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ODD-BALANCED UNIMODAL SEQUENCES AND RELATED FUNCTIONS: PARITY, MOCK MODULARITY AND QUANTUM MODULARITY

BYUNGCHAN KIM, SUBONG LIM, AND JEREMY LOVEJOY

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ABSTRACT. We define odd-balanced unimodal sequences and show that their generating function $\mathcal{V}(x,q)$ has the same remarkable features as the generating function for strongly unimodal sequences $U(x,q)$. In particular, we discuss (mixed) mock modularity, quantum modularity, and congruences modulo 2 and 4. We also study two related functions which share some of the properties of $U(x,q)$ and $\mathcal{V}(x,q)$.

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. **Strongly unimodal sequences.** A *strongly unimodal sequence* is a sequence of positive integers which are strictly increasing up to a point and strictly decreasing thereafter. The high point of the sequence is called the *peak*, the sum of the entries is called the *weight*, and the difference between the number of entries before and after the peak is called the *rank*. For example, the ten strongly unimodal sequences of weight 6 are $(6),(1,5),(5,1),(2,4),(4,2),(1,2,3),(3,2,1),(1,4,1),(2,3,1)$, and $(1,3,2)$, and their ranks are $0,1,-1,1,-1,2,-2,0,0$, and 0.

Let $u(n)$ denote the number of strongly unimodal sequences of weight n and let $u(m,n)$ denote the number of strongly unimodal sequences of weight n and rank m. It is easy to see that the generating function for $u(m,n)$ is

$$U(x,q) := \sum_{\substack{n\geq 1\\ m\in\mathbb{Z}}} u(m,n)x^m q^n = \sum_{n\geq 0}(-xq)_n(-q/x)_n q^{n+1}, \tag{1.1}$$

where as usual

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^{n-1}). \tag{1.2}$$

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This generating function is rather remarkable, having received considerable attention over the last few years because of its connections to mock theta functions and quantum modular forms [4, 5, 8, 10, 11, 14, 24]. Using results of the third author [19], it can be expressed in terms of indefinite theta series [14, 21],

$$(1.3)\quad (1+x)U(x;q) = \frac{q}{(q)_\infty}\left(\sum_{r,n\geq 0} - \sum_{r,n<0}\right)(-1)^n x^{-r} q^{n(3n+5)/2+2nr+r(r+3)/2},$$

and using a result of Ramanujan [2, Entry 3.4.7], it can be expressed in terms of Appell-Lerch-type series,

$$(1.4)\qquad (1+1/x)U(x;q) = \frac{-1}{(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{n(3n+1)/2}}{1+xq^n} + \frac{1}{(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{x^{-n}q^{\binom{n+1}{2}}}{1+xq^n}.$$

Applying work of Zwegers [29], these expressions imply that $U(x,q)$ is a mixed mock modular form (i.e., is in the tensor space of modular forms and mock modular forms) when $x \neq -1$ is a root of unity times a rational power of q. When $x = \pm i$ we recover a classical third order mock theta function $\psi(q)$ of Ramanujan,

$$(1.5)\qquad U(\pm i, q) = \sum_{n\geq 0}(-q^2;q^2)_n q^{n+1} = \sum_{n\geq 1}\frac{q^{n^2}}{(q;q^2)_n} =: \psi(q).$$

(The second equality follows from a simple combinatorial argument.) When $x = -1$ we have a quantum modular form which played a crucial role in the study of radial limits of Ramanujan's third order mock theta function [10],

$$(1.6)\qquad f(q) := \sum_{n\geq 0}\frac{q^{n^2}}{(q)_n^2},$$

and which is dual to the Kontsevich-Zagier series

$$(1.7)\qquad F(q) := \sum_{n\geq 0}(q)_n,$$

in the sense that for all roots of unity ζ_N we have [8]

$$(1.8)\qquad F(\zeta_N^{-1}) = U(-1;\zeta_N).$$

The function $U(1;q)$ also satisfies interesting congruences modulo 2. Namely, if $3 < \ell \not\equiv 23 \pmod{24}$ is prime and $\ell \nmid k$, then [8]

$$(1.9)\qquad u(\ell^2 n + k\ell + (\ell^2-1)/24) \equiv 0 \pmod{2}.$$

Similar congruences are conjectured to hold modulo 4 for $u(n)$ as well as for the coefficients of $U(\pm i, q)$ [8].

1.2. **Odd-balanced unimodal sequences.** In the first part of this paper we consider a type of unimodal sequence whose generating function has the same features as the generating function for strongly unimodal sequences. To define our unimodal sequences, we first relax the "strong" condition in the definition of $u(n)$ and allow odd parts to repeat on either side of the peak. We then add two requirements. First the peak must be even and second, the odd parts must be identical on either side of the peak. We call these *odd-balanced* unimodal sequences. For example, the 16 odd-balanced unimodal sequences of weight 10 are (10), $(1,8,1)$, $(8,2)$,

$(2,8)$, $(1,1,6,1,1)$, $(2,6,2)$, $(4,6)$, $(6,4)$, $(1,6,2,1)$, $(1,2,6,1)$, $(1,1,1,4,1,1,1)$, $(1,2,4,2,1)$, $(1,1,2,4,1,1)$, $(1,1,4,2,1,1)$, $(3,4,3)$, and $(1,1,1,1,2,1,1,1,1)$.

