New congruences modulo 4 and 8 for Ramanujan's ϕ function

Julia Q.D. Du[1], Jeremy Lovejoy[2] and Olivia X.M. Yao[3]

[1]School of Mathematical Sciences,
Hebei Key Laboratory of Computational Mathematics and Applications,
Hebei Normal University, Shijiazhuang 050024, P.R. China

[2]CNRS, Université Paris Cité,
Bâtiment Sophie Germain, Case Courier 7014,
8 Place Aurélie Nemours, 75205 Paris Cedex 13, FRANCE

[3]School of Mathematical Sciences,
Suzhou University of Science and Technology,
Suzhou, 215009, Jiangsu Province, P.R. China

e-mail: qddu@hebtu.edu.cn, lovejoy@math.cnrs.fr, yaoxiangmei@163.com

Abstract. In his lost notebook, Ramanujan defined the function $\phi(q)$, which is a mock modular form and is related to some of Ramanujan's mock theta functions. In recent years, a number of congruences for the coefficients of $\phi(q)$ have been proved by Baruah and Begum, Chan, Du and Tang, and Xia. Motivated by their works, we characterize congruences modulo 2 and 4 for the coefficients of $\phi(q)$ based on the congruences on the eighth order mock theta function established by Chen and Garvan. We also prove some congruences modulo 8 for the coefficients of $\phi(q)$ based on an identity due to Newman.

Keywords: congruences, Ramanujan's ϕ function, mock theta functions, Newman's identities.

AMS Subject Classification: 11P81, 11P83, 05A17.

1 Introduction

On page 3 of his lost notebook, Ramanujan [14] defined the function $\phi(q)$:

$$\phi(q) = \sum_{n=0}^{\infty} a(n)q^n := \sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{n+1}}{(q;q^2)_{n+1}^2}.$$

Here and throughout this paper, we use the standard q-series notation,

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \qquad (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

where q is a complex number with $|q|<1$.

The function $\phi(q)$ is related to some of Ramanujan's mock theta functions. Ramanujan [14] discovered the following identity involving $\phi(q^3)$ and the sixth order mock theta function $\rho(q)$:

$$\rho(q) = 2q^{-1}\phi(q^3) + \frac{(q^2;q^2)_\infty^4(q^6;q^6)_\infty}{(q;q)_\infty^2(q^3;q^3)_\infty^2},$$

where $\rho(q)$ is defined by

$$\rho(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+1)/2}}{(q;q^2)_{n+1}}.$$

Motivated by Ramanujan's work, Choi [9] proved analogous identities involving $\phi(q)$ and two other sixth order mock theta functions $\lambda(q)$ and $\psi(q)$ defined by

$$\lambda(q) := \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q;q)_n} q^n, \qquad \psi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q;q)_{2n+1}} q^{(n+1)^2}.$$

Recently, Mortenson [11] discovered some identities involving $\phi(q)$ and two second order mock theta functions $A(q)$ and $\mu(q)$ defined by

$$A(q) = \sum_{n=0}^{\infty} N_A(n) q^n := \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n}{(q;q^2)_{n+1}} q^{n+1}, \tag{1.1}$$

$$\mu(q) := \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q^2;q^2)_n^2} q^{n^2}.$$

The function $\phi(q)$ also appears as the mock modular form $H_2^{(4)}$ of type $2A$ in [8, (2.67)].

In recent years, a number of congruences for the coefficients $a(n)$ of $\phi(q)$ have been proved. In 2012, Chan [5] proved the following: for $n \geq 0$,

$$a(9n+4) \equiv 0 \pmod 2, \tag{1.2}$$

$$a(3n+2) \equiv a(18n+7) \equiv a(18n+13) \equiv 0 \pmod 3,$$

$$a(25n+14) \equiv a(25n+24) \equiv 0 \pmod 4, \tag{1.3}$$

$$a(10n+9) \equiv 0 \pmod 5, \tag{1.4}$$

$$a(7n+3) \equiv a(7n+4) \equiv a(7n+6) \equiv 0 \pmod 7,$$

$$a(6n+5) \equiv 0 \pmod{27}.$$

Furthermore, Chan [5, Conjecture 7.1] conjectured that for any $n \geq 0$,

$$a(50n+19) \equiv a(50n+39) \equiv a(50n+49) \equiv 0 \pmod{25}. \tag{1.5}$$

Utilizing some q-series techniques, Baruah and Begum [2] not only confirmed (1.5), but also proved the following three congruences modulo 125 satisfied by $a(n)$:

$$a(1250n+469) \equiv a(1250n+969) \equiv a(1250n+1219) \equiv 0 \pmod{125}. \tag{1.6}$$

Inspired by Chan's work, Xia [16] deduced some new congruences modulo powers of 2 and 3 for $a(n)$. For example, he proved that for $n \geq 0$,

$$a(49n+9) \equiv a(49n+23) \equiv a(49n+30) \equiv 0 \pmod 2. \tag{1.7}$$

Du and Tang [10] established several infinite families of congruences modulo arbitrary powers of 5 for $a(n)$, which contain (1.4)–(1.6) as special cases.

