ODD UNIMODAL SEQUENCES

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ABSTRACT. In this paper we study odd unimodal and odd strongly unimodal sequences. We use q-series methods to find several fundamental generating functions. Employing the Euler–Maclaurin summation formula we obtain the asymptotic main term for both types of sequences. We also find families of congruences modulo 4 for the number of odd strongly unimodal sequences.

1. INTRODUCTION AND STATEMENT OF RESULTS

A sequence is *unimodal* if it is weakly increasing up to a point and then weakly decreasing thereafter. Let $u(n)$ denote the number of unimodal sequences of natural numbers having the form

$$a_1 \leq \cdots \leq a_r \leq \overline{c} \geq b_1 \geq \cdots \geq b_s, \tag{1.1}$$

with

$$n = c + \sum_{j=1}^{r} a_j + \sum_{j=1}^{s} b_j.$$

The distinguished point c is called the *peak* of the sequence and the sum of the entries n is called the *weight*. For example, we have $u(4) = 12$, the twelve unimodal sequences of weight 4 being

$$\begin{gathered}\left(\overline{4}\right), \left(1, \overline{3}\right), \left(\overline{3}, 1\right), \left(2, \overline{2}\right), \left(1, 1, \overline{2}\right), \left(1, \overline{2}, 1\right), \left(\overline{2}, 2\right), \left(\overline{2}, 1, 1\right), \\ \left(1, 1, 1, \overline{1}\right), \left(1, 1, \overline{1}, 1\right), \left(1, \overline{1}, 1, 1\right), \left(\overline{1}, 1, 1, 1\right).\end{gathered} \tag{1.2}$$

A unimodal sequence is *strongly unimodal* if the inequalities in (1.1) are strict. Let $u^*(n)$ denote the number of strongly unimodal sequences of natural numbers with weight n. For example, we have $u^*(4) = 4$, the four strongly unimdodal sequences of weight 4 being[1]

$$(4), (1, 3), (3, 1), (1, 2, 1). \tag{1.3}$$

Unimodal sequences and strongly unimodal sequences have been the subject of a considerable amount of research, especially over the last two decades. The generating functions for these sequences are related to number-theoretic objects like mock theta functions, false theta functions, and quantum modular forms, whose theories can then be applied to deduce many interesting results (see Subsections 14.4, 15.7, and 21.4 of [11]; for some more recent work, see [13, 16]).

In this paper we initiate the study of odd unimodal sequences and odd strongly unimodal sequences, wherein all numbers must be odd.[2] Let $\mathrm{ou}(n)$ and $\mathrm{ou}^*(n)$ denote the number of odd unimodal and odd strongly unimodal sequences of weight n. Continuing the example in (1.2) and (1.3), we have $\mathrm{ou}(4) = 6$ and $\mathrm{ou}^*(4) = 2$.

We begin by computing the generating functions upon which all of our work is based. We use two-variable generating functions, where the second variable tracks the rank of the unimodal sequence.

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[1] For strongly unimodal sequences we drop the overline notation since the peak cannot repeat.

[2] These were briefly treated in [4], where they were called convex and strictly convex compositions with odd parts.

In the notation of (1.1), the rank is defined to be $r-s$, or the number of parts of the sequence to the left of the peak minus the number to the right. From the perspective of modular forms, this second variable is the "Jacobi variable". We use the standard q-hypergeometric notation,

$$(a;q)_n := \prod_{j=0}^{n-1}\left(1-aq^j\right), \qquad (a_1,\ldots,a_\ell;q)_n := (a_1;q)_n\cdots(a_\ell;q)_n \tag{1.4}$$

valid for $\ell\in\mathbb{N}$, $a,a_1,\ldots,a_\ell\in\mathbb{C}$ and $n\in\mathbb{N}_0\cup\{\infty\}$. Let $\mathrm{ou}(m,n)$ denote the number of odd unimodal sequences of weight n with rank m.

Theorem 1.1. *We have*

$$\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}}\mathrm{ou}(m,n)\zeta^m q^n = \sum_{n\geq 0}\frac{q^{2n+1}}{(\zeta q,\zeta^{-1}q;q^2)_{n+1}} \tag{1.5}$$

$$= \sum_{n\geq 0}(-1)^{n+1}\zeta^{3n+1}q^{3n^2+2n}\left(1+\zeta q^{2n+1}\right) + \frac{1}{(\zeta q,\zeta^{-1}q;q^2)_\infty}\sum_{n\geq 0}(-1)^n\zeta^{2n+1}q^{n^2+n} \tag{1.6}$$

$$= \frac{q}{(q^2;q^2)_\infty}\left(\sum_{n,r\geq 0}-\sum_{n,r<0}\right)\frac{(-1)^{n+r}q^{n^2+3n+4rn+r^2+3r}}{1-\zeta q^{2r+1}}. \tag{1.7}$$

Let $\mathrm{ou}^*(m,n)$ denote the number of odd strongly unimodal sequences of weight n with rank m.

Theorem 1.2. *We have, with* $Q(r,s) := \frac{r^2}{4}+\frac{7rs}{2}+\frac{s^2}{4}+\frac{3r}{2}+\frac{5s}{2}+1$,

$$\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}}\mathrm{ou}^*(m,n)\zeta^m q^n = \sum_{n\geq 0}\left(-\zeta q,-\zeta^{-1}q;q^2\right)_n q^{2n+1} \tag{1.8}$$

$$= -\frac{1}{(q^2;q^2)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{3n^2+3n+1}}{1+\zeta q^{2n+1}} + \frac{1}{(q^2;q^2)_\infty}\sum_{n\in\mathbb{Z}}\frac{\zeta^{-n}q^{n^2+2n+1}}{1+\zeta q^{2n+1}} \tag{1.9}$$

$$= \frac{q}{(q^2;q^2)_\infty}\left(\sum_{n,r\geq 0}-\sum_{n,r<0}\right)(-1)^n\zeta^r q^{3n^2+3n+4nr+r^2+2r} \tag{1.10}$$

$$= \frac{\left(-\zeta q,-\zeta^{-1}q;q^2\right)_\infty}{(q^2;q^2)_\infty^2}\left(\sum_{\substack{r,s\geq 0\\ r\equiv s \pmod 2}}-\sum_{\substack{r,s<0\\ r\equiv s \pmod 2}}\right)\frac{(-1)^{\frac{r-s}{2}}q^{Q(r,s)}}{1+\zeta q^{r+s+1}}. \tag{1.11}$$

Remark. *The analogue of Theorem 1.1 for ordinary unimodal sequences is* [22, Proposition 2.1]. *For strongly unimodal sequences, the corresponding generating functions are dispersed in the literature. The analogues of* (1.8) *and* (1.9) *are* [11, equation (14.18)] *and* [12, Lemma 3.1], *the analogues of* (1.10) *and* (1.11) *are Theorems 4.1 and 1.3 of* [19].

Our first use of these generating functions is to find asymptotic estimates for $\mathrm{ou}(n)$ and $\mathrm{ou}^*(n)$.

Theorem 1.3. *We have, as* $n\to\infty$,

$$\mathrm{ou}(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{2^{\frac{13}{4}}3^{\frac{1}{4}}n^{\frac{3}{4}}}. \tag{1.12}$$

Theorem 1.4. *We have, as* $n\to\infty$,

$$\mathrm{ou}^*(n) \sim \frac{e^{\pi\sqrt{\frac{n}{3}}}}{2^{\frac{5}{2}}3^{\frac{1}{4}}n^{\frac{3}{4}}}. \tag{1.13}$$

Remark. *By [7, 27], the analogue of* (1.12) *for the number of ordinary unimodal sequences* $u(n)$ *is*

$$u(n) \sim \frac{e^{\pi\sqrt{\frac{4n}{3}}}}{2^3 3^{\frac{3}{4}} n^{\frac{5}{4}}}.$$

The analogue of (1.13) *for the number of strongly unimodal sequences* $u^*(n)$ *is*

$$u^*(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{2^{\frac{13}{4}} 3^{\frac{1}{4}} n^{\frac{3}{4}}}, \tag{1.14}$$

due to [26]*. Note that the asymptotics in* (1.12) *and* (1.14) *agree.*

As a second result, we prove congruences modulo 4 for the number of odd strongly unimodal sequences of weight n. Here we are motivated by a corresponding result for strongly unimodal sequences, which says that if $\ell \equiv 7, 11, 13, 17 \pmod{24}$ is prime and $(\frac{j}{\ell}) = -1$, then

$$u^*\left(\ell^2 n + \ell j - \left(\tfrac{\ell^2-1}{24}\right)\right) \equiv 0 \pmod{4}.$$

This was conjectured by Bryson, Ono, Pitman, and Rhoades [14] and proved by Chen and Garvan [16]. Our analogue for $\mathrm{ou}^*(n)$ is as follows; see Theorem 6.2 for a more general theorem.

Theorem 1.5. *Let $\ell \geq 5$ be prime. If $\ell \equiv 7, 13 \pmod{24}$ and $(\frac{3j}{\ell}) = -1$ or if $\ell \not\equiv 7, 13 \pmod{24}$ and $\ell \nmid j$, then we have:*

(1) *If j is odd, then*

$$\mathrm{ou}^*\left(4\ell^2 n + 2j\ell + \left(\tfrac{8\ell^2+1}{3}\right)\right) \equiv 0 \pmod{4}.$$

(2) *If j is even, then*

$$\mathrm{ou}^*\left(4\ell^2 n + 2j\ell + \left(\tfrac{2\ell^2+1}{3}\right)\right) \equiv 0 \pmod{4}.$$

The paper is organized as follows: In Section 2, we gather some necessary background on asymptotic methods and indefinite theta functions. In Section 3 we prove Theorems 1.1 and 1.2. Sections 4 and 5 contain proofs of Theorems 1.3 and 1.4. In Section 6, we show Theorem 6.2, which contains the congruences in Theorem 1.5 as a special case. We close in Section 7 with some open problems.

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2. Preliminaries

2.1. A Tauberian Theorem.

Recall the following[3] special case $\alpha = 0$ of Theorem 1.1 of [10], which follows from Ingham's Theorem [20].

Proposition 2.1. *Suppose that $B(q) = \sum_{n\geq 0} b(n)q^n$ is a power series with non-negative real coefficients and radius of convergence at least one and that the $b(n)$ are weakly increasing. Assume that λ, β, $\gamma \in \mathbb{R}$ with $\gamma > 0$ exist such that*

$$B\left(e^{-t}\right) \sim \lambda t^\beta e^{\frac{\gamma}{t}} \quad \text{as } t \to 0^+, \qquad B\left(e^{-z}\right) \ll |z|^\beta e^{\frac{\gamma}{|z|}} \quad \text{as } z \to 0, \tag{2.1}$$

with $z = x + iy$ ($x, y \in \mathbb{R}, x > 0$) in each region of the form $|y| \leq \Delta x$ for $\Delta > 0$. Then

$$b(n) \sim \frac{\lambda \gamma^{\frac{\beta}{2}+\frac{1}{4}}}{2\sqrt{\pi} n^{\frac{\beta}{2}+\frac{3}{4}}} e^{2\sqrt{\gamma n}} \qquad \text{as } n \to \infty.$$

[3] The second condition is often dropped in (2.1) which makes the proposition unfortunately incorrect (see [10]).

