ON A ROGERS-RAMANUJAN TYPE IDENTITY FROM CRYSTAL BASE THEORY

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Abstract. We refine and generalise a Rogers-Ramanujan type partition identity arising from crystal base theory. Our proof uses the variant of the method of weighted words recently introduced by the first author.

1. Introduction

The *Rogers-Ramanujan identities* assert that for $i = 0$ or 1 and for all non-negative integers n, the number of partitions of n into parts differing by at least two and having at most i ones is equal to the number of partitions of n into parts congruent to $\pm(2-i)$ modulo 5. A Lie-theoretic interpretation and proof of these identities were given by Lepowsky and Wilson [13, 14]; the partitions defined by congruence conditions correspond to the principally specialised Weyl-Kac character formula for level 3 standard $A_1^{(1)}$-modules, while the partitions defined by difference conditions correspond to bases constructed from vertex operators.

The vertex operator approach of Lepowsky and Wilson was subsequently extended by many authors to treat level k and/or other affine Lie algebras, beginning a fruitful interaction between Lie theory and partition theory. For some examples of vertex operator constructions leading to partition identities, see [4, 5, 15, 16, 17, 20], and for some combinatorial approaches to such partition identities we refer to [1, 3, 7, 11].

In [19], Primc observed that the difference conditions in certain vertex operator constructions correspond to energy functions of perfect crystals, and in [18] he studied partition identities of the Rogers-Ramanujan type coming from crystal base theory. Here the Weyl-Kac character formula again gives the partitions defined by congruence conditions, while the crystal base character formula of Kang, Kashiwara, Misra, Miwa, Nakashima and Nakayashiki [12] ensures the correspondence with partitions defined by difference conditions.

In this paper we will be concerned with the following partition identity of Primc. Consider partitions $(\lambda_1, \lambda_2, \dots)$ into integers in four colours a, b, c, d, with the order

$$1_a < 1_b < 1_c < 1_d < 2_a < 2_b < 2_c < 2_d < \cdots , \tag{1.1}$$

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such that the entry (x, y) in the matrix D gives the minimal difference between λ_i of colour $c(\lambda_i) = x$ and λ_{i+1} of colour $c(\lambda_{i+1}) = y$:

$$D = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}. \tag{1.2}$$

Then under the transformations

$$\begin{aligned} k_a &\to 2k - 1, \\ k_b &\to 2k, \\ k_c &\to 2k, \\ k_d &\to 2k + 1, \end{aligned} \tag{1.3}$$

the generating function for these coloured partitions is equal to the generating function for ordinary partitions[1],

$$\frac{1}{(q;q)_\infty}.$$

Here we have used the notation

$$(a;q)_\infty = \prod_{k \geq 1} (1 - aq^{k-1}).$$

Our main result is a generalisation of Primc's identity to a refined partition identity with 3 colours at the non-dilated level.

Theorem 1.1. *Let $A(n; k, \ell, m)$ denote the number of four-coloured partitions of n with the ordering* (1.1) *and difference matrix* (1.2)*, having k parts coloured a, ℓ parts coloured c and m parts coloured d. Then*

$$\sum_{n,k,\ell,m \geq 0} A(n; k, \ell, m) q^n a^k c^\ell d^m = \frac{(-aq;q^2)_\infty (-dq;q^2)_\infty}{(q;q)_\infty (cq;q^2)_\infty}.$$

Under the dilations

$$\begin{aligned} q &\to q^2, \\ a &\to aq^{-1}, \\ c &\to c, \\ d &\to dq, \end{aligned} \tag{1.4}$$

the ordering of integers (1.1) becomes

$$1_a < 2_b < 2_c < 3_d < 3_a < 4_b < 4_c < 5_d < \cdots$$

[1]This was actually stated with a question mark by Primc, who was unsure of the application of the crystal base formula of [12] to the case of the $A_1^{(1)}$-crystal whose energy matrix is (1.2). We are indebted to K. Misra for pointing out that this case is covered by Section 1.2 of [12], rendering Primc's question mark unnecessary.

and the matrix D in (1.2) becomes

$$D_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 4 & 1 & 3 & 2 \\ 3 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 3 & 1 & 4 \end{pmatrix} \end{array}.$$

Considering the a-parts and c-parts unlabelled and the b-parts and the d-parts primed, this gives the following refinement of Primc's identity in terms of two-coloured partitions.