The aim of this paper is to characterize congruences mod 2 and 4 for $a(n)$ based on a method for proving congruences on mock theta functions and the Hurwitz class number due to Chen and Garvan [6, 7]. In addition, we deduce some congruences modulo 8 for $a(2n+1)$ by using an identity given by Newman [12].

In order to state the following theorems, we recall the Legendre symbol. Let $p \geq 3$ be a prime. The Legendre symbol $\left(\frac{a}{p}\right)_L$ is defined by

$$\left(\frac{a}{p}\right)_L := \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } p \nmid a, \\ 0, & \text{if } p \mid a, \\ -1, & \text{if } a \text{ is a nonquadratic residue modulo } p. \end{cases}$$

The main results of this paper can be stated as follows.

Theorem 1.1 *Let $n \geq 1$ be an integer.*

(1) $a(n)$ is odd if and only if $4n-1$ has the form

$$4n - 1 = p^{4\alpha+1} m^2,$$

where p is a prime, and m and α are integers satisfying $(m, p) = 1$ and $\alpha \geq 0$.

(2) $a(n) \equiv 2 \pmod 4$ if and only if $4n-1$ has the form

$$4n - 1 = p_1^{4a+1} p_2^{4b+1} m^2,$$

where p_1 and p_2 are primes such that $\left(\frac{p_1}{p_2}\right)_L = -1$, $(m, p_1 p_2) = 1$ and $a, b \geq 0$ are integers.

Theorem 1.2 *Let $p \equiv 1 \pmod 4$ be a prime and suppose $4\delta_p \equiv 1 \pmod{p^2}$ and $k, n \in \mathbb{Z}$ with $\left(\frac{k}{p}\right)_L = 1$. Then,*

$$a\left(p^2 n + (pk+1)\delta_p\right) \equiv 0 \pmod 4. \tag{1.8}$$

Taking $p = 5$ in (1.8) yields (1.3).

Based on Theorem 1.1, we can deduce the following corollary:

Corollary 1.3 *Let $p \geq 3$ be a prime. Suppose that ξ_p is a positive integer such that $4\xi_p \equiv 1 \pmod p$ with $1 \leq \xi_p \leq p^2 - 1$. If $\left(\frac{\frac{4\xi_p - 1}{p}}{p}\right)_L = -1$ or $\frac{4\xi_p-1}{p} \equiv 3 \pmod 4$ with $p^2 \nmid (4\xi_p - 1)$, then for $n \geq 0$,*

$$a(p^2 n + \xi_p) \equiv 0 \pmod 2. \tag{1.9}$$

Setting $p = 3, 7$ in (1.9), we arrive at (1.2) and (1.7), respectively. Furthermore, if we set $p = 11, 13$ in (1.9), we deduce the following corollary:

Corollary 1.4 *For $n \geq 0$,*

$$\begin{aligned} a(121n + i) &\equiv 0 \pmod 2, \\ a(169n + j) &\equiv 0 \pmod 2, \end{aligned}$$

where $i \in \{36, 47, 58, 80, 113\}$ and $j \in \{10, 23, 36, 49, 62, 75, 88, 101, 114, 140, 153, 166\}$.

We also prove some congruences modulo 8 for $a(2n+1)$. We first present some definitions. Throughout the rest of this paper, we define

$$\sum_{n=0}^{\infty} c_9(n)q^n := (q;q)_\infty^9, \tag{1.10}$$

$$\nu(p) := \begin{cases} 2, & \text{if } r \equiv 0 \pmod{8}, \\ 4, & \text{if } r \equiv 4 \pmod{8}, \\ 8, & \text{if } r \equiv 2 \pmod{4}, \end{cases} \tag{1.11}$$

and

$$g(p) := \begin{cases} -s, & \text{if } \nu(p) = 2, \\ -r^2 s + s^2, & \text{if } \nu(p) = 4, \\ -r^6 s + 5s^2 r^4 - 6r^2 s^3 + s^4, & \text{if } \nu(p) = 8, \end{cases} \tag{1.12}$$

where $p \geq 3$ is a prime, and r and s are defined by

$$r := r(p) = c_9\left(\frac{3(p^2-1)}{8}\right) + (-1)^{\frac{(p-1)(p-3)}{8}} p^3 \left(\frac{\frac{3(p^2-1)}{8}}{p}\right)_L, \qquad s := s(p) = p^7. \tag{1.13}$$

Remark 1.5 *Though it is not clear from the definition, we can prove that r is an even number. It is easy to check that*

$$\sum_{n=0}^{\infty} c_9(n)q^n \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{n(n+1)/2+m^2+m} \pmod{2}, \tag{1.14}$$

where we have used Lemma 2.4 and the following identity due to Gauss:

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}$$

We can rewrite (1.14) *as*

$$\sum_{n=0}^{\infty} c_9(n)q^{8n+3} \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{(2n+1)^2+2(2m+1)^2} \pmod{2}.$$

Therefore,

$$c_9(n) \equiv \frac{1}{4} r_2(8n+3) \pmod{2}, \tag{1.15}$$

where $r_2(n)$ denotes the number of solutions to $x^2+2y^2=n$ with $x, y \in \mathbb{Z}$. It follows from (1.15) *and [4, Proposition 2.2] that $c_9\left(\frac{3(p^2-1)}{8}\right)$ is even when $p=3$ and is odd when $p>3$. Thus, r is even.*