2.2. **The Euler–Maclaurin summation formula.** For simplicity we only state the versions of the Euler–Maclaurin summation formula that we use in this paper; see [10] for a more general version for all dimensions. Let $D_\theta := \{re^{i\alpha} : r \geq 0 \text{ and } |\alpha| \leq \theta\}$. A multivariable function f in ℓ variables is of *sufficient decay* in D if there exist $\varepsilon_1, \ldots, \varepsilon_\ell > 0$ such that (we write vectors in bold letters) $f(\boldsymbol{x}) \ll (x_1+1)^{-1-\varepsilon_1} \cdots (x_\ell+1)^{-1-\varepsilon_\ell}$ uniformly as $|x_1| + \ldots + |x_\ell| \to \infty$ in D. We first require a one-dimensional version of the Euler–Maclaurin summation formula (see [28]).

Proposition 2.2. *Suppose that $0 \leq \theta < \frac{\pi}{2}$. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic in a domain containing D_θ, so that in particular f is holomorphic at the origin, and assume that f and all of its derivatives are of sufficient decay. Then for $a \in \mathbb{R}$ and $N \in \mathbb{N}_0$, we have, uniformly as $z \to 0$ in D_θ,*

$$\sum_{m \geq 0} f((m+a)z) = \frac{1}{z} \int_0^\infty f(w)dw - \sum_{n=0}^{N-1} \frac{B_{n+1}(a) f^{(n)}(0)}{(n+1)!} z^n + O_N\left(z^N\right),$$

where $B_n(x)$ denotes the n-th Bernoulli polynomial.

We also require the two-dimensional case of the Euler–Maclaurin summation formula (see [10]).

Proposition 2.3. *Suppose that $0 \leq \theta_j < \frac{\pi}{2}$ for $1 \leq j \leq 2$, and that $f : \mathbb{C}^2 \to \mathbb{C}$ is holomorphic in a domain containing $D_{\boldsymbol{\theta}} := D_{\theta_1} \times D_{\theta_2}$. If f and all of its derivatives are of sufficient decay in $D_{\boldsymbol{\theta}}$, then for $\boldsymbol{a} \in \mathbb{R}^2$ and $N \in \mathbb{N}_0$ we have uniformly, as $w \to 0$ in $D_{\boldsymbol{\theta}}$,*

$$\begin{aligned}\sum_{\boldsymbol{m} \in \mathbb{N}_0^2} f((\boldsymbol{m}+\boldsymbol{a})z) = \frac{1}{z^2} \int_0^\infty \int_0^\infty f(\boldsymbol{w}) dw_1 dw_2 &- \frac{1}{z} \sum_{n_1=0}^{N-1} \frac{B_{n_1+1}(a_1)}{(n_1+1)!} z^{n_1} \int_0^\infty f^{(n_1,0)}(0,w_2)dw_2 \\ &- \frac{1}{z} \sum_{n_2=0}^{N-1} \frac{B_{n_2+1}(a_2)}{(n_2+1)!} z^{n_2} \int_0^\infty f^{(0,n_2)}(w_1,0)dw_1 \\ &+ \sum_{n_1+n_2<N} \frac{B_{n_1+1}(a_1) B_{n_2+1}(a_2) f^{(n_1,n_2)}(\boldsymbol{0})}{(n_1+1)!(n_2+1)!} z^{n_1+n_2} + O_N\left(z^N\right).\end{aligned}$$

2.3. **Indefinite theta functions.** In this subsection, we recall results from Zwegers' thesis [29]. Fix a quadratic form Q of signature $(n,1)$ with associated matrix A, so that $Q(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x}$. Let $B(\boldsymbol{x},\boldsymbol{y}) := Q(\boldsymbol{x}+\boldsymbol{y}) - Q(\boldsymbol{x}) - Q(\boldsymbol{y})$ denote the corresponding bilinear form. The set of vectors $\boldsymbol{c} \in \mathbb{R}^\ell$ with $Q(\boldsymbol{c}) < 0$ splits into two connected components. Two vectors $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ lie in the same component if and only if $B(\boldsymbol{c_1},\boldsymbol{c_2}) < 0$. We fix one of the components and denote it by C_Q. Picking any vector $\boldsymbol{c_0} \in C_Q$, we have

$$C_Q = \left\{\boldsymbol{c} \in \mathbb{R}^\ell : Q(\boldsymbol{c}) < 0,\ B(\boldsymbol{c},\boldsymbol{c_0}) < 0\right\}.$$

The cusps are elements from

$$S_Q := \left\{\boldsymbol{c} \in \mathbb{Z}^\ell : \gcd(c_1, c_2, \ldots, c_\ell) = 1,\ Q(\boldsymbol{c}) = 0,\ B(\boldsymbol{c},\boldsymbol{c_0}) < 0\right\}.$$

Let $\overline{C}_Q := C_Q \cup S_Q$ and define for, $\boldsymbol{c} \in \overline{C}_Q$

$$R(\boldsymbol{c}) := \begin{cases} \mathbb{R}^\ell & \text{if } \boldsymbol{c} \in C_Q, \\ \left\{\boldsymbol{a} \in \mathbb{R}^\ell : B(\boldsymbol{c},\boldsymbol{a}) \notin \mathbb{Z}\right\} & \text{if } \boldsymbol{c} \in S_Q. \end{cases}$$

Let $\boldsymbol{c_1}, \boldsymbol{c_2} \in \overline{C}_Q$. We define the *theta function with characteristic* $\boldsymbol{a} \in R(\boldsymbol{c_1}) \cap R(\boldsymbol{c_2})$ and $\boldsymbol{b} \in \mathbb{R}^\ell$ by

$$\vartheta_{\boldsymbol{a},\boldsymbol{b}}(\tau) := \sum_{\boldsymbol{n} \in \mathbb{Z}^\ell + \boldsymbol{a}} \varrho(\boldsymbol{n};\tau) e^{2\pi i B(\boldsymbol{b},\boldsymbol{n})} q^{Q(\boldsymbol{n})},$$

where

$$\varrho(\boldsymbol{n};\tau)=\varrho_Q^{\boldsymbol{c_1},\boldsymbol{c_2}}(\boldsymbol{n};\tau):=\varrho^{\boldsymbol{c_1}}(\boldsymbol{n};\tau)-\varrho^{\boldsymbol{c_2}}(\boldsymbol{n};\tau)$$

with ($\tau=u+iv$)

$$\varrho^{\boldsymbol{c}}(\boldsymbol{n};\tau):=\begin{cases}E\left(\frac{B(\boldsymbol{c},\boldsymbol{n})\sqrt{v}}{\sqrt{-Q(\boldsymbol{c})}}\right) & \text{if } \boldsymbol{c}\in C_Q,\\ \operatorname{sgn}(B(\boldsymbol{c},\boldsymbol{n})) & \text{if } \boldsymbol{c}\in S_Q.\end{cases}$$

Here the odd function E is defined as

$$E(w):=2\int_0^w e^{-\pi t^2}dt$$

with the usual convention that $\operatorname{sgn}(w):=\frac{w}{|w|}$ for $w\in\mathbb{R}\setminus\{0\}$ and $\operatorname{sgn}(0):=0$. Note that

$$E(x)=\operatorname{sgn}(x)\left(1-\beta\left(x^2\right)\right),\ \text{where } \beta(x):=\int_x^\infty w^{-\frac{1}{2}}e^{-\pi w}dw. \tag{2.2}$$

This in particular yields that $E(x)\sim\operatorname{sgn}(x)$ as $|x|\to\infty$.

The theta function satisfies the following transformation law.

Theorem 2.4. *If $\boldsymbol{a},\boldsymbol{b}\in R(\boldsymbol{c_1})\cap R(\boldsymbol{c_2})$, then*

$$\vartheta_{\boldsymbol{a},\boldsymbol{b}}\left(-\frac{1}{\tau}\right)=\frac{1}{\sqrt{-\det(A)}}(-i\tau)^{\frac{\ell}{2}}e^{2\pi iB(\boldsymbol{a},\boldsymbol{b})}\sum_{\boldsymbol{\ell}\in A^{-1}\mathbb{Z}^\ell/\mathbb{Z}^\ell}\vartheta_{\boldsymbol{b}+\boldsymbol{\ell},-\boldsymbol{a}}(\tau).$$

3. Generating functions and the proofs of Theorems 1.1 and 1.2

In this section we establish in Theorems 1.1 and 1.2. We begin with Theorem 1.1.

Proof of Theorem 1.1. Equation (1.5) is a straightforward consequence of the fact that $(\zeta q;q^2)_{n+1}^{-1}$ is the generating function for partitions into odd parts of size at most $2n+1$, with the exponent of ζ counting the number of parts. Namely, in the notation of (1.1), the term q^{2n+1} generates the peak $\overline{c}$, the term $(\zeta q;q^2)_{n+1}^{-1}$ generates the odd parts $(a_1,\dots,a_r)$ to the left of the peak, and the term $(\zeta^{-1}q;q^2)_{n+1}^{-1}$ generates the odd parts $(b_1,\dots,b_s)$ to the right of the peak. The exponent of ζ is $r-s$. Equation (1.6) is a result in Ramanujan's lost notebook [6, Entry 6.3.4].

Equation (1.7) requires more work. Fot its proof, we require so-called Bailey pairs (for background see [25]). A pair of sequences $(\alpha_n,\beta_n)_{n\geq0}$ is called a *Bailey pair relative to* (a,q) if

$$\beta_n=\sum_{k=0}^n\frac{\alpha_k}{(q;q)_{n-k}(aq;q)_{n+k}}.$$

If (α_n,β_n) is a Bailey pair relative to (a,q), then by [24, equation (1.5)]

$$\sum_{n\geq0}q^n\beta_n=\frac{1}{(aq,q;q)_\infty}\sum_{n,r\geq0}(-a)^nq^{\frac{n(n+1)}{2}+(2n+1)r}\alpha_r. \tag{3.1}$$

The following sequences form a Bailey pair relative to (q^2,q^2) [23, pp. 727–728]:

$$\alpha_n=\frac{(-1)^nq^{n^2+n}\left(1-q^{4n+2}\right)}{\left(1-q^2\right)\left(1-\zeta q^{2n+1}\right)\left(1-\zeta^{-1}q^{2n+1}\right)},\quad \beta_n=\frac{1}{\left(\zeta q,\zeta^{-1}q;q^2\right)_{n+1}}. \tag{3.2}$$

Inserting (3.2) into (3.1) and using the fact that

$$\frac{1-q^{4r+2}}{\left(1-\zeta q^{2r+1}\right)\left(1-\zeta^{-1}q^{2r+1}\right)}=\frac{1}{1-\zeta q^{2r+1}}+\frac{\zeta^{-1}q^{2r+1}}{1-\zeta^{-1}q^{2r+1}}, \tag{3.3}$$

we compute

$$\sum_{n\geq 0}\frac{q^{2n+1}}{(\zeta q,\zeta^{-1}q;q^2)_{n+1}}=\frac{q}{(q^2;q^2)_\infty^2}\sum_{n,r\geq 0}\frac{(-1)^{n+r}q^{n^2+3n+4nr+r^2+3r}\left(1-q^{4r+2}\right)}{(1-\zeta q^{2r+1})(1-\zeta^{-1}q^{2r+1})}$$
$$=\frac{q}{(q^2;q^2)_\infty^2}\left(\sum_{n,r\geq 0}\frac{(-1)^{n+r}q^{n^2+3n+4nr+r^2+3r}}{1-\zeta q^{2r+1}}+\zeta^{-1}\sum_{n,r\geq 0}\frac{(-1)^{n+r}q^{n^2+3n+4nr+r^2+5r+1}}{1-\zeta^{-1}q^{2r+1}}\right)$$
$$=\frac{q}{(q^2;q^2)_\infty^2}\left(\sum_{n,r\geq 0}-\sum_{n,r<0}\right)\frac{(-1)^{n+r}q^{n^2+3n+4rn+r^2+3r}}{1-\zeta q^{2r+1}}.$$

In the last step we let $(n,r)\mapsto(-n-1,-r-1)$ and simplify. This completes the proof. □

We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. We use the fact that $(-\zeta q;q^2)_n$ is the generating function for partitions into distinct odd parts of size at most $2n-1$, with the exponent of ζ counting the number of parts.