Corollary 1.2. *Let $\mathcal{P}_2$ denote the set of partitions where parts may appear in two colors, say ordinary and primed. Let $A_2(n; k, \ell, m)$ denote the number of partitions $(\lambda_1, \lambda_2, \dots)$ of n in $\mathcal{P}_2$ having k odd ordinary parts, ℓ even ordinary parts, and m odd primed parts, such that no part is $1'$ and*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 1, & \text{if } \lambda_i \text{ is odd and } c(\lambda_i) \neq c(\lambda_{i+1}), \\ 2, & \text{if } \lambda_i \text{ is even and } c(\lambda_i) \neq c(\lambda_{i+1}), \\ 3, & \text{if } \lambda_i \text{ is odd and } c(\lambda_i) = c(\lambda_{i+1}). \end{cases}$$

Then

$$\sum_{n,k,\ell,m \geq 0} A_2(n; k, \ell, m) q^n a^k c^\ell d^m = \frac{(-aq; q^4)_\infty (-dq^3; q^4)_\infty}{(q^2; q^2)_\infty (cq^2; q^4)_\infty}. \tag{1.5}$$

In other words, if $B_2(n; k, \ell, m)$ denotes the number of partitions of n in $\mathcal{P}_2$ such that odd parts are distinct and only parts 2 modulo 4 may be primed, having k parts congruent to 1 modulo 4, ℓ primed parts, and m parts congruent to 3 modulo 4, then

$$A_2(n; k, \ell, m) = B_2(n; k, \ell, m).$$

One recovers Primc's identity by setting $a = c = d = 1$, as the dilations in (1.4) correspond to (1.3) and the infinite product in (1.5) becomes

$$\begin{aligned} \frac{(-q; q^4)_\infty (-q^3; q^4)_\infty}{(q^2; q^2)_\infty (q^2; q^4)_\infty} &= \frac{(-q; q^2)_\infty (q; q^2)_\infty}{(q^2; q^2)_\infty (q^2; q^4)_\infty (q; q^2)_\infty} \\ &= \frac{(q^2; q^4)_\infty}{(q; q)_\infty (q^2; q^4)_\infty} \\ &= \frac{1}{(q; q)_\infty}. \end{aligned}$$

Another nice application of Theorem 1.1 is the dilation

$$\begin{aligned} q &\to q^4, \\ a &\to aq^{-3}, \\ c &\to cq^{-2}, \\ d &\to dq^3, \end{aligned}$$

where the ordering of integers (1.1) becomes

$$1_a < 2_c < 4_b < 5_a < 6_c < 7_d < 8_b < 9_a < \cdots,$$

the matrix D becomes

$$D_4 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 8 & 1 & 7 & 2 \\ 7 & 0 & 6 & 1 \\ 1 & 2 & 0 & 3 \\ 6 & 7 & 5 & 8 \end{pmatrix} \end{array},$$

and we obtain the following partition identity.

Corollary 1.3. *Let $A_4(n;k,\ell,m)$ denote the number of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of n with k, ℓ, and m parts congruent to 1, 2, and 3 modulo 4, respectively, such that $\lambda_i \neq 3$ and $\lambda_i - \lambda_{i+1} \geq 5$ if $\lambda_i \equiv 3 \pmod 4$ or if $\lambda_i \equiv 0$ or $1 \pmod 4$ and $\lambda_{i+1} \equiv 1$ or $2 \pmod 4$. Then*

$$\sum_{n,k,\ell,m \geq 0} A_4(n;k,\ell,m) q^n a^k c^\ell d^m = \frac{(-aq;q^8)_\infty (-dq^7;q^8)_\infty}{(q^4;q^4)_\infty (cq^2;q^8)_\infty}. \tag{1.6}$$

In other words, if $B_4(n;k,\ell,m)$ denotes the number of partitions of n into even parts not congruent to 6 modulo 8 and distinct odd parts congruent to ± 1 modulo 8, with k, ℓ, and m parts congruent to $1, 2$, and 7 modulo 8, respectively, then

$$A_4(n;k,\ell,m) = B_4(n;k,\ell,m).$$

The proof of Theorem 1.1 relies on the variant of the method of weighted words recently introduced by the first author [9, 11]. The difference with the original method of Alladi and Gordon [2] is that instead of using the minimal partitions and q-series identities, we use recurrences and q-difference equations (with colours) coming from the difference conditions in matrix D and we solve them directly. This is presented in the next section, and in Section 3 we give some examples and another application of Theorem 1.1.