Theorem 1.6 *Let $p \geq 3$ be a prime.*
(1) For $n, k \geq 0$, if $p \nmid n$, then

$$a\left(2p^{2\nu(p)(k+1)-1}n + \frac{3p^{2\nu(p)(k+1)}+1}{4}\right) \equiv 0 \pmod{8}. \tag{1.16}$$

(2) For $k \geq 0$,

$$a\left(2p^{2\nu(p)k} + \frac{3p^{2\nu(p)k}+1}{4}\right) \equiv g(p)^k \pmod{8}. \tag{1.17}$$

Example. Taking $p = 3$ in (1.16) and (1.17), we find $r = 4$, $s = 3^7$, $\nu(3) = 4$ and $g(3) \equiv 1 \pmod 8$. Thus, for $n, k \geq 0$, if $3 \nmid n$, then

$$a\left(2 \times 3^{8k+7}n + \frac{3^{8k+9}+1}{4}\right) \equiv 0 \pmod 8$$

and

$$a\left(2 \times 3^{8k} + \frac{3^{8k+1}+1}{4}\right) \equiv 1 \pmod 8.$$

Theorem 1.7 *Suppose that b is a positive integer such that $c_9(b) \equiv 0 \pmod 8$ and that $8b+3 = \prod_{i=1}^{u} f_i \prod_{j=1}^{v} g_j^{\alpha_j}$ with each $\alpha_j \geq 2$ is the prime factorization of $8b+3$. Then for $n \geq 1$,*

$$a\left(2bn^2 + \frac{3n^2+1}{4}\right) \equiv 0 \pmod 8, \tag{1.18}$$

where $\left(n, 2\prod_{j=1}^{v} g_j^{\alpha_j}\right) = 1$.

This paper is organized as follows. In Section 2, we recall some results on congruences modulo 2 and 4 for the coefficients of the eighth order mock theta function $V_1(q)$. In Section 3, we present proofs of Theorems 1.1, 1.2 and Corollary 1.3. Sections 4 and 5 are devoted to the proofs of Theorems 1.6 and 1.7 by using a method posed by Newman [12], and Xue and Yao [17]. Finally, we make some concluding remarks concerning future directions.

2 Preliminaries

In [7], Chen and Garvan proved some congruences modulo 4 for certain mock theta functions. In particular, they investigated congruences modulo 2 and 4 for the eighth order mock theta function $V_1(q)$ defined by

$$V_1(q) = \sum_{n=0}^{\infty} N_{V_1}(n) q^n := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q;q^2)_{n+1}} q^{(n+1)^2}. \tag{2.1}$$

Chen and Garvan [7] proved the following interesting results on the coefficients of $V_1(q)$.

Lemma 2.1 *[7, Lemma 3.4] For each positive integer n, we have*

$$N_{V_1}(n) \equiv \chi(n) H(4n-1) \pmod 4, \tag{2.2}$$

where $H(n)$ is the Hurwitz class number, and

$$\chi(n) := \begin{cases} 1, & \text{if } n \equiv 2,3 \pmod 4, \\ -1, & \text{if } n \equiv 0,1 \pmod 4. \end{cases} \tag{2.3}$$

Based on Lemma 2.1, Chen and Garvan [7] proved the following two theorems.

Theorem 2.2 *[7, Theorem 3.5] Let $n \geq 1$ be an integer.*

(1) $N_{V_1}(n)$ is odd if and only if $4n-1$ has the form

$$4n-1=p^{4\alpha+1}m^2,$$

where p is a prime, and m and α are integers satisfying $(m,p)=1$ and $\alpha\geq 0$.

(2) $N_{V_1}(n)\equiv 2 \pmod 4$ if and only if $4n-1$ has the form

$$4n-1=p_1^{4a+1}p_2^{4b+1}m^2,$$

where p_1 and p_2 are primes such that $\left(\frac{p_1}{p_2}\right)_L=-1$, $(m,p_1p_2)=1$ and $a,b\geq 0$ are integers.

Theorem 2.3 *[7, Theorem 3.6] Let $p\equiv 1 \pmod 4$ be a prime and suppose $4\delta_p\equiv 1 \pmod{p^2}$ and $k,n\in\mathbb{Z}$ with $\left(\frac{k}{p}\right)_L=1$. Then,*

$$N_{V_1}\left(p^2n+(pk+1)\delta_p\right)\equiv 0 \pmod 4. \tag{2.4}$$

Moreover, we require the following lemma which follows from the binomial theorem.

Lemma 2.4 *Let m,k be positive integers. Then*

$$(q^m;q^m)_\infty^{2^k}\equiv(q^{2m};q^{2m})_\infty^{2^{k-1}} \pmod{2^k}. \tag{2.5}$$

3 Proofs of Theorems 1.1, 1.2 and Corollary 1.3

The main goal of this section is to characterize congruences mod 2 and 4 for $a(n)$ by using some results due to Chen and Garvan [7].