For (1.9) we require two identities,

$$\sum_{n\geq 0}\frac{q^{2n^2+2n+1}}{(-\zeta q,-\zeta^{-1}q;q^2)_{n+1}}=\frac{1}{(q^2;q^2)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{3n^2+3n+1}}{1+\zeta q^{2n+1}},\tag{3.4}$$

$$\sum_{n\in\mathbb{Z}}\frac{(a,b;q)_n w^n}{(c,d;q)_n}=\frac{\left(aw,\frac{d}{a},\frac{c}{b},\frac{dq}{abw};q\right)_\infty}{\left(w,d,\frac{q}{b},\frac{cd}{abw};q\right)_\infty}\sum_{n\in\mathbb{Z}}\frac{\left(a,\frac{abw}{d};q\right)_n\left(\frac{d}{a}\right)^n}{(aw,c;q)_n}.\tag{3.5}$$

Equation (3.4) may be found in [1, p. 397], while equation (3.5) is a bilateral transformation of Bailey [18, example 5.20 (i)]. The notation in (1.4) is extended to all integers via

$$(a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

Note that we have the identity [18, (I.2)]

$$(a;q)_{-n}=\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{a^n\left(\frac{q}{a};q\right)_n}.\tag{3.6}$$

We begin by letting $(a,b,w,q)=(-\zeta q,-\zeta^{-1}q,q^2,q^2)$ in (3.5) and then letting $c,d\to 0$. Simplifying and exchanging left- and right-hand sides gives

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n\in\mathbb{Z}}\frac{\zeta^{-n}q^{n^2+2n+1}}{1+\zeta q^{2n+1}}=\sum_{n\in\mathbb{Z}}\left(-\zeta q,-\zeta^{-1}q;q^2\right)_n q^{2n+1}$$
$$=\sum_{n\geq 0}\left(-\zeta q,-\zeta^{-1}q;q^2\right)_n q^{2n+1}+\sum_{n\leq -1}\left(-\zeta q,-\zeta^{-1}q;q^2\right)_n q^{2n+1}$$
$$=\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}}\mathrm{ou}^*(m,n)\zeta^m q^n+\sum_{n\geq 1}\left(-\zeta q,-\zeta^{-1}q;q^2\right)_{-n}q^{-2n+1}$$
$$=\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}}\mathrm{ou}^*(m,n)\zeta^m q^n+\sum_{n\geq 1}\frac{q^{2n^2-2n+1}}{(-\zeta q,-\zeta^{-1}q;q^2)_n}=\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}}\mathrm{ou}^*(m,n)\zeta^m q^n+\sum_{n\geq 0}\frac{q^{2n^2+2n+1}}{(-\zeta q,-\zeta^{-1}q;q^2)_{n+1}}$$

$$= \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} \mathrm{ou}^*(m,n)\zeta^m q^n + \frac{1}{(q^2;q^2)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{3n^2+3n+1}}{1+\zeta q^{2n+1}}.$$

Here the final equality uses (3.4) and the antepenultimate equality uses (3.6). Comparing the extremes in this string of equations gives (1.9).

We now turn to (1.10). For this require the fact that if (α_n, β_n) is a Bailey pair relative to (a, q), then we use [24, Corollary 1.3]

$$\sum_{n\geq 0}(aq;q)_{2n} q^n \beta_n = \frac{1}{(q;q)_\infty}\sum_{n,r\geq 0}(-a)^n q^{\frac{3n(n+1)}{2}+(2n+1)r}\alpha_r, \tag{3.7}$$

along with the following Bailey pair relative to $(1, q)$ [3, Lemma 3]:

$$\alpha_n = \begin{cases} (-1)^n\left(w^n q^{\frac{n(n-1)}{2}} + w^{-n}q^{\frac{n(n+1)}{2}}\right) & \text{if } n\geq 1,\\ 1 & \text{if } n=0,\end{cases} \qquad \beta_n = \frac{\left(w;\frac{q}{w};q\right)_n}{(q;q)_{2n}}.$$

Using this Bailey pair with $(w, q) = (-\zeta q, q^2)$ in (3.7) we compute

$$\sum_{n\geq 0}\left(-\zeta q, -\zeta^{-1}q; q^2\right)_n q^{2n+1}$$

$$= \frac{q}{(q^2;q^2)_\infty}\left(\sum_{n,r\geq 0}(-1)^n\zeta^r q^{3n^2+3n+4nr+r^2+2r} + \sum_{\substack{n\geq 0\\ r\geq 1}}(-1)^n\zeta^{-r}q^{3n^2+3n+4nr+r^2+2r}\right)$$

$$= \frac{q}{(q^2;q^2)_\infty}\left(\sum_{n,r\geq 0}(-1)^n\zeta^r q^{3n^2+3n+4nr+r^2+2r} - \sum_{n,r<0}(-1)^n\zeta^r q^{3n^2+3n+4nr+r^2+2r}\right),$$

where the last line follows upon replacing (n, r) by $(-n-1, -r)$. This gives (1.10).

Finally, we treat (1.11). Again we use Bailey pairs. This time we require the fact that if (α, β_n) is a Bailey pair relative to (a, q), then [25, Theorem 10.1]

$$\sum_{n\geq 0}(b,c;q)_n\left(\frac{aq}{bc}\right)^n\beta_n = \frac{\left(\frac{aq}{b},\frac{aq}{c};q\right)_\infty}{\left(aq,\frac{aq}{bc};q\right)_\infty}\sum_{n\geq 0}\frac{(b,c;q)_n\left(\frac{aq}{bc}\right)^n}{\left(\frac{aq}{b},\frac{aq}{c};q\right)_n}\alpha_n, \tag{3.8}$$

along with the Bailey pair[4] relative to (q, q) from [2, equation (5.11)],

$$\alpha_n = \frac{q^{2n^2+n}\left(1-q^{2n+1}\right)}{1-q}\sum_{j=-n}^{n}(-1)^j q^{-\frac{j(3j+1)}{2}}, \qquad \beta_n = 1.$$

Using this Bailey pair from (3.8) with $(b, c, q) = (-\zeta q, -\zeta^{-1}q, q^2)$ and employing (3.3), we compute

$$\sum_{n\geq 0}\left(-\zeta q, -\zeta^{-1}q; q^2\right)_n q^{2n+1}$$

$$= \frac{q\left(-\zeta q, -\zeta^{-1}q; q^2\right)_\infty}{(q^2;q^2)_\infty^2}\sum_{n\geq 0}\frac{q^{4n^2+4n}\left(1-q^{4n+2}\right)}{\left(1+\zeta q^{2n+1}\right)\left(1+\zeta^{-1}q^{2n+1}\right)}\sum_{j=-n}^{n}(-1)^j q^{-j(3j+1)}$$

$$= \frac{q\left(-\zeta q, -\zeta^{-1}q; q^2\right)_\infty}{(q^2;q^2)_\infty^2}\left(\sum_{n\geq 0}\sum_{j=-n}^{n}\frac{(-1)^j q^{4n^2+4n-j(3j+1)}}{1+\zeta q^{2n+1}} - \zeta^{-1}\sum_{n\geq 0}\sum_{j=-n}^{n}\frac{(-1)^j q^{4n^2+6n+1-j(3j+1)}}{1+\zeta^{-1}q^{2n+1}}\right)$$

[4] We point out to the reader that the A_n in Andrews' paper are related to the α_n via $\alpha_n = a^n q^{n^2} A_n$.

$$= \frac{q\left(-\zeta q, -\zeta^{-1}q; q^2\right)_\infty}{(q^2;q^2)_\infty^2} \left(\sum_{n\geq 0}\sum_{j=-n}^{n} \frac{(-1)^j q^{4n^2+4n-j(3j+1)}}{1+\zeta q^{2n+1}} - \sum_{n\leq 0}\sum_{j=n}^{-n} \frac{(-1)^j q^{4n^2-4n-j(3j+1)}}{1+\zeta q^{2n-1}}\right).$$

Now letting $(n,j) = (\frac{r+s}{2}, \frac{r-s}{2})$ in the first sum on the right-hand side, letting $(n,j) = (\frac{r+s+2}{2}, \frac{r-s}{2})$ in the second sum, and then simplifying gives (1.11). This completes the proof of Theorem 1.2. $\square$

4. Proof of Theorem 1.3

In this section, we prove Theorem 1.3.

Proof of Theorem 1.3. From Theorem 1.1 we have that

$$\sum_{n\geq 0} \mathrm{ou}(n) q^n = \sum_{n\geq 0} \frac{q^{2n+1}}{(q;q^2)_{n+1}^2} = \sum_{n\geq 0} (-1)^{n+1} q^{n(3n+2)}\left(1+q^{2n+1}\right) + \frac{1}{(q;q^2)_\infty^2}\sum_{n\geq 0}(-1)^n q^{n^2+n}. \quad (4.1)$$

We apply Proposition 2.1. To begin, it is not hard to see that the $\mathrm{ou}(n)$ are monotonic, since

$$(1-q)\sum_{n\geq 0}\mathrm{ou}(n)q^n = \sum_{n\geq 0}\frac{q^{2n+1}}{(q^3;q^2)_n (q;q^2)_{n+1}},$$

and the right-hand side has non-negative coefficients. Alternatively, note that for any $n \in \mathbb{N}$

$$(a_1, \dots, a_r, \overline{c}, b_1, \dots, b_s) \mapsto (1, a_1, \dots, a_r, \overline{c}, b_1, \dots, b_s)$$

is an injective mapping from the set of odd unimodal sequences of weight n to the set of odd unimodal sequences of weight $n+1$.

Next, let $F_2(q)$ denote the right-hand side of (4.1). We analyze each of the terms separately with the goal of showing that as $z \to 0$,

$$F_2\left(e^{-z}\right) \sim \frac{e^{\frac{\pi^2}{6z}}}{4}.$$

The modularity of the Dedekind η-function implies that

$$\frac{1}{(e^{-z};e^{-z})_\infty} \sim \sqrt{\frac{z}{2\pi}}\, e^{\frac{\pi^2}{6z}} \qquad \text{as } z \to 0. \quad (4.2)$$

Thus, with $q = e^{-z}$, we have

$$\frac{1}{(q;q^2)_\infty^2} = \frac{\left(q^2;q^2\right)_\infty^2}{(q;q)_\infty^2} \sim \frac{e^{\frac{\pi^2}{6z}}}{2}. \quad (4.3)$$

We apply Proposition 2.2 to the two sums in (4.1) . We start by splitting the first sum according to the parity of the summation variable in order to rewrite it as

$$\sum_{n\geq 0}(-1)^{n+1}q^{n(3n+2)}\left(1+q^{2n+1}\right) = q^{-\frac{1}{3}}\sum_{n\geq 0}\left(q^{12\left(n+\frac{2}{3}\right)^2} + q^{12\left(n+\frac{5}{6}\right)^2} - q^{12\left(n+\frac{1}{6}\right)^2} - q^{12\left(n+\frac{1}{3}\right)^2}\right).$$

Now we can apply Proposition 2.2 with $f(z) := e^{-12z^2}$ and $a \in \{\frac{2}{3}, \frac{5}{6}, \frac{1}{6}, \frac{1}{3}\}$. The main terms from Proposition 2.2 cancel and using that $q^{-\frac{1}{3}} = O(1)$ we obtain that the first sum is $O(1)$.

