THE ARITHMETIC OF PARTITIONS INTO DISTINCT PARTS

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1. Introduction.

A partition of the positive integer n into distinct parts is a decreasing sequence of positive integers whose sum is n, and the number of such partitions is denoted by $Q(n)$. If we adopt the convention that $Q(0) = 1$, then we have the generating function

$$\sum_{n=0}^{\infty} Q(n)q^n = \prod_{n=1}^{\infty}(1+q^n) = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + \ldots$$

From Euler's Pentagonal Number Theorem we know that almost all values of $Q(n)$ are even. More precisely,

$$\sum_{n=0}^{\infty} Q(n)q^n \equiv \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} \pmod{2},$$

so that $Q(n)$ is odd if and only if n is a pentagonal number. This fact was generalized by Gordon and Ono [4], who demonstrated that for any positive integer k almost all values of $Q(n)$ are divisible by 2^k, and by Ono and Penniston [7], who found an exact formula for $Q(n)$ modulo 8.

Their techniques do not apply to odd primes, however, and for these primes the situation seems to be more difficult. Apart from some results of Rødseth [9] and Gordon and Hughes [3] on the distribution of $Q(n)$ modulo powers of 5 and 7, little was known. In fact, the strongest result for general primes p was due to Rickert [8], who used techniques from analytic number theory to demonstrate that the number of primes $p < X$ such that p divides at least one value of $Q(n)$ is $\gg \log\log X$.

Subsequently, the second author [6] used the theory of modular forms to significantly improve this result, establishing that for any prime $p \geq 5$ we have

$$\liminf_{X\to\infty} \frac{\{\# n \leq X : Q(n) \equiv 0 \pmod{p}\}}{X} \geq \frac{1}{p}. \tag{1.1}$$

In particular, for any given prime $p \geq 5$, it is certain that a positive proportion of the values of $Q(n)$ are divisible by p. As p grows, however, the guaranteed proportion approaches 0. With our main theorem and its corollary we give substantial improvements on the estimate in (1.1) as well as the class of moduli for which such an estimate can be obtained. For a prime $p \geq 5$, define

$$S_p := \left\{ n \in \mathbb{N} \ : \ n \equiv 0 \pmod{p} \ \text{ or } \ \left(\tfrac{n}{p}\right) = -\left(\tfrac{-2}{p}\right) \right\}. \tag{1.2}$$

Our main result is

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Theorem 1. *Suppose that $p \geq 5$ is prime and that s is a positive integer. Then for almost all $n \in S_p$ we have*

$$Q\left(\frac{n-1}{24}\right) \equiv 0 \pmod{p^s}.$$

As an immediate corollary, we obtain

Corollary 2. *Suppose that M is coprime to 3. Then*

$$\liminf_{X\to\infty} \frac{\{\#n \leq X : Q(n) \equiv 0 \pmod{M}\}}{X} \geq \prod_{\substack{p\mid M\\ p\geq 5,\ \text{prime}}} \frac{p+1}{2p}.$$

It light of such a density result, it is not surprising that congruences in arithmetic progressions are quite common. Combined with the Chinese remainder theorem and the work of Gordon and Ono in the case $p = 2$, the following theorem implies the existence of infinitely many distinct such congruences

$$Q(an+b) \equiv 0 \pmod{M}$$

for every modulus M which is coprime to 3.

Theorem 3. *Suppose that s is a positive integer and that $p \geq 5$ is prime. Then a positive proportion of the primes ℓ have the property that*

$$Q\left(\frac{n\ell-1}{24}\right) \equiv 0 \pmod{p^s}$$

for all n such that $\ell n \in S_p$ and $\ell \nmid n$.

Remark. The results in [6] imply Theorem 3 when $s = 1$ and $n \equiv 0 \pmod{p}$.

Finally, we are able to obtain estimates on the distribution of $Q(n)$ in each of the residue classes modulo M.