Let $v(n)$ denote the number of odd-balanced unimodal sequences of weight $2n+2$ and let $v(m,n)$ denote the number of such sequences having rank m. We define

$$\mathcal{V}(x,q) := \sum_{n\geq 0} \frac{(-xq,-q/x)_n q^n}{(q;q^2)_{n+1}} = \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} v(m,n)x^m q^n \tag{1.10}$$

$$= 1 + 2q + (x^{-1}+3+x)q^2 + (2x^{-1}+5+2x^2)q^3 + \cdots . \tag{1.11}$$

As with $U(x,q)$, the series $\mathcal{V}(x,q)$ can be expressed in terms of indefinite quadratic forms and Appell-Lerch series. This follows from work of the third author [19] and Hickerson and Mortenson [15], and is presented in [21]:

$$\left(1+\frac{1}{x}\right) q\mathcal{V}(x,q)$$

$$= \frac{(-q)_\infty}{(q)_\infty}\left(\sum_{n,r\geq 0} - \sum_{n,r<0}\right)(-1)^n x^r q^{n^2+2n+(2n+1)r+r(r+1)/2+1} \tag{1.12}$$

$$= -m(x,q,-1) + \frac{j(-x,q)}{J_{1,2}} m(x^2,q^2,-1) - x\frac{J_4^3}{J_2^3}\frac{j(x,q)j(x^2q,q^2)}{j(x^4,q^4)}. \tag{1.13}$$

Here we have adopted one of the usual notations for the Appell-Lerch series,

$$m(x,q,z) := \frac{1}{j(z,q)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1-xzq^{n-1}}, \tag{1.14}$$

where $j(z,q) = (z)_\infty(q/z)_\infty(q)_\infty$, $J_{a,m} = j(q^a,q^m)$, and $J_m = J_{m,3m}$. These expressions imply that $\mathcal{V}(x,q)$ is generically a mixed mock modular form for $x \neq -1$ a root of unity times a rational power of q. Certain cases have more special structure. For example, when $x = \pm i$ direct substitution in (1.10) gives

$$\mathcal{V}(\pm i,q) = \sum_{n\geq 0} \frac{(-q^2;q^2)_n q^n}{(q;q^2)_{n+1}} = q^{-1}A(q),$$

where $A(q)$ is a mock theta function of 2nd order [20]. For another example, equation (1.13) combined with the fact that $m(q,q^2,-1) = 1/2$ [15] shows that $(q+q^2)\mathcal{V}(q,q^2)$ is a modular function. For the role of $\mathcal{V}(x,q)$ in the study of radial limits of mock theta functions, see [22].

The first result we prove in this paper concerning $\mathcal{V}(x,q)$ is the following determination of the parity of $v(n)$, the number of odd-balanced unimodal sequences of weight $2n+2$.

Theorem 1.1. *For all nonnegative integers n, $v(n)$ is odd if and only if $8n+7 = p^e m^2$ with $p \equiv -1 \pmod 8$ and $e \equiv 1 \pmod 4$.*

This implies numerous Ramanujan-type congruences.

Corollary 1.2. *Let $\ell \not\equiv -1 \pmod 8$ be prime and let k be a positive integer satisfying $\ell \| 8k+7$. Then, for all non-negative integers n,*

$$v(\ell^2 n + k) \equiv 0 \pmod 2.$$

We also give a conjecture on congruences for $v(n)$ modulo 4 analogous to the conjecture for $u(n)$ modulo 4 in [8].

Conjecture 1.3. *Let $\ell \not\equiv -1 \pmod 8$ be prime and let and k be a positive integer satisfying $\ell \| 8k+7$. If $(8k+7)/\ell$ is a quadratic residue modulo ℓ, then*

$$v(\ell^2 n + k) \equiv 0 \pmod 4.$$

For example, $v(9n+4) \equiv v(25n+11) \equiv v(25n+16) \equiv 0 \pmod 4$. Moreover, the same congruences hold for the coefficients of $\mathcal{V}(\pm i, q) = q^{-1}A(q)$.

Our second result concerns the quantum modularity of $\mathcal{V}(-1,q)$. Quantum modular forms were introduced by Zagier [28], who gave some examples coming from various objects such as quadratic forms, q-series and quantum invariants of 3-manifolds and knots. Quantum modular forms have been studied in connection with numerous topics including Maass forms, mock theta functions and strongly unimodal sequences (for example, see [7, 9, 10]). Following Zagier, we define a quantum modular form as follows.