2. Proof of Theorem 1.1

2.1. **Idea of the proof.** To prove Theorem 1.1, we proceed as follows.

For $|a|, |c|, |d|, |q| < 1$, $k \in \mathbb{N}^*$, define $G_k(q) = G_k(q;a,c,d)$ (resp. $E_k(q) = E_k(q;a,c,d)$) to be the generating function for coloured partitions satisfying the difference conditions from matrix D with the added condition that the largest part is at most (resp. equal to) k.

Then we want to find $G_\infty(q) := \lim_{k \to \infty} G_k(q)$, which is the generating function for all partitions with difference conditions, as there is no more restriction on the size of the largest part.

We start by using the matrix D to give simple q-difference equations relating the $G_k(q)$'s and the $E_k(q)$'s. Then we combine them to obtain a big q-difference equation involving only $G_{k_d}(q)$'s. This is done in Section 2.2.

Then we use the technique consisting of going back and forth from q-difference equations to recurrences introduced by the first author [8, 9, 10], and conclude using Appell's lemma. This is done in Section 2.3.

2.2. **Recurrences and q-difference equations.** We use combinatorial reasoning on the largest part of partitions to state some recurrences. We have the following identities:

Lemma 2.1. *For all* $k \in \mathbb{N}^*$,

$$G_{k_d}(q) - G_{k_c}(q) = E_{k_d}(q) = dq^k(E_{k_c}(q) + E_{k_a}(q) + G_{(k-1)_c}(q)), \tag{2.1}$$

$$G_{k_c}(q) - G_{k_b}(q) = E_{k_c}(q) = cq^k(E_{k_c}(q) + E_{k_a}(q) + G_{(k-1)_c}(q)), \tag{2.2}$$

$$G_{k_b}(q) - G_{k_a}(q) = E_{k_b}(q) = q^k(E_{k_b}(q) + G_{(k-1)_d}(q)), \tag{2.3}$$

$$G_{k_a}(q) - G_{(k-1)_d}(q) = E_{k_a}(q) = aq^k(E_{(k-1)_b}(q) + G_{(k-2)_d}(q)). \tag{2.4}$$

Proof: We give details only for (2.1). The others are similar. The first equality $G_{k_d}(q) - G_{k_c}(q) = E_{k_d}(q)$ follows directly from the definitions. Next, in a partition counted by $E_{k_d}(q)$ we remove the largest part of size k and colour d, giving the factor dq^k. An examination of the difference conditions in (1.2) shows that in the partition remaining the largest part could be k_c, k_a, or a part at most $(k-1)_c$. This corresponds to the terms $E_{k_c}(q) + E_{k_a}(q) + G_{(k-1)_c}(q)$. ☐

Together with the initial conditions

$$\begin{aligned}
&E_{k_a}(q) = E_{k_b}(q) = E_{k_c}(q) = E_{k_d}(q) = 0 \text{ for all } k \leq -1,\\
&E_{0_b}(q) = 1,\\
&E_{0_a}(q) = E_{0_c}(q) = E_{0_d}(q) = 0,\\
&G_{k_a}(q) = G_{k_b}(q) = G_{k_c}(q) = G_{k_d}(q) = 0 \text{ for all } k \leq -1,\\
&G_{0_a}(q) = 0,\\
&G_{0_b}(q) = G_{0_c}(q) = G_{0_d}(q) = 1,
\end{aligned}$$

these q-difference equations completely characterise the coloured partitions with difference conditions of Theorem 1.1.

We now want to find a q-difference equation involving only G_{k_d}'s.