We first present a proof of Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2. In [11], Mortenson proved that

$$\sum_{n=0}^{\infty}a(n)q^n=\phi(q)=A(-q^2)+q\frac{(q^2;q^2)_\infty^7(q^8;q^8)_\infty^4}{(q;q)_\infty^4(q^4;q^4)_\infty^6}, \tag{3.1}$$

where $A(q)$ is defined by (1.1). It follows from [3, p. 40, Entry 25] that

$$\frac{1}{(q;q)_\infty^4}=\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4}+4q\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}. \tag{3.2}$$

Substituting (3.2) into (3.1) yields

$$\sum_{n=0}^{\infty}a(n)q^n=A(-q^2)+q\frac{(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^7}+4q^2\frac{(q^8;q^8)_\infty^8}{(q^2;q^2)_\infty^3(q^4;q^4)_\infty^4}. \tag{3.3}$$

In view of (1.1) and (3.3),

$$a(2n)\equiv(-1)^nN_A(n) \pmod 4. \tag{3.4}$$

In [7], Chen and Garvan proved that for $n\geq 1$,

$$N_A(n)\equiv(-1)^{n+1}H(8n-1) \pmod 4. \tag{3.5}$$

Combining (3.4) and (3.5) yields

$$a(2n) \equiv 3H(8n-1) \pmod 4. \tag{3.6}$$

By (2.5) and (3.3),

$$\sum_{n=0}^{\infty} a(2n+1)q^n = \frac{(q^2;q^2)_\infty^8}{(q;q)_\infty^7} \equiv \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^3} \pmod 4. \tag{3.7}$$

Recall from [15, (1.14)] that

$$3\sum_{n=0}^{\infty} H(8n+3)q^n = \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^3}. \tag{3.8}$$

Thus, from (3.7) and (3.8), we arrive at

$$a(2n+1) \equiv 3H(8n+3) \pmod 4. \tag{3.9}$$

It follows from (3.6) and (3.9) that for $n \geq 1$,

$$a(n) \equiv 3H(4n-1) \pmod 4.^1 \tag{3.10}$$

In light of (2.2) and (3.10),

$$a(n) \equiv 3\chi(n)N_{V_1}(n) \pmod 4, \tag{3.11}$$

where $\chi(n)$ is defined by (2.3). Combining (3.11) and Theorems 2.2 and 2.3, we arrive at Theorems 1.1 and 1.2. This completes the proof. ∎

To end this section, we provide a proof of Corollary 1.3.

Proof of Corollary 1.3. Suppose that $4\xi_p - 1 = pt$, where t is an integer. It is easy to check that

$$4(p^2n+\xi_p) - 1 = p(4pn+t).$$

If $\left(\frac{\frac{4\xi_p-1}{p}}{p}\right)_L = -1$ or $\frac{4\xi_p-1}{p} \equiv 3 \pmod 4$ with $p^2 \nmid (4\xi_p - 1)$, then $p||(4(p^2n+\xi_p)-1)$ and $4pn+t$ is not a square. Congruence (1.9) follows from Theorem 1.1 (1). This completes the proof. ∎

4 Proof of Theorem 1.6

In Sections 4 and 5, we apply the method given in [17] to prove Theorems 1.6 and 1.7. Throughout this section, we always suppose that $p \geq 3$ is a prime and r, s are defined by (1.13).

To prove Theorem 1.6, we require some lemmas.

Lemma 4.1 *For $n, k \geq 0$,*

$$c_9\left(p^{2k}n + \frac{3(p^{2k}-1)}{8}\right) = A_k(p)c_9\left(p^2n + \frac{3(p^2-1)}{8}\right) + B_k(p)c_9(n), \tag{4.1}$$

[1] Since $N^o(1, 1/q; 1; q) \equiv N^o(1, 1/q; -1; q) \pmod 4$ which are defined in [1], (3.10) also follows from [1, (1.6)] and the last line of p. 387 in [1].

where $c_9(n)$ is defined by (1.10), and $A_k(p)$ and $B_k(p)$ are two sequences defined by

$$A_k(p) = rA_{k-1}(p) - sA_{k-2}(p), \tag{4.2}$$

$$B_k(p) = rB_{k-1}(p) - sB_{k-2}(p) \tag{4.3}$$

with

$$A_1(p) = 1, \quad A_0(p) = 0, \quad B_1(p) = 0, \quad B_0(p) = 1. \tag{4.4}$$

Proof. We prove this lemma by induction on k. Note that this lemma is true when $k = 0$ and $k = 1$ by (4.4). Now suppose that Lemma 4.1 holds when $k = m$ and $k = m + 1$ which gives that

$$c_9\left(p^{2m}n + \frac{3(p^{2m}-1)}{8}\right) = A_m(p)c_9\left(p^2n + \frac{3(p^2-1)}{8}\right) + B_m(p)c_9(n) \tag{4.5}$$

and

$$c_9\left(p^{2m+2}n + \frac{3(p^{2m+2}-1)}{8}\right) = A_{m+1}(p)c_9\left(p^2n + \frac{3(p^2-1)}{8}\right) + B_{m+1}(p)c_9(n). \tag{4.6}$$

In [12, 13], Newman proved the following identity on $c_9(n)$:

$$c_9\left(p^2n + \frac{3(p^2-1)}{8}\right) = \kappa(n)c_9(n) - p^7 c_9\left(\frac{n - \frac{3(p^2-1)}{8}}{p^2}\right), \tag{4.7}$$

where

$$\kappa(n) := c_9\left(\frac{3(p^2-1)}{8}\right) + (-1)^{\frac{(p-1)(p-3)}{8}} p^3 \left(\left(\frac{\frac{3(p^2-1)}{8}}{p}\right)_L - \left(\frac{\frac{3(p^2-1)}{8} - n}{p}\right)_L\right).$$