Theorem 4. *Let M be an integer coprime to 3. Suppose that there exists a positive integer n_0 such that $n_0 \in S_p$ for each odd prime p dividing M and such that $(Q\left(\frac{n_0-1}{24}\right), M) = 1$. Suppose that $1 \leq t < M$. Then*

$$\#\{n < X : Q(n) \equiv t \pmod{M}\} \gg_M \begin{cases} \frac{X}{\log X} & \text{if } M \text{ is odd,} \\ \frac{\sqrt{X}}{\log X} & \text{if } M \text{ is even.} \end{cases}$$

In the second section we construct modular forms whose Fourier coefficients capture the values of $Q\left(\frac{n-1}{24}\right)$ modulo p^s for those $n \in S_p$. This construction relies on an adaptation of the methods developed in [1] for the study of the unrestricted partition function. In the third section, we use the theory of modular forms (in particular, the theory developed by Serre) to prove our results.

2. Modular Forms and $Q(n)$

Let $M_k(\Gamma_0(N), \chi)$ (respectively $S_k(\Gamma_0(N), \chi)$) denote the usual vector spaces of holomorphic modular (respectively cusp) forms of integral weight k, level N and character χ (for the relevant background on modular forms, see [5]). If $p \geq 5$ is prime, then we fix the notation

$$\delta_p := \frac{p^2-1}{24}, \tag{2.1}$$

$$\sigma_p := \frac{(p^2-1)^2}{24}. \tag{2.2}$$

If t is a squarefree integer, then denote by χ_t the usual Kronecker character for $\mathbb{Q}(\sqrt{t})$. All of our results rely on the following theorem, whose proof occupies this section.

Theorem 5. *Let $p \geq 5$ be prime, let σ_p and δ_p be defined as in (2.2) and (2.1), and let s be a positive integer. Then there exists an integer $t_p \geq s-1$ and a cusp form $H_{p,s}(z) \in S_{p^{t_p}(p^3-1)}(\Gamma_0(1152p^3), \chi_2)$ such that*

$$H_{p,s}(z) \equiv \sum_{n \equiv 0 \pmod{p}} Q(n-\sigma_p) q^{24n-24\sigma_p+1} + 2 \sum_{\left(\frac{n}{p}\right) = -\left(\frac{-3}{p}\right)} Q(n-\sigma_p) q^{24n-24\sigma_p+1} \pmod{p^s}. \tag{2.3}$$

We remark that Gordon and Ono [4] have shown that if s is a positive integer, then there exists a cusp form $H_{2,s}(z) \in S_{2^s}(\Gamma_0(1152), \chi_2)$ for which

$$H_{2,s}(z) \equiv \sum_{n=0}^{\infty} Q(n) q^{24n+1} \pmod{2^s}. \tag{2.4}$$

Although the forms $H_{p,s}$ typically lie in spaces of large dimension, it will be possible to describe them explicitly for for small values of p^s. Let Dedekind's eta function be given by

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n).$$

Then, for example, we find by applying a theorem of Sturm [11] and a finite computation that

$$\begin{aligned} 3H_{5,1}(z) &\equiv \sum_{n=0}^{\infty} Q(5n) q^{120n+1} + 3\sum_{n=0}^{\infty} Q(5n+1) q^{120n+25} + \sum_{n=0}^{\infty} Q(5n+2) q^{120n+49} \\ &\equiv \frac{\eta^{47}(24z)}{\eta^{23}(48z)} - \frac{\eta^{25}(48z)}{\eta(24z)} \pmod{5}. \end{aligned}$$

We turn to the proof of Theorem 5. Let δ_p and σ_p be defined as above, and define the eta-product

$$f_p(z) := \frac{\eta(2z)\eta^{8p\delta_p-p}(2pz)\eta^{8p\delta_p+p}(pz)}{\eta(z)}. \tag{2.5}$$

Since

$$\frac{\eta(2z)}{\eta(z)} = \sum_{n=0}^{\infty} Q(n) q^{n+1/24},$$

it follows that

$$f_p(z) = \left(\sum_{n=0}^{\infty} Q(n) q^{n+\sigma_p} \right) \cdot \prod_{n=1}^{\infty} (1-q^{2pn})^{8p\delta_p-p} (1-q^{pn})^{8p\delta_p+p}. \tag{2.6}$$