Proposition 2.2. *We have*

$$\begin{aligned}(1-cq^k)G_{k_d}(q) = {}& \frac{1-cq^{2k}}{1-q^k}G_{(k-1)_d}(q)\\
&+ \frac{aq^k + dq^k + adq^{2k}}{1-q^{k-1}}G_{(k-2)_d}(q) + \frac{adq^{2k-1}}{1-q^{k-2}}G_{(k-3)_d}(q).\end{aligned} \tag{2.5}$$

Proof: Let us first observe that

$$G_{k_b}(q) = G_{(k-1)_d}(q) + E_{k_a}(q) + E_{k_b}(q). \tag{2.6}$$

By Equation (2.3), it is clear that for all k,

$$E_{k_b}(q) = \frac{q^k}{1-q^k}G_{(k-1)_d}(q). \tag{2.7}$$

Now substituting this with k replaced by $k-1$ into Equation (2.4), we get

$$E_{k_a}(q) = \frac{aq^k}{1-q^{k-1}}G_{(k-2)_d}(q). \tag{2.8}$$

Thus combining Equations (2.6), (2.8) and (2.7), we obtain

$$G_{k_b}(q) = \frac{1}{1-q^k}G_{(k-1)_d}(q) + \frac{aq^k}{1-q^{k-1}}G_{(k-2)_d}(q). \tag{2.9}$$

Let us now turn to $E_{k_c}(q)$. By Equation (2.2), we have

$$E_{k_c}(q) = \frac{cq^k}{1-cq^k}\left(E_{k_a}(q) + G_{(k-1)_c}(q)\right).$$

Substituting (2.8), we obtain

$$E_{k_c}(q) = \frac{cq^k}{1-cq^k}\left(\frac{aq^k}{1-q^{k-1}}G_{(k-2)_d}(q) + G_{(k-1)_c}(q)\right). \tag{2.10}$$

Finally, by Equations (2.1) and (2.2) and the initial conditions, for all k, we have

$$dE_{k_c}(q) = cE_{k_d}(q).$$

Combining that with (2.10), we obtain that for all k,

$$E_{k_d}(q) = \frac{dq^k}{1-cq^k}\left(\frac{aq^k}{1-q^{k-1}}G_{(k-2)_d}(q) + G_{(k-1)_c}(q)\right). \tag{2.11}$$

Using Equations (2.9), (2.10), (2.11) and the fact that

$$G_{k_d}(q) = G_{k_b}(q) + E_{k_c}(q) + E_{k_d}(q),$$

we obtain

$$\begin{aligned} G_{k_d}(q) = {} & \frac{1}{1-q^k}G_{(k-1)_d}(q) + \frac{aq^k}{1-q^{k-1}}G_{(k-2)_d}(q) \\ & + \frac{(c+d)q^k}{1-cq^k}\left(\frac{aq^k}{1-q^{k-1}}G_{(k-2)_d}(q) + G_{(k-1)_c}(q)\right). \end{aligned}$$

Rearranging gives an expression for $G_{(k-1)_c}(q))$ in terms of $G_{k_d}(q)$'s.

$$\begin{aligned} G_{(k-1)_c}(q) = \frac{1-cq^k}{(c+d)q^k} & \left(G_{k_d}(q) - \frac{1}{1-q^k}G_{(k-1)_d}(q)\right. \\ & \left. - \frac{aq^k(1+dq^k)}{(1-q^{k-1})(1-cq^k)}G_{(k-2)_d}(q)\right). \end{aligned} \tag{2.12}$$

Substituting this into (2.10) and simplifying leads to

$$\begin{aligned} E_{k_c}(q) = {} & \frac{c}{c+d}G_{k_d}(q) - \frac{c}{(c+d)(1-q^k)}G_{(k-1)_d}(q) \\ & - \frac{acq^k}{(c+d)(1-q^{k-1})}G_{(k-2)_d}(q). \end{aligned} \tag{2.13}$$

On the other hand, using (2.9), (2.12) and the fact that

$$E_{k_c}(q) = G_{k_c}(q) - G_{k_b}(q),$$

we obtain

$$\begin{aligned} E_{k_c}(q) = {} & \frac{1-cq^{k+1}}{(c+d)q^{k+1}}G_{(k+1)_d}(q) - \frac{1-cq^{k+1}}{(c+d)q^{k+1}(1-q^{k+1})}G_{k_d}(q) \\ & - \frac{a+c+d+adq^{k+1}}{(c+d)(1-q^k)}G_{(k-1)_d}(q) - \frac{aq^k}{1-q^{k-1}}G_{(k-2)_d}(q). \end{aligned} \tag{2.14}$$

Equating (2.13) and (2.14) and replacing k by $k-1$ yields the desired recurrence equation. □

2.3. **Finding** $\lim_{k\to\infty} G_k(q;a,c,d)$**.** Equation (2.5) is a recurrence of order 3, and therefore not so easy to solve as it is. Thus we will transform it into a q-difference equation of lower order.