Definition 1.4. *A quantum modular form of weight k and multiplier system χ on Γ is a function $f(x)$ on $\mathbb{Q}$ such that for every $\gamma \in \Gamma$ the function*

$$f(x) - (f|_{k,\chi}\gamma)(x)$$

can be extended smoothly to $\mathbb{R}$ except finitely many points. Here the operator $|_{k,\chi}\gamma$ is defined by

$$(f\ |_{k,\chi}\ \gamma)(x) := \overline{\chi(\gamma)}(cx+d)^{-k} f\,(\gamma x)\,.$$

The *formal dual* of a q-hypergeometric series is obtained by setting $q = q^{-1}$ and using $(q^{-1};q^{-1})_n = (-1)^n q^{-\binom{n+1}{2}}(q)_n$. Unlike the case of $U(-1,q)$, the dual of $\mathcal{V}(-1,q)$ is well-defined,

$$\mathcal{V}(-1,q^{-1}) = q \sum_{n\geq 0} \frac{(-1)^{n+1}(q)_n^2}{(q;q^2)_{n+1}}. \tag{1.15}$$

Here convergence is defined as the average of the limits of the even and odd partial sums. We show that the function $\mathcal{V}(-1,q^{-1})$ satisfies quantum modular transformation properties at roots of unity. More precisely, we define

$$A := \{x \in \mathbb{Q} |\ x \text{ is equivalent to } i\infty \text{ under the action of } \Gamma_0(16)\},$$

which is an infinite subset of $\mathbb{Q}$. Since every element in A can be written as $\gamma(i\infty)$ for $\gamma \in \Gamma_0(16)$, one can see that A is the set of rational numbers $\frac{c}{d}$ such that d is divisible by 16 and c is an integer with $(2,c)=1$. For $x \in \mathbb{Q}$ let

$$h(x) := \lim_{z\to x} q^{-7}\mathcal{V}(-1,q^{-8}), \tag{1.16}$$

where $q = e^{2\pi i z}$ and $z \in \mathbb{H}$. Note that if we use the expression of $\mathcal{V}(-1,q^{-1})$ in (1.15), one can see that $h(x)$ is well-defined on A. This comes from the fact that $(q;q^2)_n$ is not zero for any n and $(q)_n$ is zero for sufficiently large n when $q = e^{2\pi i x}$ and $x \in A$.

Theorem 1.5. *The function $h(x)$ is well-defined on A and satisfies the modular transformation equation*

$$h(x) - (h|_{\frac{3}{2},\chi}\gamma)(x) = p_\gamma(x)$$

for $x \in A$ and $\gamma \in \Gamma_0(16)$, where $p_\gamma : \mathbb{R} \to \mathbb{C}$ is a C^∞ function which is real analytic everywhere except at $\gamma^{-1}(i\infty)$. Here, a multiplier system χ is given by

$$\chi(\gamma) = \psi(d)\left(\frac{c}{d}\right)\epsilon_d^3$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(16)$*, where* $\epsilon_d = 1$ *or* i *as* $d \equiv 1$ *or* $3 \pmod 4$*, and* ψ *is a primitive Dirichlet character modulo* 2 *defined by* $\psi(n) = 1$ *or* 0 *as* n *is odd or even.*

This shows that the function $h(x)$ (and hence $q^7\mathcal{V}(-1,q^8)$) is (nearly) an example of a quantum modular form of weight $3/2$.

Remark 1.6. *We shall see that* $h(x)$ *can be understood as the formal Eichler integral of a non-cusp form. Such an example was asked for in a recent work of Bringmann and Rolen* [6]*, where they defined Eichler quantum modular forms associated with cusp forms.*

1.3. **Related functions.** In the second part of this paper we consider two functions which have some, but not all, of the same features as $U(x,q)$ and $\mathcal{V}(x,q)$. These are

$$\mathcal{W}(x,q) := \sum_{n\geq 0} \frac{(xq;q^2)_n(q/x;q^2)_n q^{2n}}{(-q)_{2n+1}} \text{ and } \mathcal{Z}(x,q) := \sum_{n\geq 0} \frac{(-xq)_n(-q/x)_n q^n}{(q)_{2n+1}}.$$

We briefly summarize the facts about these series. Details can be found in Sections 3 and 4.

First, both $\mathcal{W}(x,q)$ and $\mathcal{Z}(x,q)$ have simple interpretations as generating functions. They are also both expressible in terms of Appell-Lerch series and indefinite theta series, though only the expressions for $\mathcal{W}(x,q)$ can be found in the literature. We compute the expressions for $\mathcal{Z}(x,q)$ in Proposition 3.1. We express each of $\mathcal{W}(x,q)$ and $\mathcal{Z}(x,q)$ in terms of known mock theta functions for two instances of (x,q) in Corollaries 3.2 and 4.2.