For the second sum we write, using Proposition 2.2

$$\begin{aligned}\sum_{n\geq 0}(-1)^n e^{-\left(n^2+n\right)z} &= \sum_{n\geq 0} e^{-\left(4n^2+2n\right)z} - \sum_{n\geq 0} e^{-\left((2n+1)^2+(2n+1)\right)z} \\ &= e^{\frac{z}{4}}\left(\sum_{n\geq 0}\left(e^{-4\left(n+\frac{1}{4}\right)^2 z} - e^{-4\left(n+\frac{3}{4}\right)^2 z}\right)\right) \sim -B_1\left(\tfrac{1}{4}\right) + B_1\left(\tfrac{3}{4}\right) = \tfrac{1}{2}.\end{aligned}$$

Combining with (4.3), Proposition 2.1 with $\lambda = \frac{1}{4}$, $\alpha = \beta = 0$ gives the claim. $\square$

5. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. As in the previous section, we wish to apply Proposition 2.1, though in this case the details are much more involved. To begin, we record the monotonicity of the sequence $\mathrm{ou}^*(n)$.

Lemma 5.1. *For $n \geq 3$ we have that* $\mathrm{ou}^*(n) \geq \mathrm{ou}^*(n-1)$.

Proof. We give two proofs, one employing q-series and one using a combinatorial argument. For the q-series proof, first observe that

$$(1-q)\sum_{n\geq 0} \mathrm{ou}^*(n)q^n = (1-q)\sum_{n\geq 0}\left(-q;q^2\right)_n^2 q^{2n+1} = q(1-q) + (1-q)\sum_{n\geq 1}\left(-q;q^2\right)_n^2 q^{2n+1}$$
$$= q(1-q) + q^3\left(1-q^2\right)(1+q)\sum_{n\geq 0}\left(-q^3;q^2\right)_n^2 q^{2n}. \tag{5.1}$$

We now require the q-binomial theorem [18, Exercise 1.2]

$$\sum_{m=0}^{n} \frac{(q;q)_n w^n q^{\frac{m(m-1)}{2}}}{(q;q)_m (q;q)_{n-m}} = (-w;q)_n \tag{5.2}$$

and a transformation of Jackson [18, Appendix (III.4)],

$$\sum_{n\geq 0} \frac{(a,b;q)_n w^n}{(c,q;q)_n} = \frac{(aw;q)_\infty}{(w;q)_\infty} \sum_{n\geq 0} \frac{\left(a,\frac{c}{b};q\right)_n (-bw)^n q^{\frac{n(n-1)}{2}}}{(c,aw,q;q)_n}. \tag{5.3}$$

Using these, we rewrite the final sum in (5.1) as follows:

$$\sum_{n\geq 0}\left(-q^3;q^2\right)_n^2 q^{2n} = \sum_{n\geq 0}\left(-q^3;q^2\right)_n q^{2n} \sum_{m=0}^{n} \frac{\left(q^2;q^2\right)_n q^{m^2+2m}}{\left(q^2;q^2\right)_m \left(q^2;q^2\right)_{n-m}}$$
$$= \sum_{m\geq 0} \frac{q^{m^2+2m}}{\left(q^2;q^2\right)_m} \sum_{n\geq m} \frac{\left(-q^3,q^2;q^2\right)_n q^{2n}}{\left(q^2;q^2\right)_{n-m}} = \sum_{m\geq 0} q^{m^2+4m}\left(-q^3;q^2\right)_m \sum_{n\geq 0} \frac{\left(-q^{2m+3},q^{2m+2};q^2\right)_n q^{2n}}{\left(q^2;q^2\right)_n}$$
$$= \sum_{m\geq 0} \frac{q^{m^2+4m}\left(-q^3;q^2\right)_m}{\left(q^2;q^2\right)_m} \sum_{n\geq 0} \frac{q^{n^2+4n+2nm}}{\left(q^2;q^2\right)_n \left(1-q^{2n+2m+2}\right)}. \tag{5.4}$$

Here the first equality follows from (5.2) and the final equality implied by (5.3) with $(a,b,c,w,q) = (q^{2m+2}, -q^{2m+3}, 0, q^2, q^2)$. Combining (5.1) and (5.4) gives

$$(1-q)\sum_{n\geq 0} \mathrm{ou}^*(n)q^n = q(1-q) + q^2\left(1-q^2\right)(1+q) \sum_{n,m\geq 0} \frac{q^{n^2+4n+m^2+4m+2nm}\left(-q^3;q^2\right)_m}{\left(q^2;q^2\right)_n \left(q^2;q^2\right)_m \left(1-q^{2n+2m+2}\right)}.$$

It is straightforward to see that the coefficient of q^n on the right-hand side is non-negative for $n \geq 3$.

Alternatively, one may deduce the montonicity using a combinatorial argument. For $n \geq 3$ we define a mapping on odd strongly unimodal sequences of weight n as follows:

$$(a_1,\ldots,a_r,\overline{c},b_1,\ldots,b_s) \mapsto \begin{cases} (1,a_1,\ldots,a_r,\overline{c},b_1,\ldots,b_s) & \text{if } a_1 \neq 1,\\ (a_2,\ldots,a_r,\overline{c+2},b_1,\ldots,b_s) & \text{if } a_1 = 1.\end{cases}$$

It is not hard to see that in either case the image is an odd strongly unimodal sequence of weight $n+1$ and that the mapping is injective. This gives the desired inequality $\mathrm{ou}^*(n) \geq \mathrm{ou}^*(n-1)$ for $n \geq 4$, and the case $n = 3$ follows from the fact that $\mathrm{ou}^*(3) = 1$ and $\mathrm{ou}^*(2) = 0$. $\square$

We now turn to the proof of Theorem 1.4.

Proof of Theorem 1.4. Lemma 5.1 implies that the $\mathrm{ou}^*(n)$ are monotonic. The rest of the proof is devoted to showing that the remaining conditions of Proposition 2.1 are satisfied. By (1.10)

$$\sum_{n\geq 0}\mathrm{ou}^*(n)q^n=\frac{q}{(q^2;q^2)_\infty}\left(\sum_{r,n\geq 0}-\sum_{r,n<0}\right)(-1)^nq^{3n^2+4nr+r^2+3n+2r}.$$

Denoting by $F_1(q)$ the right-hand side, we aim to prove that

$$F_1\left(e^{-t}\right)\sim\frac{e^{\frac{\pi^2}{12t}}}{2}\quad\text{as } t\to 0,\qquad F\left(e^{-z}\right)\ll e^{\frac{\pi^2}{12|z|}}\quad\text{as } z\to 0, \tag{5.5}$$

with $z=x+iy$ ($x,y\in\mathbb{R}$, $x>0$, $|y|\leq\Delta x$, $\Delta>0$). We first consider the outside factor. By (4.2) with $q=e^{-z}$ we have, as $z\to 0$,

$$\frac{q}{(q^2;q^2)_\infty}\sim\sqrt{\frac{z}{\pi}}e^{\frac{\pi^2}{12z}}.$$

Next define

$$G(q):=\frac{1}{2}\sum_{n,r\in\mathbb{Z}}\left(\mathrm{sgn}\left(n+\tfrac{1}{2}\right)+\mathrm{sgn}\left(r+\tfrac{1}{2}\right)\right)(-1)^nq^{3n^2+4nr+r^2+3n+2r},$$

so $F_1(q)=\frac{G(q)}{(q^2;q^2)_\infty}$. We realize G as "holomorphic part" of an indefinite theta function. For this set

$$g(\tau):=2q^{\frac{3}{4}}G(q)=i\sum_{\boldsymbol{n}\in\mathbb{Z}^2+\boldsymbol{a}}(\mathrm{sgn}(B(\boldsymbol{c_1},\boldsymbol{n}))-\mathrm{sgn}(B(\boldsymbol{c_2},\boldsymbol{n})))e^{2\pi iB(\boldsymbol{b},\boldsymbol{n})}q^{Q(\boldsymbol{n})},$$

where $Q(\boldsymbol{n}):=3n_1^2+4n_1n_2+n_2^2$, $\boldsymbol{c_1}:=(1,-2)^T$, $\boldsymbol{c_2}:=(2,-3)^T$, $\boldsymbol{b}:=(-\frac{1}{4},\frac{1}{2})^T$, and $\boldsymbol{a}:=(\frac{1}{2},0)^T$.

Using (2.2), we may decompose

$$g(\tau)=\Theta(\tau)+\Theta^-(\tau),$$

where

$$\Theta(\tau):=i\sum_{\boldsymbol{n}\in\mathbb{Z}^2+\boldsymbol{a}}\left(E\left(\frac{B(\boldsymbol{c_1},\boldsymbol{n})}{\sqrt{-Q(\boldsymbol{c_1})}}\sqrt{v}\right)-E\left(\frac{B(\boldsymbol{c_2},\boldsymbol{n})}{\sqrt{-Q(\boldsymbol{c_2})}}\sqrt{v}\right)\right)e^{2\pi iB(\boldsymbol{b},\boldsymbol{n})}q^{Q(\boldsymbol{n})},$$
$$\Theta^-(\tau):=\sum_{\boldsymbol{n}\in\mathbb{Z}^2+\boldsymbol{a}}\left(\mathrm{sgn}(n_1)\beta\left(4n_1^2v\right)+\mathrm{sgn}(n_2)\beta\left(\tfrac{4n_2^2v}{3}\right)\right)(-1)^{n_1-\frac{1}{2}}q^{3n_1^2+4n_1n_2+n_2^2}.$$

In fact the identity holds termwise and we use that

$$3n_1^2+4n_1n_2+n_2^2=Q(\boldsymbol{n}),\quad (-1)^{n_1-\frac{1}{2}}=-ie^{2\pi iB(\boldsymbol{b},\boldsymbol{n})},\quad n_1=-\tfrac{1}{2}B(\boldsymbol{c_1},\boldsymbol{n}),\quad n_2=\tfrac{1}{2}B(\boldsymbol{c_2},\boldsymbol{n}),$$
$$Q(\boldsymbol{c_1})=-1,\qquad Q(\boldsymbol{c_2})=-3.$$

We determine the asymptotic behavior of Θ and Θ^- separately: For Θ we use modularity and for Θ^- the Euler–Maclaurin summation formula.