For all $k \geq 0$, let us define

$$H_k := \frac{G_{k_d}(q)}{1-q^{k+1}}.$$

Thus (H_k) satisfies the following recurrence equation:

$$(1-cq^k-q^{k+1}+cq^{2k+1})H_k = (1-cq^{2k})H_{k-1}+(aq^k+dq^k+adq^{2k})H_{k-2}+adq^{2k-1}H_{k-3}. \tag{2.15}$$

To obtain the correct values of H_k for all $k \geq 0$ using Equation (2.15), we define the initial values $H_{-1} = 1$ and $H_k = 0$ for all $k \leq -2$.

We now define

$$f(x) := \sum_{k\geq 0} H_{k-1}x^k,$$

and convert Equation (2.15) into a q-difference equation on f:

$$(1-x)f(x) = (1+\frac{c}{q}+ax^2q+dx^2q)f(xq)-(1+xq)(\frac{c}{q}-adx^2q^2)f(xq^2), \tag{2.16}$$

together with the initial conditions

$$f(0) = H_{-1} = 1,$$
$$f'(0) = H_0 = \frac{1}{1-q}.$$

This is a q-difference of order 2, which is still not so easy to solve. But we will make a last transformation which will make things much better. Define

$$g(x) := \frac{f(x)}{(-x;q)_\infty}.$$

We obtain:

$$(1-x^2)g(x) = (1+\frac{c}{q}+ax^2q+dx^2q)g(xq)-(\frac{c}{q}-adx^2q^2)g(xq^2), \tag{2.17}$$

and

$$g(0) = f(0) = 1,$$
$$g'(0) = f'(0) - \frac{f(0)}{1-q} = \frac{1}{1-q} - \frac{1}{1-q} = 0.$$

Finally let us define (a_n) as

$$\sum_{n\geq 0} a_n x^n := g(x).$$

Then (a_n) satisfies the recurrence equation

$$\left(1-q^n-cq^{n-1}+cq^{2n-1}\right)a_n = \left(1+aq^{n-1}+dq^{n-1}+adq^{2n-2}\right)a_{n-2},$$

which simplifies as

$$a_n = \frac{\left(1+aq^{n-1}\right)\left(1+dq^{n-1}\right)}{\left(1-q^n\right)\left(1-cq^{n-1}\right)}a_{n-2}, \tag{2.18}$$

and the initial conditions

$$\begin{aligned} a_0 &= g(0) = 1, \\ a_1 &= g'(0) = 0. \end{aligned}$$

Thus for all $n \geq 0$, we have

$$a_{2n} = \frac{(-aq;q^2)_n(-dq;q^2)_n}{(q^2;q^2)_n(cq;q^2)_n} a_0 = \frac{(-aq;q^2)_n(-dq;q^2)_n}{(q^2;q^2)_n(cq;q^2)_n},$$

and

$$a_{2n+1} = \frac{(-aq^2;q^2)_n(-dq^2;q^2)_n}{(q^3;q^2)_n(cq^2;q^2)_n} a_1 = 0.$$

We now conclude using Appell's lemma [6]. We have

$$\begin{aligned} \lim_{k\to\infty} G_k(q;a,c,d) &= \lim_{k\to\infty} H_k \\ &= \lim_{x\to 1^-} (1-x) \sum_{k\geq 0} H_k \\ &= \lim_{x\to 1^-} (1-x) f(x) \\ &= \lim_{x\to 1^-} g(x) \prod_{k\geq 0} (1+xq^k) \\ &= (-q;q)_\infty \lim_{x\to 1^-} (1-x^2) \sum_{n\geq 0} a_{2n} x^{2n} \\ &= (-q;q)_\infty \lim_{n\to\infty} a_{2n} \\ &= \frac{(-q;q)_\infty(-aq;q^2)_\infty(-dq;q^2)_\infty}{(q^2;q^2)_\infty(cq;q^2)_\infty} \\ &= \frac{(-aq;q^2)_\infty(-dq;q^2)_\infty}{(q;q)_\infty(cq;q^2)_\infty}. \end{aligned}$$

On the second line we used Appell's lemma and on the sixth we used it with x replaced by x^2.