If we replace n by $p^2n + \frac{3(p^2-1)}{8}$ in (4.7), we obtain

$$c_9\left(p^4n + \frac{3(p^4-1)}{8}\right) = rc_9\left(p^2n + \frac{3(p^2-1)}{8}\right) - sc_9(n). \tag{4.8}$$

Replacing n by $p^{2m}n + \frac{3(p^{2m}-1)}{8}$ in (4.8) and utilizing (4.5) and (4.6) yields

$$\begin{aligned}
& c_9\left(p^{2m+4}n + \frac{3(p^{2m+4}-1)}{8}\right) \\
&= rc_9\left(p^{2m+2}n + \frac{3(p^{2m+2}-1)}{8}\right) - sc_9\left(p^{2m}n + \frac{3(p^{2m}-1)}{8}\right) \\
&= r\left(A_{m+1}(p)c_9\left(p^2n + \frac{3(p^2-1)}{8}\right) + B_{m+1}(p)c_9(n)\right) \\
&\quad - s\left(A_m(p)c_9\left(p^2n + \frac{3(p^2-1)}{8}\right) + B_m(p)c_9(n)\right) \\
&= \left(rA_{m+1}(p) - sA_m(p)\right)c_9\left(p^2n + \frac{3(p^2-1)}{8}\right) + \left(rB_{m+1}(p) - sB_m(p)\right)c_9(n) \\
&= A_{m+2}(p)c_9\left(p^2n + \frac{3(p^2-1)}{8}\right) + B_{m+2}(p)c_9(n),
\end{aligned}$$

which implies that (4.1) holds when $k = m + 2$. Lemma 4.1 is proved by induction. ∎

Lemma 4.2 *For $k \geq 0$,*

$$rA_{\nu(p)k+\nu(p)-1}(p) + B_{\nu(p)k+\nu(p)-1}(p) \equiv 0 \pmod{8}, \tag{4.9}$$

where $\nu(p)$ is defined by (1.11) *and $A_k(p)$ and $B_k(p)$ are defined by* (4.2) *and* (4.3)*, respectively.*

Proof. Lemma 4.2 will be proved by induction on k. In view of (4.2)–(4.4), it is easy to check that

$$rA_{\nu(p)-1}(p) + B_{\nu(p)-1}(p) = h(p), \tag{4.10}$$

where

$$h(p) := \begin{cases} r, & \text{if } \nu(p) = 2, \\ r^3 - 2rs, & \text{if } \nu(p) = 4, \\ r^7 - 6r^5 s + 10s^2 r^3 - 4rs^3, & \text{if } \nu(p) = 8. \end{cases} \tag{4.11}$$

Based on (1.11) and (4.11), one can check that

$$h(p) \equiv 0 \pmod{8}. \tag{4.12}$$

So (4.9) is true when $k = 0$. Suppose that (4.9) holds when $k = m$ ($m \geq 0$) which implies that

$$rA_{\nu(p)m+\nu(p)-1}(p) + B_{\nu(p)m+\nu(p)-1}(p) \equiv 0 \pmod{8}. \tag{4.13}$$

In light of (4.2) and (4.3) and the values of $\nu(p)$,

$$\begin{aligned} &rA_{\nu(p)m+2\nu(p)-1}(p) + B_{\nu(p)m+2\nu(p)-1}(p) \\ &= h(p)\left(rA_{\nu(p)m+\nu(p)}(p) + B_{\nu(p)m+\nu(p)}(p)\right) \\ &\quad + g(p)\left(rA_{\nu(p)m+\nu(p)-1}(p) + B_{\nu(p)m+\nu(p)-1}(p)\right), \end{aligned} \tag{4.14}$$

where $g(p)$ and $h(p)$ are defined by (1.12) and (4.11), respectively. Thanks to (4.12), (4.13) and (4.14),

$$rA_{\nu(p)m+2\nu(p)-1}(p) + B_{\nu(p)m+2\nu(p)-1}(p) \equiv 0 \pmod{8},$$

which implies that (4.9) is true when $k = m + 1$ and so Lemma 4.2 is proved by induction. ∎

Lemma 4.3 *For $k \geq 0$,*

$$A_{\nu(p)k}(p) \equiv 0 \pmod{8} \tag{4.15}$$

and

$$B_{\nu(p)k}(p) \equiv g(p)^k \pmod{8}, \tag{4.16}$$

where $\nu(p)$ and $g(p)$ are defined by (1.11) *and* (1.12)*, respectively.*

Proof. We prove (4.15) and (4.16) by induction on k. Note that (4.15) and (4.16) hold when $k = 0$ because $A_0(p) = 0$ and $B_0(p) = 1$. Suppose that (4.15) and (4.16) hold when $k = m$, so that