We start with Θ. We have that (in the notation of Subsection 2.2)

$$\Theta(\tau)=i\vartheta_{\boldsymbol{a},\boldsymbol{b}}(\tau).$$

We apply Theorem 2.4 to obtain

$$\vartheta_{\boldsymbol{a},\boldsymbol{b}}(\tau)=-\frac{1}{2\tau}\sum_{\boldsymbol{\ell}\in\{\boldsymbol{0},(\frac{1}{2},0),(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2})\}}\vartheta_{\boldsymbol{b}+\boldsymbol{\ell},-\boldsymbol{a}}\left(-\frac{1}{\tau}\right).$$

Now write

$$\vartheta_{\boldsymbol{b}+\boldsymbol{\ell},-\boldsymbol{a}}(\tau)=-\sum_{\boldsymbol{n}\in\mathbb{Z}^2+\boldsymbol{b}+\boldsymbol{\ell}}\left(E\left(2n_1\sqrt{v}\right)+E\left(\frac{2n_2\sqrt{v}}{\sqrt{3}}\right)\right)e^{2\pi iB(-\boldsymbol{a},\boldsymbol{n})}q^{Q(\boldsymbol{n})}. \tag{5.6}$$

Using that $E(x)\sim\operatorname{sgn}(x)$ as $|x|\to\infty$ the terms in (5.6) exponentially decay.

We next investigate Θ^- and write it as

$$\begin{aligned}
\Theta^-(\tau) &= \sum_{\boldsymbol{n}\in\mathbb{Z}^2+\boldsymbol{a}}\left(\operatorname{sgn}(n_1)\beta\left(4n_1^2v\right)+\operatorname{sgn}(n_2)\beta\left(\frac{4n_2^2v}{3}\right)\right)(-1)^{n_1-\frac12}q^{3n_1^2+4n_1n_2+n_2^2}\\
&= \sum_{\boldsymbol{n}\in\mathbb{Z}^2}\left(\operatorname{sgn}\left(n_1+\tfrac12\right)\beta\left(4\left(n_1+\tfrac12\right)^2v\right)+\operatorname{sgn}(n_2)\beta\left(\frac{4n_2^2v}{3}\right)\right)(-1)^{n_1}\\
&\qquad\times q^{3\left(n_1+\frac12\right)^2+4\left(n_1+\frac12\right)n_2+n_2^2}\\
&= \sum_{\substack{n_1\geq 0\\ n_2\geq 1}}(-1)^{n_1}\left(\beta\left(4\left(n_1+\tfrac12\right)^2v\right)+\beta\left(\tfrac43 n_2^2v\right)\right)q^{3\left(n_1+\frac12\right)^2+4\left(n_1+\frac12\right)n_2+n_2^2}\\
&+ \sum_{\substack{n_1\geq 0\\ n_2\geq 1}}(-1)^{-n_1-1}\left(-\beta\left(4\left(-n_1-1+\tfrac12\right)^2v\right)+\beta\left(\tfrac43 n_2^2v\right)\right)q^{3\left(-n_1-1+\frac12\right)^2+4\left(-n_1-1+\frac12\right)n_2+n_2^2}\\
&+ \sum_{\substack{n_1\geq 0\\ n_2\geq 1}}(-1)^{n_1}\left(\beta\left(4\left(n_1+\tfrac12\right)^2v\right)-\beta\left(\tfrac43(-n_2)^2v\right)\right)q^{3\left(n_1+\frac12\right)^2+4\left(n_1+\frac12\right)(-n_2)+(-n_2)^2}\\
&+ \sum_{n_1\geq 0}(-1)^{-n_1-1}\left(-\beta\left(4\left(-n_1-1+\tfrac12\right)^2v\right)-\beta\left(\tfrac43(-n_2)^2v\right)\right)\\
&\qquad\times q^{3\left(-n_1-1+\frac12\right)^2+4\left(-n_1-1+\frac12\right)(-n_2)+(-n_2)^2}\\
&= 2\sum_{\substack{n_1\geq 0\\ n_2\geq 1}}(-1)^{n_1}\left(\beta\left(4\left(n_1+\tfrac12\right)^2v\right)+\beta\left(\frac{4n_2^2v}{3}\right)\right)q^{3\left(n_1+\frac12\right)^2+4\left(n_1+\frac12\right)n_2+n_2^2}\\
&+ 2\sum_{\substack{n_1\geq 0\\ n_2\geq 1}}(-1)^{n_1}\left(\beta\left(4\left(n_1+\tfrac12\right)^2v\right)-\beta\left(\frac{4n_2^2v}{3}\right)\right)q^{3\left(n_1+\frac12\right)^2-4\left(n_1+\frac12\right)n_2+n_2^2}\\
&= 2\sum_{\pm}\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{n_1,n_2\geq 0}\left(\beta\left(16\left(n_1+\tfrac{\delta}{2}+\tfrac14\right)^2v\right)\pm\beta\left(\tfrac43(n_2+1)^2v\right)\right)\\
&\qquad\times q^{12\left(n_1+\frac{\delta}{2}+\frac14\right)^2\pm 8\left(n_1+\frac{\delta}{2}+\frac14\right)(n_2+1)+(n_2+1)^2}.
\end{aligned}$$

We now first show the first asymptotic in (5.5). For this, let $\tau=\frac{it}{2\pi}$. Then

$$\Theta^-\left(\tfrac{it}{2\pi}\right)=2\sum_{\pm}\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{n_1,n_2\geq 0}f_\pm\left(\left(n_1+\tfrac{\delta}{2}+\tfrac14,n_2+1\right)\sqrt{t}\right),$$

where

$$f_\pm(x_1,x_2):=\left(\beta\left(\frac{8x_1^2}{\pi}\right)\pm\beta\left(\frac{2x_2^2}{3\pi}\right)\right)e^{-12x_1^2\mp 8x_1x_2-x_2^2}.$$

We now use Proposition 2.3. The term with the double integral term vanishes (the two δ-terms cancel). The second term contributes

$$-\frac{2}{\sqrt{t}}\sum_{\pm}\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{n_1=0}^{N-1}\frac{B_{n_1+1}\left(\frac{\delta}{2}+\frac{1}{4}\right)t^{\frac{n_1}{2}}}{(n_1+1)!}\int_0^\infty f_\pm^{(n_1,0)}(0,w_2)dw_2.$$

By combining the term for $\delta=0$ and $\delta=1$, using properties of Bernoulli polynomials it is not hard to see that only n_1 even survive. The terms from $n_1\geq 1$ yield a contribution overall that is $O(t)$. Using that $\beta(0)=1$, the term $n_1=0$ gives

$$-\frac{4}{\sqrt{t}}\sum_{\pm}B_1\left(\tfrac{1}{4}\right)\int_0^\infty\left(1\pm\beta\left(\tfrac{2x_2^2}{3\pi}\right)\right)e^{-x_2^2}dx_2=\frac{2}{\sqrt{t}}\int_0^\infty e^{-w_2^2}dw_2=\sqrt{\tfrac{\pi}{t}}.$$

For the third term, we have

$$-\frac{2}{\sqrt{t}}\sum_{\pm}\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{n_2=0}^{N-1}\frac{B_{n_2+1}(1)}{(n_2+1)!}t^{\frac{n_2}{2}}\int_0^\infty f_\pm^{(0,n_2)}(w_1,0)dw_1=0,$$

because the δ-terms cancel. The final term in Proposition 2.3 is in $O(t)$. This gives that the first asymptotic in (5.5) holds.

We next need to show that the second asymptotic in (5.5) holds. For this, we need to prove that

$$\sum_{\pm}\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{n_1,n_2\geq 0}\left(\beta\left(\tfrac{8}{\pi}\left(n_1+\tfrac{\delta}{2}+\tfrac{1}{4}\right)^2x\right)\pm\beta\left(\tfrac{2}{3\pi}(n_2+1)^2x\right)\right)$$
$$\times e^{-\left(12\left(n_1+\frac{\delta}{2}+\frac{1}{4}\right)^2\pm 8\left(n_1+\frac{\delta}{2}+\frac{1}{4}\right)(n_2+1)+(n_2+1)^2\right)z}\ll|z|^{\frac{1}{2}}. \tag{5.7}$$

The proof follows by a lengthy calculation from the following refinement of Proposition 2.3 in the one-dimensional case, namely (see (5.8) of [10])

$$\sum_{n\geq 0}f((n+a)z)=\frac{1}{z}\int_0^\infty f(w)dw-\sum_{n=0}^{N-1}\frac{B_{n+1}(a)f^{(n)}(0)}{(n+1)!}z^n+\mathcal{E}(a;z), \tag{5.8}$$

where ($C_R(0)$ is the circle around 0 with radius R, $\widetilde{B}_N(x):=B_N(x-\lfloor x\rfloor)$)

$$\mathcal{E}(a;z):=-\sum_{k\geq N}\frac{f^{(k)}(0)a^{k+1}}{(k+1)!}z^k-\frac{z^N}{2\pi i}\sum_{n=0}^{N-1}\frac{B_{n+1}(0)a^{N-n}}{(n+1)!}\int_{C_R(0)}\frac{f^{(n)}(w)}{w^{N-n}(w-az)}dw$$
$$-\frac{(-1)^Nz^{N-1}}{N!}\int_{az}^{z\infty}f^{(N)}(w)\widetilde{B}_N\left(\frac{w}{z}-a\right)dw.$$

For the reader's convenience we defer the full proof of (5.7) to Appendix A.

Combining and using Proposition 2.1 with $\gamma=\frac{\pi^2}{12}$, $\beta=0$, and $\lambda=\frac{1}{2}$ gives the claim. $\square$

6. Congruences for $\mathrm{ou}^*(n)$ modulo 4 and the proof of Theorem 1.5

In this section we prove Theorem 6.2 below. Note that this reduces to Theorem 1.5 for $k=0$. First, we determine the parity of $\mathrm{ou}^*(n)$.

Proposition 6.1. *For $n\in\mathbb{N}$, we have that $\mathrm{ou}^*(n)$ is odd if and only if $6n-2$ is a square.*

Proof. We require a classical q-series identity from Ramanujan's lost notebook [5, Entry 9.5.2],

$$\sum_{n\geq 0}\left(q;q^2\right)_nq^n=\sum_{n\geq 0}(-1)^nq^{3n^2+2n}\left(1+q^{2n+1}\right).$$

Using this along with (1.8), we have

$$\sum_{n\geq 0}\mathrm{ou}^*(n)q^n=\sum_{n\geq 0}\left(-q;q^2\right)_n^2 q^{2n+1}\equiv\sum_{n\geq 0}\left(q^2;q^4\right)_n q^{2n+1}=\sum_{n\geq 0}(-1)^n q^{6n^2+4n+1}\left(1+q^{4n+2}\right)$$
$$\equiv\sum_{n\in\mathbb{Z}} q^{6n^2+4n+1} \pmod 2.$$

Now in the extremes of this string of equations we replace q by q^6 and multiply by q^{-2} to obtain

$$\sum_{n\geq 0}\mathrm{ou}^*(n)q^{6n-2}\equiv\sum_{n\in\mathbb{Z}} q^{(6n+2)^2} \pmod 2,$$

and the result follows. □

We now state the main result of this section.

Theorem 6.2. *Let $k\in\mathbb{N}$ and for r with $1\leq r\leq k+1$, let $p_r\geq 5$ be prime. For any $j\not\equiv 0\pmod{p_{k+1}}$, if $p_{k+1}\not\equiv 7,13\pmod{24}$ or $p_{k+1}\equiv 7,13\pmod{24}$ and $(\frac{3j}{p_{k+1}})=-1$, then we have:*

(1) *If j is odd, then we have*

$$\mathrm{ou}^*\left(4p_1^2\cdots p_{k+1}^2 n+2p_1^2\cdots p_k^2 p_{k+1}j+\tfrac{8p_1^2\cdots p_{k+1}^2+1}{3}\right)\equiv 0 \pmod 4.$$

(2) *If j is even, then we have*

$$\mathrm{ou}^*\left(4p_1^2\cdots p_{k+1}^2 n+2p_1^2\cdots p_k^2 p_{k+1}j+\tfrac{2p_1^2\cdots p_{k+1}^2+1}{3}\right)\equiv 0 \pmod 4.$$

To make the proof smoother, we first prove a simple lemma.