We examine the formal duals of $\mathcal{W}(1,q)$ and $\mathcal{Z}(-1,q)$, showing in Section 4.3 and Proposition 3.4 that the former is a quantum modular form and the latter is a weight 1 modular form. The congruences modulo 2 and 4 for $U(1;q)$ and $\mathcal{V}(1;q)$ do not appear to have any analogue in the case of $\mathcal{W}(1,q)$, but there are elegant congruences for $\mathcal{Z}(1,q)$ modulo 2. These are presented in Theorem 3.3.

The rest of this article is organized as follows. In Section 2, we prove the results on the function $\mathcal{V}(x,q)$ and discuss its properties in more detail. We study the functions $\mathcal{Z}(x,q)$ and $\mathcal{W}(x,q)$ in Sections 3 and 4, respectively. In particular, we prove the properties sketched above.

2. Odd-balanced unimodal sequences

2.1. **Congruences.** In the first part of this section we prove Theorem 1.1. Before getting started we recall a lemma of Andrews, Dyson, and Hickerson.

Lemma 2.1. [3, Lemma 3] *Let* (x_1,y_1) *be the fundamental solution of* $x^2-Dy^2=1$*; i.e. the solution in which* x_1 *and* y_1 *are minimal positive. If* $m>0$*, then each equivalence class of solutions of* $u^2-Dv^2=m$ *contains a unique* (u,v) *with* $u>0$ *and*

$$-\frac{y_1}{x_1+1}u < v \leq \frac{y_1}{x_1+1}u.$$

If $m<0$*, the corresponding conditions are* $v>0$ *and*

$$-\frac{Dy_1}{x_1+1}v < u \leq \frac{Dy_1}{x_1+1}v.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By setting $x = 1$ in (1.12), we have

$$2\sum_{n\geq 0}\frac{(-q)_n^2 q^n}{(q;q^2)_{n+1}} = \frac{(-q)_\infty}{(q)_\infty}\left(\sum_{n,r\geq 0} - \sum_{n,r<0}\right)(-1)^n q^{n^2+2n+(2n+1)r+r(r+1)/2}.$$

Let

$$f_1(q) := \left(\sum_{n,r\geq 0} - \sum_{n,r<0}\right)(-1)^n q^{n^2+2n+(2n+1)r+r(r+1)/2}.$$

By noting that

$$(n+r+1)^2 - 2\left(\frac{r}{2}+\frac{1}{4}\right)^2 = n^2+2n+(2n+1)r+\frac{r(r+1)}{2}+\frac{7}{8},$$

we see that

$$q^7 f_1(q^8) = 2\sum_{\substack{n\geq 0\\ 0\leq r\leq n}}(-1)^{n+r}q^{-((2r+1)^2-2(2n+2)^2)}.$$

Therefore, we conclude that

$$q^7\mathcal{V}(1,q^8) = \frac{(-q^8;q^8)_\infty}{(q^8;q^8)_\infty}\sum_{\substack{n\geq 0\\ 0\leq r\leq n}}(-1)^{n+r}q^{-((2r+1)^2-2(2n+2)^2)}.$$

From Lemma 2.1, we observe that

$$\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K\\ N(\mathfrak{a})\equiv 7 \pmod 8}} q^{N(\mathfrak{a})} = \sum_{\substack{n\geq 0\\ -n-1\leq r\leq n}} q^{-((2r+1)^2-2(2n+2)^2)} = 2\sum_{\substack{n\geq 0\\ 0\leq r\leq n}} q^{-((2r+1)^2-2(2n+2)^2)},$$

where $K := \mathbb{Q}(\sqrt{2})$. Therefore, we obtain that

$$\frac{1}{2}q^7 f_1(q^8) \equiv \frac{1}{2}\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K\\ N(\mathfrak{a})\equiv 7 \pmod 8}} q^{N(\mathfrak{a})} \pmod 2. \tag{2.1}$$

Since $\frac{(-q)_\infty}{(q)_\infty} \equiv 1 \pmod 2$, we have proven that

$$q^7\mathcal{V}(1,q^8) \equiv \frac{1}{2}\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K\\ N(\mathfrak{a})\equiv 7 \pmod 8}} q^{N(\mathfrak{a})} \pmod 2. \tag{2.2}$$

To precisely determine $v(n)$ modulo 2, we will cite a formula for the number of ideals of norm m in $\mathcal{O}_K$ from [18]. Such a formula depends on the prime factorization of m and whether such primes split, ramify, or remain inert in $\mathcal{O}_K$.