3. Examples and further results

We begin this section by illustrating Theorems 1.2 and 1.3. First, the eleven two-coloured partitions of 6 satisfying the difference conditions in Theorem 1.2 are

$$(6), (6'), (5,1), (5',1), (4,2), (4',2), (4,2'), (4',2'),$$
$$(3',2,1), (2,2,2), (2',2',2'),$$

while the eleven two-coloured partitions with distinct odd parts where only parts 2 modulo 4 can occur primed are

$$(6), (6'), (5,1), (4,2), (4,2'), (3,2,1), (3,2',1),$$
$$(2,2,2), (2,2,2'), (2,2',2'), (2',2',2').$$

One may then easily verify that $A_2(6;k,\ell,m) = B_2(6;k,\ell,m)$ for a given choice of (k,ℓ,m). For example, $A_2(6;1,0,1) = B_2(6;1,0,1) = 1$, the relevant partitions being $(5',1)$ and $(3,2,1)$, respectively.

Next, the thirteen partitions of 14 satisfying the difference conditions in Theorem 1.3 are

$$(14),(13,1),(12,2),(11,2,1),(10,4),(10,2,2),(9,2,2,1),$$
$$(8,2,2,2),(7,2,2,2,1),(6,6,2),(6,4,4),(6,2,2,2,2),(2,2,2,2,2,2,2),$$

while the thirteen partitions of 14 satisfying the congruence conditions are

$$(12,2),(10,4),(10,2,2),(9,4,1),(9,2,2,1),(8,4,2),(8,2,2,2),$$
$$(7,4,2,1),(7,2,2,2,1),(4,4,4,2),(4,4,2,2,2),(4,2,2,2,2,2),(2,2,2,2,2,2,2).$$

Again, one easily verifies that $A_4(13;k,\ell,m)=B_4(13;k,\ell,m)$ for a given choice of (k,ℓ,m).

We close with one more application of Theorem 1.1. Here parts divisible by 3 may appear in two colours. Performing the dilation

$$\begin{aligned} q &\to q^3, \\ a &\to aq^{-1}, \\ c &\to 1, \\ d &\to dq, \end{aligned}$$

the ordering of integers (1.1) becomes

$$2_a < 3_b < 3_c < 4_d < 5_a < 6_b < 6_c < 7_d < 8_a < 9_b < 9_c < \cdots$$

and the matrix D in (1.2) becomes

$$D_3 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 6 & 2 & 5 & 4 \\ 4 & 0 & 3 & 2 \\ 1 & 3 & 0 & 5 \\ 2 & 4 & 1 & 6 \end{pmatrix} \end{array}.$$

Letting b-parts and c-parts be ordinary and primed multiples of 3, respectively, we obtain the following partition identity.

Corollary 3.1. *Let $\mathcal{P}_3$ denote the set of partitions where parts divisible by 3 may appear in two colours, say ordinary and primed. Let $A_3(n;k,m)$ denote the number of partitions of n in $\mathcal{P}_3$ with k and m parts congruent to 2 and 1 modulo 3, respectively, such that $\lambda_i \neq 1$ and*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 3, & \text{if } (\lambda_i,\lambda_{i+1}) \pmod 3 \subset (\{0,2\},\{0',2\}) \text{ or } (\{0',1\},\{0,1\}), \\ 4, & \text{if } 3 \nmid \lambda_i,\lambda_{i+1} \text{ and } \lambda_i - \lambda_{i+1} \not\equiv 2 \pmod 3. \end{cases}$$

Then

$$\sum_{n,k,m\geq 0} A_3(n;k,m) q^n a^k d^m = \frac{(-aq^2;q^6)_\infty (-dq^4;q^6)_\infty (-q^3;q^3)_\infty}{(q^3;q^3)_\infty}. \tag{3.1}$$

In other words, if $B_3(n;k,m)$ denotes the number of partitions of n in $\mathcal{P}_3$ with k and m parts congruent to 2 and 4 modulo 6, respectively, such that primed multiples of 3 may not repeat, then

$$A_3(n;k,m) = B_3(n;k,m).$$

Note that the generating function in (3.1) differs only slightly from the infinite product appearing in the Alladi-Andrews-Gordon generalisation of Capparelli's identity [1],

$$(-aq^2;q^6)_\infty(-bq^4;q^6)_\infty(-q^3;q^3)_\infty.$$

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