Lemma 6.3. *For $n\in\mathbb{N}$ we have that modulo 4,*

$$\left(-q;q^2\right)_n^2-\left(-q^2;q^4\right)_n$$

is an odd polynomial.

Proof. We prove the claim by induction. The case $n=1$ is clear.

Now assume that the claim holds for $n\in\mathbb{N}$. Then

$$\left(-q;q^2\right)_{n+1}^2-\left(-q^2;q^4\right)_{n+1}=\left(1+q^{4n+2}\right)\left(\left(-q;q^2\right)_n^2-\left(-q^2;q^4\right)_n\right)+2q^{2n+1}\left(-q;q^2\right)_n^2.$$

By the induction assumption the first term is an odd polynomial $\pmod 4$. Thus we are left to show that $(-q;q^2)_n^2$ is an even polynomial $\pmod 2$. Now we have the even polynomial

$$\left(-q;q^2\right)_n^2\equiv\left(-q^2;q^4\right)_n \pmod 2.$$

□

Next we use a result of Chen and Chen [15], who employed the theory of class numbers to prove congruences modulo 4 for $\mathcal{EO}(n)$, defined via the infinite product

$$\sum_{n\geq 0}\mathcal{EO}(n)q^n:=\frac{\left(q^4;q^4\right)_\infty^3}{\left(q^2;q^2\right)_\infty^2}. \tag{6.1}$$

Theorem 6.4 ([15]). *Let $k\in\mathbb{N}_0$. For i with $1\leq i\leq k+1$, let $p_i\geq 5$ be prime. For $j\not\equiv 0\pmod{p_{k+1}}$, if $p_{k+1}\not\equiv 7,13\pmod{24}$ or $p_{k+1}\equiv 7,13\pmod{24}$ and $(\frac{3j}{p_{k+1}})=-1$, then for $n\in\mathbb{N}_0$*

$$\mathcal{EO}\left(p_1^2\cdots p_{k+1}^2 n+p_1^2\cdots p_k^2 p_{k+1}j+\tfrac{p_1^2\cdots p_{k+1}^2-1}{3}\right)\equiv 0 \pmod 4. \tag{6.2}$$

We are now ready to prove Theorem 6.2.

Proof of Theorem 6.2. First recall the third order mock theta function,

$$\nu(q) := \sum_{n\geq 0} \left(q;q^2\right)_n (-q)^n =: \sum_{n\geq 0} c(n)q^n.$$

Using Lemma 6.3 we have that

$$\sum_{n\geq 0} \mathrm{ou}^*(n)q^n - \sum_{n\geq 0}(-1)^n c(n)q^{2n+1} = \sum_{n\geq 0} \left(-q;q^2\right)_n^2 q^{2n+1} - \sum_{n\geq 0} \left(-q^2;q^4\right)_n q^{2n+1}$$

is supported on even exponents of q modulo 4. Therefore

$$\mathrm{ou}^*(2n+1) \equiv (-1)^n c(n) \pmod 4. \tag{6.3}$$

Next we note that by [17, equation (26.88)] we have

$$\sum_{n\geq 0} c(2n)q^{2n} = \frac{\left(q^4;q^4\right)_\infty^3}{\left(q^2;q^2\right)_\infty^2}.$$

This is the same product as in (6.1), and so $c(n) = \mathcal{EO}(n)$ if n is even. Therefore the congruences for $\mathcal{EO}(n)$ in Theorem 6.4 imply congruences for $c(n)$. The argument on the left-hand side of (6.2) is even if and only if $n \equiv j \pmod 2$. So for j even, we let $n \mapsto 2n$ and for j odd, we let $n \mapsto 2n+1$ to obtain:

(1) If j is odd, then

$$c\left(2p_1^2\cdots p_{k+1}^2 n + p_1^2\cdots p_k^2 p_{k+1} j + \frac{4p_1^2\cdots p_{k+1}^2 - 1}{3}\right) \equiv 0 \pmod 4.$$

(2) If j is even, then

$$c\left(2p_1^2\cdots p_{k+1}^2 n + p_1^2\cdots p_k^2 p_{k+1} j + \frac{p_1^2\cdots p_{k+1}^2 - 1}{3}\right) \equiv 0 \pmod 4.$$

By (6.3) the theorem follows. □

7. Conclusion

This paper contains a preliminary investigation of odd unimodal sequences, establishing generating functions, basic asymptotics, and some congruence properties modulo 4. While other variants of unimodal sequences have arisen in the literature (e.g. [8, 9, 21]), odd unimodal sequences are perhaps the most natural. Below we leave two ideas for further study. The motivated reader will surely find many more.

First, improve upon the asymptotics in Theorems 1.3 and 1.4 to find Rademacher-type formulas for $\mathrm{ou}(n)$ and $\mathrm{ou}^*(n)$. Up to simple terms the generating function for $\mathrm{ou}(n)$ and is a mixed false theta function while the generating function for $\mathrm{ou}^*(n)$ is a mixed mock theta function. In both cases the weight is $\frac{1}{2}$. One could now use the Circle Method to deduce asymptotic formulas for $\mathrm{ou}(n)$ and $\mathrm{ou}^*(n)$. However, an exact formula is out of reach with this method because the weight is too large. To obtain an exact formula one would need to find new methods (like Poincaré type series).

Second, it appears that the arithmetic progressions in some of the congruences in Theorem 6.2 can be enlarged. For example, with $k = 0$ (i.e., in the case of Theorem 1.5) the congruences corresponding to the primes $5, 7$, and 11 are

$$\mathrm{ou}^*(100n + r) \equiv 0 \pmod 4 \text{ for } r \in \{37, 57, 77, 97\}, \tag{7.1}$$

$$\mathrm{ou}^*(196n + r) \equiv 0 \pmod 4 \text{ for } r \in \{61, 89, 145\}, \tag{7.2}$$

$$\mathrm{ou}^*(484n + r) \equiv 0 \pmod 4 \text{ for } r \in \{125, 169, 213, 257, 301, 345, 389, 433, 477, 521\}. \tag{7.3}$$

Computations suggest that the cases $r \in \{37, 97\}$ of (7.1) are special cases of the congruences

$$\mathrm{ou}^*(50n + r) \equiv 0 \pmod 4 \text{ for } r \in \{37, 47\},$$

the cases $r \in \{61, 145\}$ of (7.2) are special cases of the congruences

$$\mathrm{ou}^*(98n + r) \equiv 0 \pmod 4 \text{ for } r \in \{47, 61\},$$

and all of the congruences in (7.3) are special cases of congruences in the corresponding arithmetic progressions modulo 242. We leave it as an open problem to establish exactly which of the congruences in Theorem 6.2 (or Theorem 1.5) can be strengthened in this way.

Appendix A

Here we prove the second bound from (5.7). We require the relation $\beta(x) = \mathrm{erfc}(\sqrt{\pi x})$ and use the bound $\mathrm{erfc}(x) \ll 1$. To use (5.8), we write

$$\begin{aligned}\Theta^-\left(\tfrac{iz}{2\pi}\right) = &-2\sum_{\pm}\sum_{\delta\in\{0,1\}}(-1)^\delta \\ &\left(\sum_{n_2\ge 0} e^{-(n_2+1)^2 z}\sum_{n_1\ge 0}\beta\left(\tfrac{8}{\pi}\left(n_1+\tfrac{\delta}{2}+\tfrac14\right)^2 x\right)e^{-12\left(n_1+\frac{\delta}{2}+\frac14\right)^2 z}e^{\mp 8\left(n_1+\frac{\delta}{2}+\frac14\right)(n_2+1)z}\right. \\ &\left.\pm\sum_{n_2\ge 0}\beta\left(\tfrac{2}{3\pi}(n_2+1)^2 x\right)e^{-(n_2+1)^2 z}\sum_{n_1\ge 0}e^{-12\left(n_1+\frac{\delta}{2}+\frac14\right)^2 z\mp 8\left(n_1+\frac{\delta}{2}+\frac14\right)(n_2+1)z}\right). \quad (A.1)\end{aligned}$$

We write the first term as

$$-2\sum_{\pm}\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{n_2\ge 0}e^{-(n_2+1)^2 z}\sum_{n_1\ge 0}H_{n_2,\pm}\left(\left(n_1+\tfrac{\delta}{2}+\tfrac14\right)\sqrt{x}\right),$$

where

$$H_{n_2,\pm}(w_1) := \beta\left(\frac{8w_1^2}{\pi}\right)e^{-12\frac{z}{x}w_1^2\mp 8(n_2+1)\frac{z}{\sqrt{x}}w_1}.$$

Now (5.8) with $N = 0$, gives that

$$\sum_{n_1\ge 0}H_{n_2,\pm}\left(\left(n_1+\tfrac{\delta}{2}+\tfrac14\right)\sqrt{x}\right) = \frac{1}{\sqrt{x}}\int_0^\infty H_{n_2,\pm}(w_1)dw_1 + \mathcal{E}\left(\tfrac{\delta}{2}+\tfrac14; x\right),$$

where

$$\mathcal{E}(a;x) := -\sum_{k_1\ge 0}\frac{H_{n_2,\pm}^{(k_1)}(0)}{(k_1+1)!}a^{k_1+1}x^{\frac{k_1}{2}} - \frac{1}{\sqrt{x}}\int_{a\sqrt{x}}^\infty H_{n_2,\pm}(w_1)\widetilde{B}_0\left(\frac{w_1}{\sqrt{x}}-a\right)dw_1. \quad (A.2)$$

Note that $\widetilde{B}_0(x) = 1$. The contribution from the main term vanishes (because of the $(-1)^\delta$).

