modular forms in $M_4(\Gamma_0(48))$. Hence, by Proposition 2.5 we only need to check that the q-expansions agree up to $n = 32$. $\square$

Equipped with these results we now proceed to prove Theorems 1.1 and 1.4.

Proof of Theorem 1.1. From (3.7) and a little manipulation using the fact that

$$f_k^2 \equiv f_{2k} \pmod{2}, \tag{3.11}$$

we find that

$$\begin{aligned}\sum_{n=0}^{\infty} S(2n+1)q^n &\equiv \frac{f_1 f_6^2}{f_2 f_3} \pmod{2} \\ &= (q;q^6)_\infty (q^5;q^6)_\infty (q^6;q^6)_\infty \\ &\equiv \sum_{n\in\mathbb{Z}} q^{3n^2+2n} \pmod{2},\end{aligned}$$

where the last line follows from (2.8). Therefore

$$\begin{aligned}\sum_{n\geq 0} S(2n+1)q^{3n+1} &\equiv \sum_{n\in\mathbb{Z}} q^{9n^2+6n+1} \pmod{2} \\ &= \sum_{n\in\mathbb{Z}} q^{(3n+1)^2} \\ &= \sum_{\substack{n\geq 0 \\ 3\nmid n}} q^{n^2},\end{aligned}$$

and the first part of Theorem 1.1 follows.

For the second part, we begin by observing that equations (3.9), (2.9), and (2.10) imply that

$$\begin{aligned}\sum_{n=0}^{\infty} S(4n+2)q^n &\equiv f_1 f_3^3 \pmod{2} \\ &\equiv \sum_{n\in\mathbb{Z}} q^{n(3n+1)/2} \sum_{m\geq 0} q^{3m(m+1)/2} \pmod{2}.\end{aligned}$$

Replacing q by q^{24}, multiplying by q^{10}, and simplifying the exponents on the right-hand side gives

$$\begin{aligned}\sum_{n\geq 0} S(4n+2)q^{24n+10} &\equiv \sum_{n\in\mathbb{Z}} q^{(6n+1)^2} \sum_{m\geq 0} q^{(6m+3)^2} \pmod{2} \\ &\equiv \sum_{n\geq 0} (q^{(6n+1)^2} + q^{(6n+5)^2}) \sum_{m\geq 0} q^{(6m+3)^2} \pmod{2} \\ &\equiv \sum_{n,m\geq 0} (q^{(6n+1)^2+(6m+3)^2} + q^{(6n+5)^2+(6m+3)^2}) \pmod{2}.\end{aligned} \tag{3.12}$$

Now, it is not hard to deduce that if $x^2+y^2 = 24n+10$ then $(x,y) \equiv (\pm 1, 3) \pmod{6}$ or $(x,y) \equiv (3, \pm 1) \pmod{6}$. Therefore the right-hand side of (3.12) generates one eighth of the number of solutions to this equation, or $\frac{1}{8}r_1(24n+10)$ using the notation of Proposition 2.1. Using this proposition, we have $r_1(24n+10) = r_1(12n+5)$, and so if $12n+5 = p_1^{a_1}\cdots p_k^{a_k} q_1^{b_1}\cdots q_\ell^{b_\ell}$, where each $p_i \equiv 1 \pmod{4}$ and each $q_j \equiv 3 \pmod{4}$, then

$$S(4n+2) \equiv \begin{cases} 0 \pmod{2}, & \text{if any } b_j \text{ is odd}, \\ \frac{1}{2}(a_1+1)\cdots(a_k+1) \pmod{2}, & \text{otherwise}. \end{cases} \tag{3.13}$$

From this, it is clear that $S(4n+2)$ is odd if and only if all b_j are even and all a_i are even except one, which must be 1 modulo 4. The second claim of Theorem 1.1 now follows.

For the third part, we first note that

$$f_k^4 \equiv f_{2k}^2 \pmod 4. \tag{3.14}$$

Using this, along with (3.9) and (3.2), we have that

$$\begin{aligned}\sum_{n\geq 0} S(4n)q^n &= \frac{f_4^3 f_6^7}{f_1 f_2^2 f_3^5 f_{12}^3} \\ &\equiv \frac{f_2^2 f_4 f_6}{f_{12}} \times \frac{1}{f_1 f_3} \pmod 4 \\ &= \frac{f_2^2 f_4 f_6}{f_{12}} \left(\frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}} \right),\end{aligned}$$

and so

$$\begin{aligned}\sum_{n\geq 0} S(8n+4)q^n &\equiv \frac{f_2^6 f_{12}^2}{f_1^2 f_3 f_4^2 f_6^2} \pmod 4 \\ &\equiv \frac{f_2^2 f_6^2}{f_1^2 f_3} \pmod 4 \\ &\equiv f_1^2 f_3^3 \pmod 4.\end{aligned}$$