Proposition 2.2 (See Theorem 1.1 of [18] and its proof)**.** *Suppose that m has the prime factorization $m = 2^a p_1^{e_1}\cdots p_j^{e_j} q_1^{f_1}\cdots q_k^{f_k}$, where the p_i are congruent to ± 1 modulo 8 and the q_i are congruent to ± 3 modulo 8. Then the number of ideals of norm m in $\mathcal{O}_K$ is equal to 0, if some f_i is odd, and $(e_1+1)(e_2+1)\cdots(e_j+1)$, otherwise.*

Now, if $m = 8n+7$, then by the above proposition the number of ideals of norm m will be divisible by 4 if: (i) some f_i is odd, (ii) at least two of the e_i's are odd, or (iii) one of the e_i's is odd and congruent to -1 modulo 4. Appealing to (2.2), this finishes the proof of Theorem 1.1. □

We now prove the Ramanujan-type congruences in Corollary 1.2.

Proof of Corollary 1.2. Note that $8(\ell^2 n+k)+7$ is a multiple of ℓ, but not a multiple of ℓ^2. Therefore, $8\ell^2 n+8k+7$ cannot be of the form $p^e m^2$, where $p \equiv -1 \pmod 8$. Therefore, $v(\ell^2 n+k)$ is even for all nonnegative integers n. □

2.2. **Quantum Modularity.** We turn to the proof of Theorem 1.5. From the lost notebook [2, Entry 5.4.3], we know that the identity

$$\left(1+\frac{1}{a}\right)\sum_{n\geq 0}\frac{(-1)^n(-aq,-q/a)_n}{(q;q^2)_{n+1}} = \frac{1}{2}\sum_{n\geq 0}(-1)^n(a^n+a^{-n-1})q^{n(n+1)/2} \tag{2.3}$$

holds "formally", or where convergence on the left is defined by taking the average of the limits of the even and odd partial sums. By taking the limit $a \to -1$, we find that

$$\mathcal{V}(-1,q^{-1}) = -\frac{q}{2}\sum_{n\geq 0}(2n+1)q^{n(n+1)/2},$$

and thus

$$q^{-7}\mathcal{V}(-1,q^{-8}) = -\frac{1}{2}\sum_{n\geq 0}(2n+1)q^{(2n+1)^2}.$$

This can be understood as a formal Eichler integral of a theta series

$$g(q) := \sum_{n\geq 0} q^{(2n+1)^2} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\psi(n)q^{n^2},$$

where ψ is a primitive Dirichlet character modulo 2 defined by $\psi(n)$ is 1 or 0 as n is odd or even. This is a modular form in $M_{\frac{1}{2}}(\Gamma_0(16),\psi)$, and it vanishes only at the cusp $i\infty$ (for details see [25]).

The proof of Theorem 1.5 is based on the argument in [27, Section 6] (see also [13, Proposition 8]). Consider

$$\tilde{g}(z) := \int_z^{i\infty} g(q)(\tau-\bar{z})^{-\frac{3}{2}}\,d\tau,$$

where $q = \exp 2\pi i\tau$. Since g decays exponentially on $A\cup\{i\infty\}$, $\tilde{g}$ is well-defined on $\mathbb{H}\cup A$. By the standard calculation of the Eichler integral,

$$\tilde{g}(z) = e^{-\frac{\pi}{4}i}\sqrt{2\pi}\sum_{n\geq 0}(2n+1)\Gamma\left(-\frac{1}{2},4\pi(2n+1)^2 y\right)e^{2\pi i(2n+1)^2\bar{z}},$$

where $\Gamma(a,x)$ is the incomplete gamma function and y is the imaginary part of z. Then we find that

$$h(x) = -\frac{e^{\frac{\pi}{4}i}}{2\sqrt{2\pi}\Gamma\left(-\frac{1}{2}\right)}\tilde{g}(x)$$

for $x \in A$, where $h(x)$ is the function defined by (1.16).

Recall that a multiplier system χ is given by $\chi(\gamma) = \psi(d)\left(\frac{c}{d}\right)\epsilon_d^3$ for $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(16)$, where ϵ_d is 1 or i as $d \equiv 1$ or $3 \pmod 4$. We can compute its period function for $\gamma \in \Gamma_0(16)$ (see [16])

$$\begin{aligned} h(x)-(h|_{\frac{3}{2},\chi}\gamma)(x) &= \frac{e^{\frac{\pi}{4}i}}{4\sqrt{2\pi}}\left(\tilde{g}(x)-(\tilde{g}|_{\frac{3}{2},\chi}\gamma)(x)\right)\\ &= \frac{e^{\frac{\pi}{4}i}}{4\sqrt{2\pi}}\int_{\gamma^{-1}(i\infty)}^{i\infty} g(q)(\tau-x)^{-\frac{3}{2}}\,d\tau, \end{aligned}$$

where we use $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$. Note that g decreases exponentially at $i\infty$ and $\gamma^{-1}(i\infty)$, and we have a C^∞ function on $\mathbb{R}$ which is real analytic everywhere except at $\gamma^{-1}(i\infty)$. $\square$