$$A_{\nu(p)m}(p) \equiv 0 \pmod{8} \tag{4.17}$$

and

$$B_{\nu(p)m}(p) \equiv g(p)^m \pmod{8}. \tag{4.18}$$

In light of the values of $\nu(p)$, (4.2) and (4.3),

$$A_{\nu(p)m+\nu(p)}(p) = h(p)A_{\nu(p)m+1}(p) + g(p)A_{\nu(p)m}(p) \tag{4.19}$$

and

$$B_{\nu(p)m+\nu(p)}(p) = h(p)B_{\nu(p)m+1}(p) + g(p)B_{\nu(p)m}(p), \tag{4.20}$$

where $g(p)$ and $h(p)$ are defined by (1.12) and (4.11), respectively. By (4.12), (4.17) and (4.19), we deduce that (4.15) is true when $k = m + 1$. It follows from (4.12), (4.18) and (4.20) that

$$B_{\nu(p)m+\nu(p)}(p) \equiv g(p)^{m+1} \pmod 8,$$

which implies that (4.16) holds when $k = m + 1$. Therefore, Lemma 4.3 is proved by induction. ∎

Now, we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. Substituting (4.7) into (4.1) yields

$$c_9\left(p^{2k}n + \frac{3(p^{2k}-1)}{8}\right) =(A_k(p)\kappa(n) + B_k(p))c_9(n) - sA_k(p)c_9\left(\frac{n - \frac{3(p^2-1)}{8}}{p^2}\right). \tag{4.21}$$

Replacing n by $pn + \frac{3(p^2-1)}{8}$ in (4.21) yields

$$c_9\left(p^{2k+1}n + \frac{3(p^{2k+2}-1)}{8}\right) =(rA_k(p) + B_k(p))c_9\left(pn + \frac{3(p^2-1)}{8}\right) - sA_k(p)c_9\left(\frac{n}{p}\right), \tag{4.22}$$

where r and s are defined by (1.13). Replacing k by $\nu(p)k + \nu(p) - 1$ in (4.22) and using (4.9) yields

$$c_9\left(p^{2\nu(p)(k+1)-1}n + \frac{3(p^{2\nu(p)(k+1)}-1)}{8}\right) \equiv - sA_{\nu(p)k+\nu(p)-1}(p)c_9\left(\frac{n}{p}\right) \pmod 8,$$

which implies that if $p \nmid n$, then

$$c_9\left(p^{2\nu(p)(k+1)-1}n + \frac{3(p^{2\nu(p)(k+1)}-1)}{8}\right) \equiv 0 \pmod 8. \tag{4.23}$$

By (2.5) and (3.3),

$$\sum_{n=0}^{\infty} a(2n+1)q^n \equiv (q;q)_\infty^9 \pmod 8, \tag{4.24}$$

from which, together with (1.10), we deduce that for $n \geq 0$,

$$a(2n+1) \equiv c_9(n) \pmod 8. \tag{4.25}$$

Congruence (1.16) follows after replacing n by $p^{2\nu(p)(k+1)-1}n + \frac{3(p^{2\nu(p)(k+1)}-1)}{8}$ in (4.25) and utilizing (4.23).

Replacing k by $\nu(p)k$ in (4.1) and utilizing (4.15) and (4.16), we get

$$c_9\left(p^{2\nu(p)k}n + \frac{3(p^{2\nu(p)k}-1)}{8}\right) \equiv g(p)^k c_9(n) \pmod 8, \tag{4.26}$$

where $g(p)$ is defined by (1.12). Setting $n = 1$ in (4.26) yields

$$c_9\left(p^{2\nu(p)k} + \frac{3(p^{2\nu(p)k}-1)}{8}\right) \equiv g(p)^k \pmod 8. \tag{4.27}$$

Replacing n by $p^{2\nu(p)k} + \frac{3(p^{2\nu(p)k}-1)}{8}$ in (4.25) and using (4.27), we arrive at (1.17). This completes the proof. ∎

5 Proof of Theorem 1.7

In this section, we give a proof of Theorem 1.7.

Proof of Theorem 1.7. We prove Theorem 1.7 by induction on the total number of prime factors of n. Suppose that b is a nonnegative integer such that $c_9(b) \equiv 0 \pmod 8$. By (4.25), we have

$$a(2b+1) \equiv 0 \pmod{8},$$

which implies (1.18) holds when $n = 1$ (n has no prime factors).

Suppose that the prime factorization of $8b+3$ is $8b+3 = \prod_{i=1}^{u} f_i \prod_{j=1}^{v} g_j^{\alpha_j}$ with each $\alpha_j \geq 2$. Let $p_1 \geq 3$ be a prime with $(p_1, \prod_{j=1}^{v} g_j^{\alpha_j}) = 1$. Replacing (n,p) by (b, p_1) in (4.7) and utilizing the hypothesis that $c_9(b) \equiv 0 \pmod 8$ yields

$$c_9\left(bp_1^2 + \frac{3(p_1^2-1)}{8}\right) \equiv -sc_9\left(\frac{b - \frac{3(p_1^2-1)}{8}}{p_1^2}\right) \pmod{8}. \tag{5.1}$$

It is easy to check that

$$\frac{b - \frac{3(p_1^2-1)}{8}}{p_1^2} = \frac{8b+3-3p_1^2}{8p_1^2} = \frac{\prod_{i=1}^{u} f_i \prod_{j=1}^{v} g_j^{\alpha_j} - 3p_1^2}{8p_1^2}$$

is not an integer since $\gcd(p_1, \prod_{j=1}^{v} g_j^{\alpha_j}) = 1$. Thus,

$$c_9\left(\frac{b - \frac{3(p_1^2-1)}{8}}{p_1^2}\right) = 0. \tag{5.2}$$

Thanks to (5.1) and (5.2),

$$c_9\left(bp_1^2 + \frac{3(p_1^2-1)}{8}\right) \equiv 0 \pmod{8},$$

from which with (4.25), we see that

$$a\left(2bp_1^2 + \frac{3p_1^2+1}{4}\right) \equiv 0 \pmod{8}.$$

Therefore, (1.18) holds when $n = p_1$ (n has only one prime factor).