Now recall that if $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(N), \chi)$, and ψ is a Dirichlet character modulo M, then we have

$$f(z) \otimes \psi := \sum_{n=1}^{\infty} \psi(n)a(n)q^n \in S_k(\Gamma_0(NM^2), \chi\psi^2). \tag{2.7}$$

If t is a non-negative integer, we define $F_{p,t}(z)$ by

$$F_{p,t}(z) := \left(f_p(z) - \left(\frac{-3}{p}\right) \cdot f_p(z) \otimes \left(\frac{\bullet}{p}\right) \right) \cdot \left(\frac{\eta^{p^3}(z)}{\eta(p^3 z)} \right)^{2p^t}. \tag{2.8}$$

Using standard criteria for eta-products (see, for example, [2]), we see that

$$f_p(z) \in S_{8p\delta_p}(\Gamma_0(2p))$$

and

$$\left(\frac{\eta^{p^3}(z)}{\eta(p^3 z)} \right)^{2p^t} \in M_{p^t(p^3-1)}(\Gamma_0(p^3)).$$

Using (2.7), we conclude that

$$F_{p,t}(z) \in S_{8p\delta_p + p^t(p^3-1)}(\Gamma_0(2p^3)). \tag{2.9}$$

It is easy to see that

$$\frac{\eta^{p^3}(z)}{\eta(p^3 z)} \equiv 1 \pmod{p},$$

and it follows that

$$\left(\frac{\eta^{p^3}(z)}{\eta(p^3 z)} \right)^{2p^t} \equiv 1 \pmod{p^s} \quad \text{if } t \geq s-1.$$

A computation using this fact together with (2.6) and (2.8) shows that for $t \geq s-1$ we have

$$\frac{F_{p,t}(24z)}{\eta^{8p\delta_p - p}(48pz)\eta^{8p\delta_p + p}(24pz)} \equiv \sum_{n \equiv 0 \pmod{p}} Q(n-\sigma_p)q^{24n-24\sigma_p+1} + 2 \sum_{\left(\frac{n}{p}\right) = -\left(\frac{-3}{p}\right)} Q(n-\sigma_p)q^{24n-24\sigma_p+1} \pmod{p^s}. \tag{2.10}$$

The form $H_{p,s}(z)$ will be defined by

$$H_{p,s}(z) := \frac{F_{p,t}(24z)}{\eta^{8p\delta_p - p}(48pz)\eta^{8p\delta_p + p}(24pz)} \tag{2.11}$$

for any sufficiently large value of t. We have

$$\eta^{8p\delta_p - p}(48pz)\eta^{8p\delta_p + p}(24pz) \in S_{8p\delta_p}(\Gamma_0(1152p), \chi_2),$$

and therefore, using (2.9), we conclude that $H_{p,s}(z)$ is a modular form of weight $p^t(p^3-1)$ and character χ_2 on $\Gamma_0(1152p^3)$. $H_{p,s}$ is clearly holomorphic on the upper half-plane; therefore to finish the proof of Theorem 5, we need only to show that if t is sufficiently large, then $H_{p,s}(z)$

vanishes at each cusp. To accomplish this goal, it will suffice to show that if t is sufficiently large, then

$$G_{p,t}(z) := \left(\frac{F_{p,t}(z)}{\eta^{8p\delta_p - p}(2pz)\eta^{8p\delta_p+p}(pz)} \right)^{24}$$

vanishes at each cusp. We note that $G_{p,t}(z)$ is a modular form on $\Gamma_0(2p^3)$. There is a standard formula (see [2] for example) to compute the order of an eta-product at a cusp. Using this formula, we find that $\frac{\eta^{p^3}(z)}{\eta(p^3z)}$ vanishes at all cusps of $\Gamma_0(2p^3)$ with the exception of ∞ and $\frac{1}{p^3}$. From (2.5), (2.6), and (2.8), it is clear that $G_{p,t}(z)$ vanishes at ∞. So if t is sufficiently large, we need only to show that $G_{p,t}(z)$ vanishes at $\frac{1}{p^3}$. The remainder of the section is devoted to this task.