The formal dual of $\mathcal{V}(1,q)$ is also well-defined. We have

$$\mathcal{V}(1,q^{-1}) = -q\sum_{n\geq 0}\frac{(-1)^n(-q)_n^2}{(q;q^2)_{n+1}} = \frac{q}{2}\sum_{n\geq 0}(-1)^{n+1}q^{n(n+1)/2}. \tag{2.4}$$

In (2.4), we obtain the last equality by putting $a = 1$ into (2.3). Therefore, we find that

$$q^{-7}\mathcal{V}(1,q^{-8}) = -\frac{1}{2}\sum_{n\geq 0}\chi(n)q^{n^2},$$

where $\chi(n)$ is the Dirichlet character mod 4 defined by $\chi(n) = 1$ or -1 as $n \equiv 1$ or $3 \pmod 4$. Since this is an odd character,

$$\sum_{n\geq 0} n\chi(n)q^{n^2} \tag{2.5}$$

is a weight 3/2 cusp form, and thus we can expect its Eichler integral will give the quantum modularity of $q^{-7}\mathcal{V}(1,q^{-8})$. Note that $\mathcal{V}(1,q^{-1})$ is well-defined at even roots of unity. For $x \in \mathbb{Q}$ let

$$\tilde{h}(x) := \lim_{z\to x} q^{-7}\mathcal{V}(1,q^{-8}),$$

where $z \in \mathbb{H}$. Note that if we define $\nu(\gamma) = \chi(d)\left(\frac{-4c}{d}\right)\epsilon_d$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(64)$, then ν is a multiplier system of weight $\frac{1}{2}$ on $\Gamma_0(64)$.

Theorem 2.3. *The function $\tilde{h}(x)$ is a quantum modular form of weight $\frac{1}{2}$ and multiplier system ν on $\Gamma_0(64)$.*

Proof. Note that the theta function in (2.5) is a cusp form in $S_{\frac{3}{2}}(\Gamma_0(64))$ since χ is the nontrivial Dirichlet character modulo 4 (for example, see [23, Theorem 1.44]). Arguing as in the proof of Theorem 1.5, one can see that $\tilde{h}$ is a quantum modular form of weight $\frac{1}{2}$ since $q^{-7}\mathcal{V}(1,q^{-8})$ is an Eichler integral of a cusp form given in (2.5). $\square$

3. Over-balanced unimodal sequences

3.1. **Combinatorics.** We define a third type of unimodal-type sequence, using overpartitions for the parts before and after the peak. By overpartitions, we mean that we may overline the first occurrence of each part. We make three additional requirements. First, the peak must be even. Second, only even parts can be overlined. Finally, the number of non-overlined parts must be identical on either side of the peak. We call these *over-balanced* unimodal sequences. There are 10 over-balanced unimodal sequences of weight 8: (8), $(1,6,1)$, $(\overline{2},6)$, $(6,\overline{2})$, $(2,4,2)$, $(1,1,4,1,1)$, $(\overline{2},4,\overline{2})$, $(1,\overline{2},4,1)$, $(1,4,\overline{2},1)$, $(1,1,1,2,1,1,1)$. Let $z(n)$ be the number of such unimodal sequences of weight $2n+2$ and let $z(m,n)$ be the number of such sequences having rank m. Then, we find that

$$q^2\mathcal{Z}(x,q^2) = \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} z(m,n)x^mq^n = q^2 + 2q^4 + (x+3+x^{-1})q^6 + \cdots.$$

3.2. **Identities.** We express $\mathcal{Z}(x,q)$ using indefinite theta series and Apell-Lerch sums as follows.

Proposition 3.1. *We have*

$$(1+x)\mathcal{Z}(x,q) = \frac{1}{(q)_\infty^2}\left(\sum_{n,r\geq 0} - \sum_{n,r<0}\right)(-1)^n x^{-r} q^{n(n+3)/2+(2n+1)r+r(r+1)/2} \tag{3.1}$$

$$= \frac{(1+x)j(-q^2/x,q)}{(q)_\infty^2} m(x^2,q^3,-q/x). \tag{3.2}$$

Proof. We use the fact [19, Eq. (1.5)] that if (α_n,β_n) is a Bailey pair to a, then

$$\sum_{n\geq 0} q^n \beta_n = \frac{1}{(aq)_\infty (q)_\infty}\sum_{n,r\geq 0}(-a)^n q^{\binom{n+1}{2}+(2n+1)r}\alpha_r. \tag{3.3}$$

Inserting the Bailey pair relative to q [1, Lemma 6],

$$\alpha_n = (-1)^n q^{\binom{n+1}{2}} x^{-n}(1-x^{2n+1}) \text{ and } \beta_n = \frac{(x)_{n+1}(q/x)_n}{(q^2)_{2n+1}}, \tag{3.4}$$

and simplifying we obtain the indefinite theta series. To obtain the Appell-Lerch sum we apply Proposition 8.1 of [15]. $\square$