The first term in the error contributes

$$\begin{aligned}&2\sum_{\pm}\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{n_2\ge 0}e^{-(n_2+1)^2 z}\sum_{k_1\ge 0}\left[\left(\frac{\partial}{\partial w_1}\right)^{k_1}H_{n_2,\pm}(w_1)\right]_{w_1=0}\frac{\left(\frac{\delta}{2}+\frac14\right)^{k_1+1}}{(k_1+1)!}x^{\frac{k_1}{2}} \\ &= 2\sum_{\pm}\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{k_1\ge 0}\frac{\left(\frac{\delta}{2}+\frac14\right)^{k_1+1}}{(k_1+1)!}x^{\frac{k_1}{2}}\sum_{\substack{0\le \ell_1,\ell_2\le k_1\\ \ell_1+\ell_2=k_1}}\binom{k_1}{\ell_1}\left[\left(\frac{\partial}{\partial w_1}\right)^{\ell_1}\left(\beta\left(\frac{8w_1^2}{\pi}\right)e^{-12\frac{z}{x}w_1^2}\right)\right]_{w_1=0} \\ &\qquad\times\sum_{n_2\ge 0}\left(\mp 8(n_2+1)\frac{z}{\sqrt{x}}\right)^{\ell_2}e^{-(n_2+1)^2 z}. \quad (A.3)\end{aligned}$$

Now only ℓ_2 even survive (otherwise the $\pm$ cancels). We now determine the asymptotic behaviors of (ℓ_2 even)

$$\sum_{n_2\geq 0}(n_2+1)^{\ell_2}e^{-(n_2+1)^2z}=(-1)^{\frac{\ell_2}{2}}\left(\frac{\partial}{\partial z}\right)^{\frac{\ell_2}{2}}\sum_{n\geq 1}e^{-n^2z}.$$

For this recall the modular theta function

$$\vartheta(\tau):=\sum_{n\in\mathbb{Z}}e^{\pi in^2\tau}.$$

It satisfies

$$\vartheta(\tau)=(-i\tau)^{-\frac{1}{2}}\vartheta\left(-\tfrac{1}{\tau}\right).$$

Thus

$$\sum_{n\geq 1}e^{-n^2z}=\tfrac{1}{2}\sum_{n\in\mathbb{Z}}e^{-n^2z}-\tfrac{1}{2}=\tfrac{1}{2}\vartheta\left(\tfrac{iz}{\pi}\right)-\tfrac{1}{2}=\tfrac{1}{2}\left(\tfrac{z}{\pi}\right)^{-\frac{1}{2}}\vartheta\left(\tfrac{\pi i}{z}\right)-\tfrac{1}{2}. \tag{A.4}$$

The second term contributes to (A.3) (it only survives if $\ell_2=0$)

$$-\tfrac{1}{2}\sum_{\pm}\sum_{\delta\in\{0,1\}}(-1)^{\delta}\sum_{k_1\geq 0}\frac{\left(\frac{\delta}{2}+\frac{1}{4}\right)^{k_1+1}}{(k_1+1)!}x^{\frac{k_1}{2}}\left[\left(\tfrac{\partial}{\partial w_1}\right)^{k_1}\left(\beta\left(\tfrac{8w_1^2}{\pi}\right)e^{-12\frac{z}{x}w_1^2}\right)\right]_{w_1=0}$$
$$\ll\sum_{k_1\geq 0}\frac{x^{\frac{k_1}{2}}}{(k_1+1)!}\left(1+\left|\frac{z}{x}\right|^{k_1}\right)\left[\left(\tfrac{\partial}{\partial w_1}\right)^{k_1}\left(\beta\left(\tfrac{8w_1^2}{\pi}\right)e^{-12w_1^2}\right)\right]_{w_1=0}\ll 1.$$

The first term from (A.4) contributes to (A.3) (noting that ℓ, k_1 need to be even)

$$\ll\sum_{k_1\geq 0}\frac{x^{k_1}}{(2k_1+1)!}\sum_{0\leq\ell\leq k_1}\binom{2k_1}{2\ell}\left[\left(\tfrac{\partial}{\partial w_1}\right)^{2k_1-2\ell}\left(\beta\left(\tfrac{8w_1^2}{\pi}\right)e^{-12\frac{z}{x}w_1^2}\right)\right]_{w_1=0}\left(\tfrac{8z}{\sqrt{x}}\right)^{2\ell}\frac{\partial^{2\ell}}{\partial z^{2\ell}}\frac{\vartheta\left(\frac{\pi i}{z}\right)}{z^{\frac{1}{2}}}. \tag{A.5}$$

Now assume that $\frac{x}{|z|^2}>1$ (which is true as $x\to 0$). Then

$$z^{2\ell}\frac{\partial^{2\ell}}{\partial z^{2\ell}}\frac{\vartheta\left(\frac{\pi i}{z}\right)}{z^{\frac{1}{2}}}\ll\left[\frac{\partial^{2\ell}}{\partial w_2^{2\ell}}\frac{\vartheta\left(\frac{\pi i}{w_2}\right)}{w_2{}^{\frac{1}{2}}}\right]_{w_2=1}.$$

Moreover, as above,

$$\left[\left(\tfrac{\partial}{\partial w_1}\right)^{2k_1-2\ell}\left(\beta\left(\tfrac{8w_1^2}{\pi}\right)e^{-12\frac{z}{x}w_1^2}\right)\right]_{w_1=0}\ll\left(1+\Delta^{2k_1-2\ell}\right)\left[\left(\tfrac{\partial}{\partial w_1}\right)^{2k_1-2\ell}\left(\beta\left(\tfrac{8w_1^2}{\pi}\right)e^{-12w_1^2}\right)\right]_{w_1=0}.$$

This yields that (A.5) can be bound against $O(1)$. Thus the first term in (A.2) is overall $O(1)$.

The second term of (A.2) contributes

$$\tfrac{2}{\sqrt{x}}\sum_{\pm}\sum_{\delta\in\{0,1\}}(-1)^{\delta}\sum_{n_2\geq 0}e^{-(n_2+1)^2z}\int_{\left(\frac{\delta}{2}+\frac{1}{4}\right)\sqrt{x}}^{\infty}H_{n_2,\pm}(w_1)dw_1$$
$$=\tfrac{2}{\sqrt{x}}\sum_{\pm}\sum_{n_2\geq 0}e^{-(n_2+1)^2z}\left(\int_{\frac{\sqrt{x}}{4}}^{\infty}-\int_{\frac{3\sqrt{x}}{4}}^{\infty}\right)H_{n_2,\pm}(w_1)dw_1$$
$$=\tfrac{2}{\sqrt{x}}\sum_{\pm}\int_{\frac{\sqrt{x}}{4}}^{\frac{3\sqrt{x}}{4}}\beta\left(\tfrac{8}{\pi}w_1^2\right)e^{-12\frac{z}{x}w_1^2}\sum_{n_2\geq 0}e^{-(n_2+1)^2z\mp 8(n_2+1)\frac{z}{\sqrt{x}}w_1}dw_1. \tag{A.6}$$

The sum on n_2 may be written as

$$\sum_{n_2\geq 0} h_{[1]}\left((n_2+1)\sqrt{x}\right),$$

where $h_{[1]}(w_2) := e^{-\frac{z}{x}w_2^2 \mp 8\frac{z}{x}w_1w_2}$. Using (5.8), with $N=0$, we have

$$\sum_{n_2\geq 0} h_{[1]}\left((n_2+1)\sqrt{x}\right) = \frac{1}{\sqrt{x}}\int_0^\infty h_{[1]}(w_2)dw_2 + \mathcal{E}^{[1]}(x),$$

where

$$\mathcal{E}^{[1]}(x) := -\sum_{k_2\geq 0}\frac{h_{[1]}^{(k_2)}(0)}{(k_2+1)!}x^{\frac{k_2}{2}} - \frac{1}{\sqrt{x}}\int_{\sqrt{x}}^\infty h_{[1]}(w_2)dw_2.$$

The main term contributes to (A.6) as

$$\frac{2}{x}\sum_{\pm}\int_{\frac{\sqrt{x}}{4}}^{\frac{3\sqrt{x}}{4}}\beta\left(\tfrac{8w_1^2}{\pi}\right)e^{-12\frac{z}{x}w_1^2}\int_0^\infty h_{[1]}(w_2)dw_2dw_1 = \frac{2}{x}\sum_{\pm}\int_{\frac{\sqrt{x}}{4}}^{\frac{3\sqrt{x}}{4}}\beta\left(\tfrac{8w_1^2}{\pi}\right)e^{4\frac{z}{x}w_1^2}\int_0^\infty e^{-\frac{z}{x}(w_2\pm 4w_1)^2}dw_2.$$

Using that $\int_{4w_1}^\infty + \int_{-4w_1}^\infty = 2\int_0^\infty$, the above is in $O(\frac{1}{\sqrt{z}})$.

We next consider the first term in $\mathcal{E}^{[1]}(x)$ which contributes

$$-\frac{2}{\sqrt{x}}\sum_{\pm}\int_{\frac{\sqrt{x}}{4}}^{\frac{3\sqrt{x}}{4}}\beta\left(\tfrac{8w_1^2}{\pi}\right)e^{-12\frac{z}{x}w_1^2}\sum_{k_2\geq 0}\left[\frac{\partial^{k^2}}{\partial w_2^{k^2}}e^{-\frac{z}{x}w_2^2\mp 8\frac{z}{x}w_1w_2}\right]_{w_2=0}dw_1\frac{x^{\frac{k_2}{2}}}{(k_2+1)!}.$$

We have

$$\left[\frac{\partial^{k_2}}{\partial w_2^{k_2}}e^{-\frac{z}{x}w_2^2\mp 8\frac{z}{x}w_1w_2}\right]_{w_2=0} = \sum_{\ell=0}^{k}\binom{k_2}{\ell}\left(\mp 8\frac{z}{x}w_1\right)^\ell\left(\frac{\sqrt{z}}{\sqrt{x}}\right)^{k_2-\ell}\left[\frac{\partial^{k_2-\ell}}{\partial w_2^{k_2-\ell}}e^{-w_2^2}\right]_{w_2=0}.$$

Now the $\pm$ enforces ℓ, $k_2-\ell$ to be even because of the $\pm$ and bound overall against $O(\frac{1}{\sqrt{z}})$.

We next consider the second term in $\mathcal{E}^{[1]}(x)$. This contributes

$$\frac{1}{x}\int_{\frac{\sqrt{x}}{4}}^{\frac{3\sqrt{x}}{4}}\beta\left(\tfrac{8w_1^2}{\pi}\right)e^{-12\frac{z}{x}w_1^2}\int_{\sqrt{x}}^\infty e^{-\frac{z}{x}w_2^2\mp 8\frac{z}{x}w_1w_2}dw_2dw_1. \tag{A.7}$$

We write

$$e^{-12\frac{z}{x}w_1^2}\int_{\sqrt{x}}^\infty e^{-\frac{z}{x}w_2^2\mp 8\frac{z}{x}w_1w_2}dw_2 \ll e^{4\frac{z}{x}w_1^2}\int_{\sqrt{x}}^\infty e^{-\frac{z}{x}(w_2\pm 4w_1)^2}dw_2 \ll e^{4w_1^2}\int_{\mathbb{R}} e^{-w_2^2}dw_2.$$

Thus (A.7) may be bound against $O(\frac{1}{\sqrt{z}})$. Combining gives that the first term is $O(\frac{1}{\sqrt{z}})$.