If $f \in M_k(\Gamma_0(2p^3))$, then f has an expansion at $\frac{1}{p^3}$ in powers of $q^{1/2}$. The form $f_p(z)$ given in (2.5) is on $\Gamma_0(2p^3)$ and, by the formula mentioned above, vanishes to even order $\frac{p^4-1}{24}$ at $\frac{1}{p^3}$. Let $j := 8p\delta_p$ be the weight of $f_p(z)$, and write $\frac{p^4-1}{24} = 2n_0$ for convenience. Then at $\frac{1}{p^3}$, $f_p(z)$ has an expansion

$$f_p(z) \mid_j \begin{pmatrix} 1 & 0 \\ p^3 & 1 \end{pmatrix} = \sum_{n=2n_0}^{\infty} b(n) q^{n/2}. \tag{2.12}$$

Let $g := \sum_{v=0}^{p-1} (\frac{v}{p}) e^{2\pi i v/p}$ be the usual Gauss sum. Arguing as in the proof of [6, Ch. III, Prop. 17], we see that

$$f_p \otimes \left(\frac{\bullet}{p}\right)(z) = \frac{g}{p} \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) f_p(z) \mid_j \begin{pmatrix} 1 & -v/p \\ 0 & 1 \end{pmatrix}. \tag{2.13}$$

We say that matrices γ_1 and γ_2 are $\Gamma_0(N)$-equivalent if $\gamma_1\gamma_2^{-1} \in \Gamma_0(N)$. A straightforward computation shows that

$$\begin{pmatrix} 1 & -v/p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p^3 & 1 \end{pmatrix} \quad \text{is } \Gamma_0(2p^3)\text{-equivalent to} \begin{cases} \begin{pmatrix} 1 & 0 \\ p^3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -v/p \\ 0 & 1 \end{pmatrix} & \text{if } v \text{ is even,} \\ \begin{pmatrix} 1 & 0 \\ p^3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1-v/p \\ 0 & 1 \end{pmatrix} & \text{if } v \text{ is odd.} \end{cases}$$

Using this fact together with (2.12) and (2.13), we obtain

$$f_p \otimes \left(\frac{\bullet}{p}\right)(z) \mid_j \begin{pmatrix} 1 & 0 \\ p^3 & 1 \end{pmatrix} =$$
$$\frac{g}{p} \sum_{v \text{ even}} \left(\frac{v}{p}\right) \sum_{n=2n_0}^{\infty} b(n) q^{n/2} \cdot e^{-\pi i n v/p} + \frac{g}{p} \sum_{v \text{ odd}} \left(\frac{v}{p}\right) \sum_{n=2n_0}^{\infty} b(n) q^{n/2} \cdot e^{-\pi i n(1+v/p)}.$$

Since $n_0 = (p^4-1)/48$ and $g^2 = (\frac{-1}{p}) \cdot p$, the first term in the expansion of $f_p \otimes \left(\frac{\bullet}{p}\right)(z)$ at $\frac{1}{p^3}$ is

$$\left(\frac{g}{p} \sum_{v=0}^{p-1} \left(\frac{v}{p}\right) e^{-2\pi i n_0 v/p} \right) b(2n_0) q^{n_0} = \frac{g^2}{p} \left(\frac{-n_o}{p}\right) b(2n_0) q^{n_0} = \left(\frac{-3}{p}\right) b(2n_0) q^{n_0}.$$

Using (2.8), we see that $F_{p,t}(z)$ has order $\geq 2n_0 + 1 = \frac{p^4+23}{24}$ at $\frac{1}{p^3}$. Finally, since $(\eta^{8p\delta_p - p}(2pz)\eta^{8p\delta_p+p}(pz))^{24}$ has order p^4 at $\frac{1}{p^3}$, it follows that the form $G_{p,t}(z)$ has order ≥ 23 at $\frac{1}{p^3}$. This shows that, for sufficiently large t, the form $H_{p,s}(z)$ given in (2.11) is a cusp form. □

3. Proof of the main results

Proof of Theorem 1 and Corollary 2. Suppose that M is coprime to 3 and that $p \geq 5$ is a prime with $s := \mathrm{ord}_p(M) \geq 1$. Let $H_{p,s}(z)$ be the cusp form given by Theorem 5. After rewriting the expression in that theorem, we find that

$$H_{p,s}(z) \equiv \sum_{n \equiv 0 \pmod{p}} Q\left(\tfrac{n-1}{24}\right) q^n + 2 \sum_{\left(\frac{n}{p}\right) = -\left(\frac{-2}{p}\right)} Q\left(\tfrac{n-1}{24}\right) q^n \pmod{p^s}. \tag{3.1}$$