The "classical" mock theta functions are all expressible in terms of the Appell-Lerch series [15]. Using these expressions we obtain identities involving instances of $\mathcal{Z}(x,q)$ and the sixth order mock theta functions

$$\psi(q) := \sum_{n\geq 0}\frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q)_{2n+1}} \text{ and } \sigma(q) := \sum_{n\geq 0}\frac{q^{\binom{n+2}{2}}(-q)_n}{(q;q^2)_{n+1}}. \tag{3.5}$$

Corollary 3.2. *We have*

$$\mathcal{Z}(1,q) = \frac{q^{-1}(-q)_\infty^2}{(q)_\infty}\psi(q) \text{ and } (1-q)\mathcal{Z}(-q,q^2) = \frac{q^{-1}(q;q^2)_\infty^2}{(q^2;q^2)_\infty}\sigma(q). \tag{3.6}$$

Proof. We use the identities $\psi(q) = m(1,q^3,-q)$ and $\sigma(q) = -m(q^2,q^6,q)$ from [15] together with Proposition 3.1. $\square$

3.3. **Congruences.** In this section we prove the following congruence.

Theorem 3.3. *For all nonnegative integers n,*

$$z(2n+1) \equiv 0 \pmod 2. \tag{3.7}$$

Proof. By setting $x=1$ in (3.1), we find that

$$2\mathcal{Z}(1,q) = \frac{1}{(q)_\infty^2}\left(\sum_{n,r\geq 0} - \sum_{n,r<0}\right)(-1)^n q^{n(n+3)/2+(2n+1)r+r(r+1)/2}.$$

Let

$$f_2(q) := \left(\sum_{n,r\geq 0} - \sum_{n,r<0}\right)(-1)^n q^{n(n+3)/2+(2n+1)r+r(r+1)/2}.$$

Since

$$\frac{1}{2}\left(n+2r+\frac{3}{2}\right)^2 - \frac{3}{2}\left(r+\frac{1}{2}\right)^2 = \frac{1}{2}n(n+3)+(2n+1)r+\frac{1}{2}r(r+1)+\frac{3}{4},$$

we have that

$$q^6 f_2(q^8) = \left(\sum_{n,r\geq 0} - \sum_{n,r<0} \right) (-1)^n q^{(2n+4r+3)^2-3(2r+1)^2}.$$

The first sum can be written as

$$\sum_{n,r\geq 0} (-1)^n q^{(2n+4r+3)^2-3(2r+1)^2} = \sum_{\substack{n\geq 0 \\ 0\leq r\leq \frac{1}{2}n}} (-1)^n q^{(2n+3)^2-3(2r+1)^2},$$

and the second sum as

$$\sum_{n,r<0} (-1)^n q^{(2n+4r+3)^2-3(2r+1)^2} = - \sum_{\substack{n\geq 0 \\ -\frac{1}{2}n-1\leq r<0}} (-1)^n q^{(2n+3)^2-3(2r+1)^2}.$$

Therefore, we have

$$q^6 f_2(q^8) = \sum_{\substack{n\geq 0 \\ -\frac{1}{2}n-1\leq r\leq \frac{1}{2}n}} (-1)^n q^{(2n+3)^2-3(2r+1)^2} = 2 \sum_{\substack{n\geq 0 \\ 0\leq r\leq \frac{1}{2}n}} (-1)^n q^{(2n+3)^2-3(2r+1)^2}.$$

In summary, we have shown that

$$q^6 \mathcal{Z}(1,q^8) = \frac{1}{(q^8;q^8)_\infty^2} \sum_{\substack{n\geq 0 \\ 0\leq r\leq \frac{1}{2}n}} (-1)^n q^{(2n+3)^2-3(2r+1)^2}.$$

Since $\frac{1}{(q)_\infty^2} \equiv \frac{1}{(q^2;q^2)_\infty} \pmod 2$, the congruence (3.7) will follow if we show that we can pair the solutions (n,r) of

$$(2n+3)^2 - 3(2r+1)^2 \equiv 14 \pmod{16} \tag{3.8}$$

with $0 \leq r \leq \frac{1}{2}n$. Let M be the set of such pairs (n,r). Note that n can not be equal to $3r$ since if $n = 3r$, then

$$(2n+3)^2 - 3(2r+1)^2 = 24r^2 + 24r + 6 = 24r(r+1) + 6 \equiv 6 \pmod{16}.$$

Therefore, we split the set M into two parts $M_1 = \{(n,r) | 0 \leq r < \frac{1}{3}n\}$ and $M_2 = \{(n,r) | \frac{1}{3}n < r \leq \frac{1}{2}n\}$. Now we assume that $(n,r) \in M_1$. Our goal is to find the corresponding $(n',r') \in M_2$. To this end, we consider

$$\left((2n+3) + (2r+1)\sqrt{3}\right)(2-\sqrt{3}) = (2(2n-3r)+3) - (2(n-2r)+1)\sqrt{3}.$$