This implies that

$$\begin{aligned}\sum_{n\geq 0} S(8n+4)q^n &\equiv f_2 f_3^3 \pmod 2 \\ &\equiv \sum_{k\in\mathbb{Z}} q^{k(3k+1)} \sum_{m\geq 0} q^{3m(m+1)/2} \pmod 2.\end{aligned}$$

If we replace q by q^{24} and multiply by q^{11}, we obtain:

$$\begin{aligned}\sum_{n\geq 0} S(8n+4)q^{24n+11} &\equiv \sum_{k\in\mathbb{Z}} q^{2(6k+1)^2} \sum_{m\geq 0} q^{(6m+3)^2} \pmod 2 \\ &\equiv \sum_{k\geq 0} (q^{2(6k+1)^2} + q^{2(6k+5)^2}) \sum_{m\geq 0} q^{(6m+3)^2} \pmod 2 \\ &\equiv \sum_{k,m\geq 0} (q^{2(6k+1)^2+(6m+3)^2} + q^{2(6k+5)^2+(6m+3)^2}) \pmod 2.\end{aligned} \tag{3.15}$$

Now we observe that if $2x^2+y^2 = 24n+11$, then $(x,y) \equiv (\pm 1, 3) \mod 6$, and therefore the right hand side of (3.15) counts one fourth of the number of solutions to $2x^2+y^2 = 24n+11$. Applying Proposition 2.2 then gives that if if $24n+11 = p_1^{a_1}\dots p_k^{a_k} q_1^{b_1}\dots q_\ell^{b_\ell}$, with $p_i \equiv 1,3 \pmod 8$ and $q_j \equiv 5,7 \pmod 8$, then

$$S(8n+4) \equiv \begin{cases} 0 \pmod 2, & \text{if any } b_j \text{ is odd,} \\ \frac{1}{2}(a_1+1)\cdots(a_k+1) \pmod 2, & \text{otherwise.}\end{cases} \tag{3.16}$$

The result follows. □

We now turn to the proof of Theorem 1.4.

Proof of Theorem 1.4. To begin, we have

$$\begin{aligned}\sum_{n\geq 0} S(4n+1)q^n &= \frac{f_2 f_4 f_6^4}{f_1^2 f_3^4 f_{12}} \\ &\equiv \frac{f_2 f_4 f_6^2}{f_1^2 f_{12}} \pmod 4 \\ &\equiv \frac{f_4 f_1^2 f_6^2}{f_2 f_{12}} \pmod 4.\end{aligned}$$

Then using (3.3) twice gives

$$\sum_{n\geq 0} S(4n+1)q^n \equiv f_4\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right)\left(\frac{f_{48}^5}{f_{24}^2 f_{96}^2} - 2q^6\frac{f_{96}^2}{f_{48}}\right) \pmod 4.$$

Picking off the odd exponents and reducing mod 4 we obtain

$$\begin{aligned}\sum_{n\geq 0} S(8n+5)q^n &\equiv 2\frac{f_2 f_8^2}{f_4} \pmod 4 \\ &\equiv 2f_8 f_2^3 \pmod 4.\end{aligned}$$

Hence

$$S(16n+13) \equiv 0 \pmod 4 \tag{3.17}$$

and

$$\sum_{n\equiv 0 \pmod 2} S(8n+5)q^{\frac{n}{2}} \equiv 2f_4 f_1^3 \pmod 4.$$

Using (2.9) and (2.10) in the above equation, setting $q = q^{24}$ and multiplying by q^7 gives

$$\begin{aligned}&\sum_{n\geq 0} S(16n+5)q^{24n+7} \\ &\quad\equiv 2\sum_{m,n\geq 0}\left(q^{(12m+2)^2+3(2n+1)^2} + q^{(12m+10)^2+3(2n+1)^2}\right) \pmod 4.\end{aligned}$$

Now, it is easy to check that any solution to $x^2+3y^2 = 24n+7$ has $x \equiv \pm 2 \pmod{12}$ and y odd, and so the double sum on the right-hand side above generates one fourth of the number of such solutions. Using Proposition 2.3, we conclude that if $24n+7 = p_1^{a_1}\cdots p_k^{a_k} q_1^{b_1}\cdots q_\ell^{b_\ell}$, with $p_i \equiv 1 \pmod 3$ and $q_j \equiv 2 \pmod 3$, then

$$S(16n+5) \equiv \begin{cases} 0 \pmod 4, & \text{if any } b_j \text{ is odd,} \\ (a_1+1)\cdots(a_k+1) \pmod 4, & \text{if all } b_j \text{ are even.}\end{cases} \tag{3.18}$$

The first part of Theorem 1.4 now follows from (3.17) and (3.18).