Since $H_{p,s}$ is an integral weight cusp form, a theorem of Serre [10] implies that almost all of the coefficients of $H_{p,s}$ are divisible by p^s. Recall the definition (1.2) of the set S_p. By Serre's result, we see that for almost all $n \in S_p$ we have

$$Q\left(\tfrac{n-1}{24}\right) \equiv 0 \pmod{p^s}.$$

This proves Theorem 1. For the corollary, we note that each set S_p is a union of arithmetic progressions with modulus p. It follows that for almost all $n \in \cap_{\substack{p|M \\ p \geq 5}} S_p$ we have $Q\left(\frac{n-1}{24}\right) \equiv 0 \pmod{M}$. Corollary 2 follows since each set S_p is comprised of $\frac{p+1}{2}$ progressions modulo p, and since for any s, almost all values of $Q\left(\frac{n-1}{24}\right)$ are divisible by 2^s. □

Proof of Theorem 3. Suppose that $p \geq 5$ is prime and that s is a positive integer, and let $H_{p,s}(z) := \sum_{n=1}^{\infty} a_{p,s}(n) q^n$ be the form of integral weight k and level $1152p^3$ given in Theorem 5. If ℓ is prime, then let $T(\ell)$ be the usual Hecke operator of index ℓ on the space $S_k(\Gamma_0(1152p^3), \chi_2)$. A result of Serre (see section 6.4 of [10]) implies that a positive proportion of the primes $\ell \equiv -1 \pmod{p}$ have the property that $H_{p,s}(z) \mid T(\ell) \equiv 0 \pmod{p^s}$. In other words, we have

$$\sum_{n=1}^{\infty} \Big(a_{p,s}(\ell n) + \chi_2(\ell)\ell^{k-1} a_{p,s}(n/\ell) \Big) q^n \equiv 0 \pmod{p^s}.$$

By (3.1), we see that if $\ell n \in S_p$, and $\ell \nmid n$, then $Q\left(\frac{\ell n - 1}{24}\right) \equiv 0 \pmod{p^s}$. □

Proof of Theorem 4. Let the prime factorization of M be given by

$$M = \prod_{p|M} p^{s_p}.$$

For each $p \geq 5$ appearing in this product, let $\Omega_p := p^{t_p}(p^3 - 1)$ be the weight of the form $H_{p,s_p}(z)$ given by Theorem 5. Also, if M is even, then let $H_{2,s_2}(z)$ be the form of weight 2^{s_2} discussed in the remark following Theorem 5; in this case, define $\Omega_2 := 2^{s_2}$. Then define

$$W := 2 \cdot \prod_{p|M} \Omega_p.$$

A simple calculation shows that for each $p \mid M$ with $p \geq 5$, the quantity $W - \Omega_p$ has the form $k_p(p-1)/2$, where $k_p \geq s_p - 1$ is an even integer. Notice that the form

$$\left(\frac{\eta^p(z)}{\eta(pz)}\right)^{k_p} \in S_{k_p(p-1)/2}\left(\Gamma_0(p)\right)$$

is congruent to 1 modulo p^{s_p}. It follows that for each $p \geq 5$, the form

$$H_{p,s_p}(z) \cdot \left(\frac{\eta^p(z)}{\eta(pz)}\right)^{k_p} \in S_W\left(\Gamma_0(1152p^3), \chi_2\right)$$

is congruent modulo p^{s_p} to $H_{p,s_p}(z)$. Moreover, if M is even, then $W - \Omega_2 = k_2 2^{s_2}$ for some integer k_2. Therefore

$$H_{2,s_2}(z) \cdot \left(\frac{\eta^2(24z)}{\eta(48z)}\right)^{k_2 \cdot 2^{s_2+1}} \in S_W\left(\Gamma_0(1152), \chi_2\right)$$