Now assume that (1.18) holds for all integers with no more than k prime factors. To prove Theorem 1.7, it suffices to show that (1.18) holds when n has $k+1$ prime factors. Write n as $n = p_1 p_2 \cdots p_k p_{k+1}$ where $3 \leq p_1 \leq p_2 \leq \cdots \leq p_k \leq p_{k+1}$ with $\left(p_1 \cdots p_{k-1} p_k p_{k+1}, 2\prod_{j=1}^{v} g_j^{\alpha_j}\right) = 1$.

By (4.25) and the hypothesis that (1.18) holds for all integers with no more than k prime factors, we have

$$\begin{aligned} c_9&\left(bp_1^2p_2^2\cdots p_{k-1}^2 + \frac{3(p_1^2p_2^2\cdots p_{k-1}^2 - 1)}{8}\right) \\ &\equiv a\left(2bp_1^2p_2^2\cdots p_{k-1}^2 + \frac{3p_1^2p_2^2\cdots p_{k-1}^2+1}{4}\right) \equiv 0 \pmod{8} \end{aligned} \tag{5.3}$$

and

$$c_9\left(bp_1^2p_2^2\cdots p_{k-1}^2p_k^2 + \frac{3(p_1^2p_2^2\cdots p_{k-1}^2p_k^2 - 1)}{8}\right)$$

$$\equiv a\left(2bp_1^2p_2^2\cdots p_{k-1}^2p_k^2+\frac{3p_1^2p_2^2\cdots p_{k-1}^2p_k^2+1}{4}\right)\equiv 0 \pmod 8. \tag{5.4}$$

If we replace (n,p) by $(bp_1^2p_2^2\cdots p_{k-1}^2p_k^2+\frac{3(p_1^2p_2^2\cdots p_{k-1}^2p_k^2-1)}{8}, p_{k+1})$ in (4.7) and utilize (5.4), we have

$$c_9\left(bp_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2+\frac{3(p_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2-1)}{8}\right)$$
$$\equiv -sc_9\left(\frac{bp_1^2p_2^2\cdots p_{k-1}^2p_k^2+\frac{3(p_1^2p_2^2\cdots p_{k-1}^2p_k^2-p_{k+1}^2)}{8}}{p_{k+1}^2}\right) \pmod 8. \tag{5.5}$$

If $p_{k+1}=p_k$, then (5.5) can be rewritten as

$$c_9\left(bp_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2+\frac{3(p_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2-1)}{8}\right)$$
$$\equiv -sc_9\left(bp_1^2p_2^2\cdots p_{k-1}^2+\frac{3(p_1^2p_2^2\cdots p_{k-1}^2-1)}{8}\right)\equiv 0 \pmod 8. \qquad \text{(by (5.3))} \tag{5.6}$$

If $p_{k+1}>p_k$, then $p_{k+1}\notin\{p_1,p_2,\ldots,p_k\}$. Note that

$$\frac{bp_1^2p_2^2\cdots p_{k-1}^2p_k^2+\frac{3(p_1^2p_2^2\cdots p_{k-1}^2p_k^2-p_{k+1}^2)}{8}}{p_{k+1}^2}=\frac{(8b+3)p_1^2p_2^2\cdots p_{k-1}^2p_k^2-3p_{k+1}^2}{8p_{k+1}^2}$$
$$=\frac{p_1^2p_2^2\cdots p_{k-1}^2p_k^2\prod_{i=1}^u f_i\prod_{j=1}^v g_j^{\alpha_j}-3p_{k+1}^2}{8p_{k+1}^2}$$

is not an integer since $\left(p_{k+1},\prod_{j=1}^v g_j^{\alpha_j}\right)=1$ and $\left(p_{k+1},p_1^2p_2^2\cdots p_k^2\right)=1$. Thus,

$$c_9\left(\frac{bp_1^2p_2^2\cdots p_{k-1}^2p_k^2+\frac{3(p_1^2p_2^2\cdots p_{k-1}^2p_k^2-p_{k+1}^2)}{8}}{p_{k+1}^2}\right)=0. \tag{5.7}$$

In light of (5.5)–(5.7),

$$c_9\left(bp_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2+\frac{3(p_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2-1)}{8}\right)\equiv 0 \pmod 8,$$

from which with (4.25), we deduce that (1.18) holds when $n=p_1p_2\cdots p_kp_{k+1}$. Therefore, Theorem 1.7 is proved by induction. This completes the proof. ∎