For the second part, we first use (3.9) and reduce modulo 4 to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} S(4n+3)q^n &\equiv \frac{f_2^7 f_{12}^3}{f_1^4 f_3^2 f_4^3 f_6^2} \pmod 4 \\ &\equiv \frac{f_2^5 f_{12} f_6^2}{f_3^2 f_4^3} \pmod 4 \\ &\equiv \frac{f_2 f_{12} f_3^2}{f_4} \pmod 4.\end{aligned}$$

Applying (3.3) yields

$$\sum_{n\geq 0} S(4n+3)q^n \equiv \frac{f_2 f_{12}}{f_4}\left(\frac{f_6 f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3\frac{f_6 f_{48}^2}{f_{24}}\right) \pmod 4,$$

and so we have

$$\begin{aligned}\sum_{n\geq 0} S(8n+7)q^n &\equiv 2q\frac{f_1 f_6 f_3 f_{24}^2}{f_2 f_{12}} \pmod 4\\ &\equiv 2q\frac{f_1 f_6^2 f_{12}^3}{f_2 f_3} \pmod 4\\ &= 2q(q;q^6)_\infty (q^5;q^6)_\infty (q^6;q^6)_\infty f_{12}^3.\end{aligned}$$

Using (2.8) and (2.10) to expand the products, then setting $q = q^6$ and multiplying by q, we have

$$\sum_{n\geq 0} S(8n+7)q^{6n+5} \equiv 2\left(\sum_{k,m\geq 0} q^{2(3k+1)^2+(6m+3)^2} + q^{2(3k+2)^2+(6m+3)^2}\right) \pmod 4. \tag{3.19}$$

Since any solution to $2x^2+y^2 = 6n+5$ has $x \equiv \pm 1 \pmod 3$ and $y \equiv 0 \pmod 3$, the double sum on the right hand side of the above equation generates $\frac{1}{4}r_2(6n+5)$, in the notation of Proposition 2.2. Using this proposition, we then conclude that if $6n+5 = p_1^{a_1}\dots p_k^{a_k} q_1^{b_1}\dots q_\ell^{b_\ell}$, with $p_i \equiv 1,3 \pmod 8$ and $q_j \equiv 5,7 \pmod 8$, then

$$S(8n+7) \equiv \begin{cases} 0 \pmod 4, & \text{if any } b_j \text{ is odd,}\\ (a_1+1)\cdots(a_k+1) \pmod 4, & \text{if all } b_j \text{ are even.}\end{cases} \tag{3.20}$$

The result follows.

□

We close this section by sketching the proofs of Corollaries 1.2 – 1.5.

Proof of Corollaries 1.2 - 1.5. We give complete details only for Corollary 1.2. With part (2) of Theorem 1.1 in mind, let $p \equiv 1 \pmod{12}$ be prime and set

$$n = pN + \frac{5(p-1)}{12}.$$

Then

$$\begin{aligned}12n+5 &= 12pN + 5p - 5 + 5\\ &= p(12N+5),\end{aligned}$$

and so if $p \nmid 12N+5$, then $12n+5$ cannot have a factorization of the form $\ell^{4a+1}m^2$ for some prime $\ell \nmid m$. We conclude that

$$S\left(4pN + \frac{5(p-1)}{3} + 2\right) \equiv 0 \pmod 2 \tag{3.21}$$

if $p \nmid 12N+5$, or equivalently, $N \not\equiv \frac{5(p-1)}{12} \pmod p$.

The other two corollaries are proved similarly. For Corollary 1.3 we let $p \equiv 1 \pmod{24}$ be prime, set

$$n = pN + \frac{11(p-1)}{24},$$

and then apply part (3) of Theorem 1.1. For Corollary 1.5 we let $p \equiv 1 \pmod{24}$ be prime, set

$$n = 2pN + \frac{7(p-1)}{12},$$

and then apply part (1) of Theorem 1.4. $\square$

4. $S(n)$ modulo 5

We begin with an identity between modular forms.