is congruent modulo 2^{s_2} to $H_{2,s_2}(z)$. If we define

$$N := 1152 \prod_{\substack{p \mid M \\ p \geq 5}} p^3,$$

then, with a slight abuse of notation, we may assume that for each $p \mid M$ we have

$$H_{p,s_p}(z) \in S_W\left(\Gamma_0(N), \chi_2\right).$$

Lemma 6. *Adopt the hypotheses of Theorem 4. If $1 \leq t < M$, then there exists an integer n_t such that $n_t \in S_p$ for all $p \mid M$ and such that $Q\left(\frac{n_t-1}{24}\right) \equiv t \pmod{M}$.*

Proof. A theorem of Serre (§6.4 of [10]) implies that a positive proportion of the primes $\ell \equiv 1 \pmod{N}$ have the property that if $f(z) := \sum_{n=1}^{\infty} a(n)q^n$ is any form in $S_W\left(\Gamma_0(N), \chi_2\right)$ with integer coefficients, then for every n such that $\ell \nmid n$, and for every $r \geq 0$, we have

$$a(n\ell^r) \equiv (r+1)a(n) \pmod{M}.$$

For $p \mid M$, write the form $H_{p,s_p}(z) := \sum_{n=1}^{\infty} a_{p,s_p}(n)q^n$ as above, and let n_0 be the distinguished integer given in the hypothesis of Theorem 4. Then for a positive proportion of the primes $\ell \equiv 1 \pmod{N}$, we have, for each p, and for $0 \leq r \leq M-2$,

$$a_{p,s_p}(n_0\ell^r) \equiv (r+1)a_{p,s_p}(n_0) \pmod{M}. \tag{3.2}$$

Using (2.4) and (3.1) (note that $n_0\ell^r \equiv n_0 \pmod{p}$), this gives, for each p,

$$Q\left(\frac{n_0\ell^r-1}{24}\right) \equiv (r+1)Q\left(\frac{n_0-1}{24}\right) \pmod{p^s}.$$

Since this holds for every $p \mid M$, we obtain

$$Q\left(\frac{n_0\ell^r-1}{24}\right) \equiv (r+1)Q\left(\frac{n_0-1}{24}\right) \pmod{M}. \tag{3.3}$$

The lemma follows by letting r range over the integers $0, 1, \ldots, M-2$. □

We turn to the proof of Theorem 4. Suppose first that M is odd. For each t with $1 \leq t < M$, let n_t be the integer given by Lemma 6. Arguing as (3.2) and (3.3) with $r = 1$, we find that a positive proportion of the primes $\ell \equiv 1 \pmod{N}$ have

$$Q\left(\frac{n_t\ell-1}{24}\right) \equiv 2Q\left(\frac{n_t-1}{24}\right) \equiv 2t \pmod{M}. \tag{3.4}$$

The quantitative estimate in Theorem 4 follows by Dirichlet's Theorem. When M is even the situation is similar. However, we must take $r = 2$ in (3.2) and (3.3); as a result we obtain a quantitative bound of the form $\sqrt{X}/\log X$. $\square$

4. Closing Remarks

Although we now have answers to many questions about the divisibility and distribution of the number of partitions into distinct parts, there remain some unresolved problems. For instance, the techniques developed here and in [6] do not seem to be useful when $p = 3$. One also naturally wonders about the correct value of

$$\liminf_{X\to\infty} \frac{\#\{n < X : Q(n) \equiv 0 \pmod{M}\}}{X}. \tag{4.5}$$

For any modulus p^s and for any r, it is in fact possible to construct a modular form $F_{p,s,r}(z)$ on $\Gamma_1(1152p^3)$ such that

$$F_{p,s,r}(z) \equiv \sum_{n\equiv r \pmod{p}} Q\left(\frac{n-1}{24}\right) q^n \pmod{p^s}.$$

If for any $r \notin S_p$ we could realize $F_{p,s,r}(z)$ as a holomorphic cusp form $\pmod{p^s}$, then we would immediately obtain an improvement of the estimate in Corollary 2 for (4.5). However, it is not clear that the arguments in Theorem 5 can be extended to forms on $\Gamma_1(N)$.

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