We next turn to the second term in (A.1) and proceed as before, again first considering the sum on n_1. We have

$$\sum_{n_1\geq 0} e^{-12\left(n_1+\frac{\delta}{2}+\frac{1}{4}\right)^2 z\mp 8\left(n_1+\frac{\delta}{2}+\frac{1}{4}\right)(n_2+1)z} = \sum_{n_1\geq 0} G_{n_2,\pm}\left(\left(n_1+\tfrac{\delta}{2}+\tfrac{1}{4}\right)\sqrt{z}\right),$$

where $G_{n_2,\pm}(w_1) := e^{-12w_1^2\mp 8(n_2+1)\sqrt{z}w_1}$. Now (5.8) gives (with $N=0$) that

$$\sum_{n_1\geq 0} G_{n_2,\pm}\left(\left(n_1+\tfrac{\delta}{2}+\tfrac{1}{4}\right)\sqrt{z}\right) = \frac{1}{\sqrt{z}}\int_0^\infty G_{n_2,\pm}(w_1)dw_1 + E\left(\tfrac{\delta}{2}+\tfrac{1}{4};z\right),$$

where

$$E(a;z) := -\sum_{k_1\geq 0}\frac{G^{(k_1)}_{n_2,\pm}(0)}{(k_1+1)!}a^{k_1+1}z^{\frac{k_1}{2}} - \frac{1}{\sqrt{z}}\int_{a\sqrt{z}}^{\sqrt{z}\infty} G_{n_2,\pm}(w_1)\widetilde{B}_0\left(\tfrac{w_1}{\sqrt{z}}-a\right)dw_1.$$

The main term contributes overall

$$-\tfrac{2}{\sqrt{z}}\sum_{\pm}\pm\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{n_2\geq 0}\beta\left(\tfrac{2}{3\pi}(n_2+1)^2x\right)e^{-(n_2+1)^2z}\int_0^\infty G_{n_2,\pm}(w_1)dw_1 = 0.$$

The first term in the error $E(\frac{\delta}{2}+\frac{1}{4};z)$ contributes

$$\begin{aligned}2\sum_{\pm}\pm\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{n_2\geq 0}\beta\left(\tfrac{2}{3\pi}(n_2+1)^2x\right)e^{-(n_2+1)^2z}\sum_{k_1\geq 0}\left[\left(\frac{\partial}{\partial w_1}\right)^{k_1}G_{n_2,\pm}(w_1)\right]_{w_1=0}\\ \times\frac{\left(\frac{\delta}{2}+\frac{1}{4}\right)^{k_1+1}}{(k_1+1)!}z^{\frac{k_1}{2}} = 2\sum_{\pm}\pm\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{k_1\geq 0}\left(\tfrac{\delta}{2}+\tfrac{1}{4}\right)^{k_1+1}\frac{z^{\frac{k_1}{2}}}{(k_1+1)!}\\ \times\left[\left(\frac{\partial}{\partial w_1}\right)^{k_1}\left(e^{-12w_1^2}\sum_{n_2\geq 0}\beta\left(\tfrac{2}{3\pi}(n_2+1)^2x\right)e^{-(n_2+1)^2z\mp 8(n_2+1)\sqrt{z}w_1}\right)\right]_{w_1=0}.\end{aligned}$$

We write the sum on n_2 as

$$\sum_{n_2\geq 0}g_{[1],\pm}\left((n_2+1)\sqrt{x}\right),$$

where $g_{[1],\pm}(w_2) := \beta(\frac{2w_2^2}{3\pi})e^{-\frac{z}{x}w_2^2\mp 8\frac{\sqrt{z}}{\sqrt{x}}w_1w_2}$. Now (5.8), with $N=0$, gives that

$$\sum_{n_2\geq 0}g_{[1],\pm}\left((n_2+1)\sqrt{x}\right) = \frac{1}{\sqrt{x}}\int_0^\infty g_{[1],\pm}(w_2)dw_2 + E^{[1]}(x),$$

where

$$E^{[1]}(x) := -\sum_{k_2\geq 0}\frac{g^{(k_2)}_{[1],\pm}(0)}{(k_2+1)!}x^{\frac{k_2}{2}} - \frac{1}{\sqrt{x}}\int_{\sqrt{x}}^\infty g_{[1],\pm}(w_2)\widetilde{B}_0\left(\tfrac{w_2}{\sqrt{x}}-1\right)dw_2.$$

The main term contributes

$$\frac{2}{\sqrt{x}}\sum_{\pm}\pm\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{k_1\geq 0}\left(\tfrac{\delta}{2}+\tfrac{1}{4}\right)^{k_1+1}\frac{z^{\frac{k_1}{2}}}{(k_1+1)!}\left[\left(\tfrac{\partial}{\partial w_1}\right)^{k_1}\left(e^{-12w_1^2}\int_0^\infty g_{[1],\pm}(w_2)dw_2\right)\right]_{w_1=0}. \tag{A.8}$$

We rewrite

$$\begin{aligned}&\left[\left(\tfrac{\partial}{\partial w_1}\right)^{k_1}\left(e^{-12w_1^2}\int_0^\infty g_{[1],\pm}(w_2)dw_2\right)\right]_{w_1=0}\\ &\qquad = \sum_{j=0}^{k_1}\binom{k_1}{j}\left[\left(\tfrac{\partial}{\partial w_1}\right)^{k_1-j}e^{-12w_1^2}\right]_{w_1=0}\int_0^\infty\beta\left(\tfrac{2w_2^2}{3\pi}\right)e^{-\frac{z}{x}w_2^2}\left(\mp 8\sqrt{\tfrac{z}{x}}w_2\right)^j dt_2.\end{aligned}$$

The $\pm$ forces j to be even. Also k_1 is even. Thus (A.8) is $O(\frac{1}{\sqrt{z}})$.

The first term from $E^{[1]}$ contributes

$$2\sum_{\pm}\pm\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{k_1\geq 0}\left(\tfrac{\delta}{2}+\tfrac{1}{4}\right)^{k_1+1}\left[\left(\tfrac{\partial}{\partial w_1}\right)^{k_1}\right.$$

$$\left(e^{-12w_1^2}\sum_{k_2\geq 0}\left[\left(\frac{\partial}{\partial w_2}\right)^{k_2}\left(g_{[1],\pm}(w_2)\right)\right]_{w_2=0}\right)\Bigg]_{w_1=0}\frac{z^{\frac{k_1}{2}}}{(k_1+1)!}\frac{x^{\frac{k_2}{2}}}{(k_2+1)!}\ll 1.$$

The second term from $E^{[1]}$ contributes

$$\frac{2}{\sqrt{x}}\sum_{\pm}\pm\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{k_1\geq 0}\left(\tfrac{\delta}{2}+\tfrac{1}{4}\right)^{k_1+1}\frac{z^{\frac{k_1}{2}}}{(k_1+1)!}\left[\left(\frac{\partial}{\partial w_1}\right)^{k_1}\left(e^{-12w_1^2}\int_{\sqrt{x}}^{\infty}g_{[1],\pm}(w_2)dw_2\right)\right]_{w_1=0}.$$

We compute

$$\left[\left(\frac{\partial}{\partial w_1}\right)^{k_1}e^{-12w_1^2\mp 8\frac{\sqrt{z}}{\sqrt{x}}w_1w_2}\right]_{w_1=0}=\sum_{\ell=0}^{k_1}\binom{k_1}{\ell}\left[\left(\frac{\partial}{\partial w_1}\right)^{k_1-\ell}e^{-12w_1^2}\right]_{w_1=0}\left(\mp 8\frac{\sqrt{z}}{\sqrt{x}}w_2\right)^{\ell}.$$

Arguing as above we need ℓ, k_1 to be even and obtain overall $O(\frac{1}{\sqrt{z}})$.

The second term in the error $E(\frac{\delta}{2}+\frac{1}{4};z)$ contributes

$$\frac{2}{\sqrt{z}}\sum_{\pm}\pm\sum_{\delta\in\{0,1\}}(-1)^\delta\sum_{n_2\geq 0}\beta\left(\tfrac{2}{3\pi}(n_2+1)^2x\right)e^{-(n_2+1)^2z}\int_{\left(\frac{\delta}{2}+\frac{1}{4}\right)\sqrt{z}}^{\sqrt{z}\infty}e^{-12w_1^2\mp 8(n_2+1)\sqrt{z}w_1}dw_1$$

$$=\frac{2}{\sqrt{z}}\sum_{\pm}\pm\sum_{n_2\geq 0}\beta\left(\tfrac{2}{3\pi}(n_2+1)^2x\right)e^{-(n_2+1)^2z}\int_{\frac{\sqrt{z}}{4}}^{\frac{3\sqrt{z}}{4}}e^{-12w_1^2\mp 8(n_2+1)\sqrt{z}w_1}dw_1.$$

Now the sum on n_2 is

$$\sum_{n_2\geq 0}g_{[2]}\left((n_2+1)\sqrt{x}\right),$$

where

$$g_{[2]}(w_2):=\beta\left(\frac{2w_2^2}{3\pi}\right)e^{-\frac{z}{x}w_2^2\mp 8\frac{\sqrt{z}}{\sqrt{x}}w_1w_2}.$$

Thus (5.8), with $N=0$, gives

$$\sum_{n_2\geq 0}g_{[2]}\left((n_2+1)\sqrt{x}\right)=\frac{1}{\sqrt{x}}\int_0^\infty g_{[2]}(t_2)dt_2+E^{[4]}(x),$$

where

$$E^{[4]}(x):=-\sum_{k_2\geq 0}\frac{g_{[2]}^{(k_2)}(0)}{(k_2+1)!}x^{\frac{k_2}{2}}-\frac{1}{\sqrt{x}}\int_{\sqrt{x}}^{\infty}g_{[2]}(w_2)\widetilde{B}_0\left(\tfrac{w_2}{\sqrt{x}}-1\right)dw_2.$$

The main term contributes

$$\frac{2}{\sqrt{z}\sqrt{x}}\sum_{\pm}\pm\int_{\frac{\sqrt{z}}{4}}^{\frac{3\sqrt{z}}{4}}e^{-12w_1^2}\int_0^\infty g_{[2]}(w_2)dw_2dw_1.$$

We write

$$e^{-12w_1^2}\int_0^\infty g_{[2]}(w_2)dw_2=e^{4w_1^2}\int_0^\infty\beta\left(\frac{2w_2^2}{3\pi}\right)e^{-\frac{z}{x}\left(w_2\mp 4\frac{\sqrt{x}}{\sqrt{z}}\right)^2}dw_2\ll e^{4w_1^2}.$$

Thus we have overall $O(\frac{1}{\sqrt{z}})$.

The first term in the error $E^{[4]}$ contributes

$$-\frac{2}{\sqrt{z}}\sum_{\pm}\pm\int_{\frac{\sqrt{z}}{4}}^{\frac{3\sqrt{z}}{4}}e^{-12w_1^2}\sum_{k_2\geq 0}\left[\left(\frac{\partial}{\partial w_2}\right)^{k_2}\beta\left(\frac{2w_2^2}{3\pi}\right)e^{-\frac{z}{x}w_2^2\mp 8\frac{\sqrt{z}}{\sqrt{x}}w_1w_2}\right]_{w_2=0}\frac{x^{\frac{k_2}{2}}}{(k_2+1)!}.$$

We bound, since $|z| \leq (1+\Delta)x$,

$$\left[\left(\frac{\partial}{\partial w_2}\right)^{k_2}\left(\beta\left(\frac{2w_2^2}{3\pi}\right)e^{-\frac{z}{x}w_2^2\mp 8\frac{\sqrt{z}}{\sqrt{x}}w_1w_2}\right)\right]_{w_2=0} \ll \left(1+\Delta^{\frac{k_2}{2}}\right)\left[\left(\frac{\partial}{\partial w_2}\right)^{k_2}\beta\left(\frac{2w_2^2}{3\pi}\right)e^{-w_2^2\mp 8w_1w_2}\right]_{w_2=0}.$$

Then overall we have

$$\ll \frac{1}{\sqrt{z}}\sum_{k_2\geq 0}\left(1+\Delta^{\frac{k_2}{2}}\right)\left[\left(\frac{\partial}{\partial w_2}\right)^{k_2}\left(\beta\left(\frac{2w_2^2}{3\pi}\right)e^{-w_2^2}\int_{\frac{\sqrt{z}}{4}}^{\frac{3\sqrt{z}}{4}}e^{-12w_1^2\mp 8w_1w_2}dw_1\right)\right]_{w_2=0}\frac{x^{\frac{k_2}{2}}}{(k_2+1)!} \ll 1.$$

The second term in the error $E^{[4]}$ can be estimated as before as $O(\frac{1}{\sqrt{z}})$.

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