Lemma 4.1. *The functions*

$$f(z) = \frac{\eta^6(6z)\eta^{11}(9z)}{\eta(3z)\eta^2(18z)} \tag{4.1}$$

and

$$g(z) = \eta^5(9z)\eta(3z)\frac{\eta^{10}(z)}{\eta^2(5z)} \tag{4.2}$$

are both holomorphic modular forms of weight 7 *and level* 1080 *for the character* $\chi(d) = \left(\frac{-3}{d}\right)$, *and we have*

$$f(z) \mid U(5) \equiv g(z) \pmod{5}. \tag{4.3}$$

Proof. By Proposition 2.4, both $f(z)$ and $g(z)$ are modular forms in $M_7(\Gamma_0(1080), \chi)$, and hence by Proposition 2.6 so is $f(z) \mid U(5)$. Using Proposition 2.5, equation (4.3) is verified by checking the first 1512 coefficients in the q-expansion. $\square$

Lemma 4.1 can be used to deduce the following simple generating function for $S(5n+2)$ modulo 5.

Theorem 4.2. *We have*

$$\sum_{n\geq 0} S(5n+2)q^n \equiv \frac{f_1 f_3^3}{f_2} \pmod{5}. \tag{4.4}$$

Proof. First note that if $f(q)$ and $g(q)$ are power series in q, we have

$$\big(f(q)g(q^5)\big) \mid U(5) = (f(q) \mid U(5)) \times g(q). \tag{4.5}$$

Using this, we have on one hand

$$\begin{aligned}
\frac{\eta^6(6z)\eta^{11}(9z)}{\eta(3z)\eta^2(18z)} \mid U(5) &= \left(q^4 \frac{f_6 f_9}{f_3 f_{18}^2} \times f_9^{10} f_6^5\right) \mid U(5) \\
&\equiv \left(q^4 \frac{f_6 f_9}{f_3 f_{18}^2} \times f_{45}^2 f_{30}\right) \mid U(5) \pmod{5} \\
&= \sum_{n\geq 0} S(n) q^{3n+4} \mid U(5) \times f_9^2 f_6 \\
&= \sum_{n\geq 0} S(5n+2) q^{3n+2} \times f_9^2 f_6.
\end{aligned}$$

On the other hand, using Lemma 4.1 we have

$$\begin{aligned}
\frac{\eta^6(6z)\eta^{11}(9z)}{\eta(3z)\eta^2(18z)} \mid U(5) &\equiv \eta^5(9z)\eta(3z)\frac{\eta^{10}(z)}{\eta^2(5z)} \pmod{5} \\
&\equiv \eta^5(9z)\eta(3z) \pmod{5} \\
&= q^2 f_9^5 f_3.
\end{aligned}$$

Comparing the last lines in each of the two strings of equations above, we have

$$\sum_{n\geq 0} S(5n+2)q^{3n+2} \equiv q^2 \frac{f_9^3 f_3}{f_6} \pmod{5}, \tag{4.6}$$

which gives the result. □

Next we give the 4−dissection of the generating function for $S(5n+2)$ modulo 5.

Proposition 4.3.

$$\sum_{n=0}^{\infty} S(5n+2)q^n \equiv \frac{f_8^9 f_{24}^2}{f_4^4 f_{16}^3 f_{48}} - q\frac{f_8^2 f_{24}^9}{f_4 f_{12}^3 f_{16} f_{48}^3} + 4q^6 \frac{f_{16} f_{48}^3}{f_8} - 4q^3 \frac{f_{12} f_{16}^3 f_{48}}{f_4 f_{24}} \pmod{5}. \tag{4.7}$$

Proof. In light of (4.6), this is a consequence of the identity

$$\begin{aligned} \eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2 = &\frac{\eta(z)\eta(2z)^3\eta(6z)^2\eta(8z)^9\eta(24z)^2}{\eta(3z)\eta(4z)^4\eta(16z)^3\eta(48z)} \\ &- \frac{\eta(z)\eta(2z)^3\eta(6z)^2\eta(8z)^2\eta(24z)^9}{\eta(3z)\eta(4z)\eta(12z)^3\eta(16z)\eta(48z)^3} \\ &+ 4\frac{\eta(z)\eta(2z)^3\eta(6z)^2\eta(16z)\eta(48z)^3}{\eta(3z)\eta(8z)} \\ &- 4\frac{\eta(z)\eta(2z)^3\eta(6z)^2\eta(12z)\eta(16z)^3\eta(48z)}{\eta(3z)\eta(4z)\eta(24z)}. \end{aligned} \tag{4.8}$$

By Proposition 2.4, this is an identity between modular forms in $M_4(\Gamma_0(48))$, and hence is confirmed by Proposition 2.5 after checking the first 32 coefficients in the q-expansion. □

We are now ready to prove our main results on congruences modulo 5.