Remark 5.1 In [12], Newman proved that if $3\le r\le 23$ is an odd integer and m and b are integers such that $c_r(b)\equiv 0 \pmod m$ and $24b+r$ is square-free, then

$$c_r(bn^2+r(n^2-1)/24)\equiv 0 \pmod m, \tag{5.8}$$

where $(n,2)=1$ if $3|r$; $(n,6)=1$, otherwise. He also mentioned that it can be strengthened by discarding the condition that $24b+r$ is square-free and restricting n to be divisible only by primes p such that $p^2\nmid 24b+r$, and $p>2$ when $3|r$; $p>3$ when $(r,3)=1$. Due to (4.25) established in Section 4, we notice that if the integer b with $c_9(b)\equiv 0 \pmod 8$ satisfies that $(3,8b+3)=1$ or $9|8b+3$, Newman's result and Theorem 1.7 are consistent. However, when $3||8b+3$, Theorem 1.7 can imply Newman's congruences (5.8) when $r=9$, but the reverse does not hold. For example, setting $b=72$ in Theorem 1.7, we get $c_9(72)\equiv 0 \pmod 8$ and $8\times 72+3=3\times 193$. By (1.18), we deduce that for $n\ge 1$ with $(n,2)=1$,

$$a((579n^2+1)/4)\equiv c_9(3(193n^2-1)/8)\equiv 0 \pmod 8. \tag{5.9}$$

However, from Newman's strengthening congruences (5.8), we only obtain that (5.9) holds for all n with $(n,6)$=1.

6 Concluding remarks

As seen in the introduction, a number of congruences for the coefficients of Ramanujan's function $\phi(q)$ have been established in recent years. In this paper, we characterize congruences modulo 2 and 4 for the coefficients $a(n)$ of Ramanujan's function $\phi(q)$ by utilizing some results proved by Chen and Garvan [7]. Furthermore, we prove infinite families of congruences modulo 8 for $a(2n+1)$ based on an identity proved by Newman [12]. A natural question is to find congruences for $a(2n)$ modulo 8, 16, 32, etc. However, the proof of congruences for $a(2n)$ modulo 8, 16 will likely require a different approach, for the method for proving congruences for $a(2n)$ used in this paper runs into serious limitations beyond the modulus of 4.

Acknowledgements

Julia Q. D. Du was supported by the National Natural Science Foundation of China (No. 12201177), the Hebei Natural Science Foundation (Nos. A2024205012 and A2023205045), the Science and Technology Project of Hebei Education Department (No. BJK2023092), the China Scholarship Council, the Program for Foreign Experts of Hebei Province. Olivia X.M. Yao was supported by the Natural Science Foundation of Jiangsu Province of China (No. BK20221383).

References

[1] C. Alfes, K. Bringmann and J. Lovejoy, Automorphic properties of generating functions for generalized odd rank moments and odd Durfee symbols, Math. Proc. Cambridge Phil. Soc. 151 (2011) 385–406.

[2] N.D. Baruah and N.M. Begum, Proofs of some conjectures of Chan on Appell–Lerch sums, Ramanujan J. 51(1) (2019) 99–115.

[3] B.C. Berndt, *Ramanujan's Notebooks*, Part III, Springer, New York, 1991.

[4] I.A. Broudy and J. Lovejoy, Arithmetic properties of Schur-type overpartitions, Involve 15 (2022) 489–505.

[5] S.H. Chan, Congruences for Ramanujan's ϕ function, Acta Arith. 153 (2012) 161–189.

[6] R. Chen and F.G. Garvan, Congruences modulo 4 for weight 3/2 eta-products, Bull. Austr. Math. Soc. 103 (2021) 405–417.

[7] R. Chen and F.G. Garvan, A proof of the mod 4 unimodal sequence conjectures and related mock theta functions, Adv. Math. 398 (2022) 108235.

[8] M.C.N. Cheng, J.F.R. Duncan and J.A. Harvey, Umbral moonshine, Commun. Number Theory Phys. 8 (2014) 101–242.

[9] Y.-S. Choi, Identities for Ramanujan's sixth-order mock theta functions, Quart. J. Math. 53 (2002) 147–159.

[10] J.Q.D. Du and D. Tang, Congruences modulo powers of 5 for Ramanujan's ϕ function, J. Math. Anal. Appl. 537 (2024) 128260.

[11] E.T. Mortenson, On Ramanujan's lost notebook and new tenth-order like identities for second-, sixth-, and eighth-order mock theta functions, Bull. Lond. Math. Soc. 56(3) (2024) 1029–1053.

[12] M. Newman, Further identities and congruences for the coefficients of modular forms, Canadian J. Math. 10 (1958) 577–586.

[13] M. Newman, Periodicity modulo m and divisibility properties of the partition function, Trans. Amer. Math. Soc. 97 (1960) 225–236.

[14] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[15] G.N. Watson, Generating functions of class-numbers, Compositio Math. 1 (1935) 39–68.

[16] E.X.W. Xia, New congruence properties for Ramanujan's ϕ function, Proc. Amer. Math. Soc. 149 (2021) 4985–4999.

[17] F.G. Xue and O.X.M. Yao, Explicit congruences modulo 2048 for overpartitions, Ramanujan J. 54 (2021) 63–77.