Proof of Theorem 1.6. We first note that

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}.$$

Using this, along with Theorem 4.2 and Proposition 4.3, we have

$$\begin{aligned} \sum_{n\geq 0} S(5n+2)(-1)^{n+1}q^n &= -\sum_{n\geq 0} S(5n+2)(-q)^n \\ &\equiv -\frac{f_2^3}{f_1 f_4} \times \left(\frac{f_6^3}{f_3 f_{12}}\right)^3 \times \frac{1}{f_2} \pmod{5} \\ &\equiv -\frac{f_2^2 f_6^9}{f_1 f_3^3 f_4 f_{12}^3} \pmod{5} \\ &\equiv \sum_{n\geq 0} S(20n+7)q^n \pmod{5}, \end{aligned}$$

which establishes the result. □

Proof of Theorem 1.7. First, by Proposition 4.3 we have that

$$\sum_{n\geq 0} S(20n+12)q^n \equiv 4q\frac{f_4 f_{12}^3}{f_2} \pmod{5}. \tag{4.9}$$

From this we deduce (1.18), which is the case $\alpha = 0$ of (1.19). We now proceed by induction. Suppose that (1.19) is true for some $\alpha \geq 0$. Then

$$S(5N+2) \equiv 0 \pmod{5},$$

where

$$N = 2^{2\alpha+3}n + \frac{7 \cdot 4^{\alpha} - 1}{3}.$$

By Theorem 1.6 we then have

$$S(20N+7) \equiv 0 \pmod{5},$$

and a quick calculation gives

$$20N + 7 = 5 \cdot 2^{2(\alpha+1)+3} \cdot n + \frac{35 \cdot 4^{\alpha+1} + 1}{3}. \tag{4.10}$$

Hence (1.19) holds for $\alpha + 1$, and the result follows by induction. □

5. Concluding Remarks

We close with several remarks. First, as with any proper study of congruences for partition functions, ours included an extensive computer search. Using Mathematica, we computed the values of $S(n)$ up to $n = 40,000$ and searched for (likely) congruences of the form

$$S(An + B) \equiv 0 \pmod{M} \tag{5.1}$$

for $M \in [2, 20] \cup \{32, 64\}$ and $A \leq 5,000$. The results in this paper do not cover all of the congruences we detected. For example, we observed probable congruences for $S(n)$ modulo 4 within every subprogression $4n + b$, for $0 \leq b \leq 3$, and these do not all fall under the purview of Theorem 1.4. We also observed some apparently sporadic congruences modulo $8, 16$, and 32. We leave it to the motivated reader to explain these congruences modulo powers of 2.

Second, since the proportion of natural numbers which have factorizations of the types in Theorems 1.1 and 1.4 is asymptotically 0, we have density results like

$$\liminf_{X \to \infty} \frac{|\{n \leq X : S(n) \equiv 0 \pmod{2}\}|}{X} \geq \frac{7}{8}. \tag{5.2}$$

We do not know the correct value of this limit. It would be interesting to have results on the parity of $S(n)$ outside of the progressions $2n+1$, $4n+2$, and $8n+4$.

Third, it can be shown that after replacing q by q^3 and then multiplying by q, the infinite product in Theorem 4.2 becomes a holomorphic modular form of weight $3/2$. In fact, Theorem 1.6 is equivalent to the statement that this holomorphic modular form is an eigenform for the $U(4)$ operator (after replacing q by $-q$). It is quite possible that the theory of modular forms can be used to deduce much more about $S(5n + 2)$ modulo 5.

Finally, while our primary motivation for looking at $S(n)$ was the similarity to Schur's theorem, a secondary motivation is the fact that the generating function for $S(n)$ is the reciprocal of a weight $1/2$ theta function. Indeed, by (1.10) and (2.8), we have

$$\sum_{n \geq 0} S(n) q^n = \left(\sum_{n \in \mathbb{Z}} (-1)^n q^{3n^2+2n} \right)^{-1}. \tag{5.3}$$

This is reminiscent of two of the most important functions in the theory of partitions, the partition function $p(n)$ and the overpartition function $\overline{p}(n)$, whose generating functions are

$$\sum_{n\geq 0} p(n)q^n = \left(\sum_{n\in\mathbb{Z}}(-1)^n q^{n(3n+1)/2}\right)^{-1} \tag{5.4}$$

and

$$\sum_{n\geq 0} \overline{p}(n)q^n = \left(\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}\right)^{-1}. \tag{5.5}$$

The similarity between the generating function in (5.3) and those in (5.4) and (5.5) suggests that the combinatorial and arithmetic properties of the Schur-type overpartitions counted by $S(n)$ might well be worth further investigation.

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