Since $2-\sqrt{3}$ is a unit in $\mathcal{O}_{\mathbb{Q}(\sqrt{3})}$, one can check that

$$(2(2n-3r)+3)^2 - 3(2(n-2r)+1)^2 = (2n+3)^2 - 3(2r+1)^2.$$

We claim that the pair $(2n-3r, n-2r)$ is in M_2. Since $n > 3r$ by assumption, we see that

(1) $2n - 3r \geq 0$ since $n > 3r \geq \frac{3}{2}r$,
(2) $\frac{1}{3}(2n-3r) < (n-2r)$ due to $2n - 3r < 3n - 6r$,
(3) $n - 2r \leq \frac{1}{2}(2n-3r)$ because $r \geq 0$, $2n - 4r \leq 2n - 3r$.

Therefore, $(2n-3r, n-2r)$ is in M_2 whenever $(n,r) \in M_1$, which completes the proof since this coupling is surely bijective. $\square$

3.4. **The formal dual.** We finish our treatment of $\mathcal{Z}(x,q)$ with a brief discussion of the formal dual of $\mathcal{Z}(\pm 1,q)$. This time the dual is a theta function rather than a quantum modular form. We have

$$\mathcal{Z}(\pm 1, q^{-1}) = -q\sum_{n\geq 0}\frac{q^{n^2+n}(\mp q)_n^2}{(q;q)_{2n+1}}.$$

Applying [17, Theorem 1.3, corrected] and [26, equation (22)] gives the following result.

Proposition 3.4. *We have*

$$\mathcal{Z}(-1,q^{-1}) = -q\frac{(q^3;q^3)_\infty^3}{(q)_\infty} \text{ and } \mathcal{Z}(1,q^{-1}) = -q\frac{(q^6;q^6)_\infty^2}{(q)_\infty(q^3;q^3)_\infty}.$$

We remark that the first product is the generating function for 3-core partitions [12].

4. Partitions without repeated odd parts

4.1. **Combinatorics.** Unlike the previous cases, the coefficients of $\mathcal{W}(x,q)$ can be given a simple partition-theoretic interpretation. We define $pod_{3,4}(n)$ to be the number of partitions without repeated odd parts where the largest part is of the form $4x+3$ and $4x+1$ doesn't occur, and define the rank of such partition to be the the number of parts $\equiv 3 \pmod 4$ $-$ the number of parts $\equiv 1 \pmod 4$ -1. We also define $pod^{e}_{3,4}(m,n)$ (resp. $pod^{o}_{3,4}(m,n)$) to be the number of such partitions with rank m and such that the number of parts is even (resp. odd). Then, we see that

$$\begin{aligned} q^3\mathcal{W}(xq,q^2) &= \sum_{n\geq 0}\frac{(xq^3,q/x;q^4)_n q^{4n+3}}{(-q^2;q^2)_{2n+1}} \\ &= \sum_{n\geq 0}\sum_{m\in\mathbb{Z}}\left(pod^{o}_{3,4}(m,n) - pod^{e}_{3,4}(m,n)\right)x^m q^n \\ &= q^3 - q^5 + 2q^7 - x^{-1}q^8 - 2q^9 + (-x+x^{-1})q^{10} + \cdots . \end{aligned}$$

4.2. **Identities.** Indefinite theta series and Apell-Lerch sum expression for $\mathcal{W}(x,q)$ are given by Mortenson [21, Theorem 4.3 and eqn. (4.25)].

Proposition 4.1 ([21])**.** *We have*

(4.1)

$$(1+x^{-1})q\mathcal{W}(x,q) = 2m(x,q^2,-1) - \frac{j(xq,q^2)}{\overline{J}_{1,4}}m(x,q,-1) - \frac{1}{2}\frac{J_1^5}{J_2^4}\frac{j(xq,q^2)}{j(-x,q)}$$

and

(4.2)

$$(1+x)\mathcal{W}(x,q) = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n,r\geq 0} - \sum_{n,r<0}\right)(-1)^n x^{-n} q^{n^2+2n+(2n+1)r+r(r+1)/2}.$$

Comparing (4.1) with Appell-Lerch series for classical mock theta functions, we find identities involving

$$\mu(q) := \sum_{n\geq 0} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2} = 4m(-q,q^4,-1) - \frac{J_{2,4}^4}{J_1^3}, \tag{4.3}$$

$$\omega(q) := \sum_{n\geq 0} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2} = -2q^{-1}m(q,q^6,q^2) + \frac{J_6^3}{J_2J_{3,6}}, \tag{4.4}$$

and

$$\phi_-(q) := \sum_{n\geq 1} \frac{q^n(-q)_{2n-1}}{(q;q^2)_n} = -m(q,q^3,q) - q\frac{\overline{J}_{3,12}^3}{J_1\overline{J}_{1,4}}, \tag{4.5}$$

which are mock theta functions of order 2, 3, and 6, respectively. Here $\overline{J}_{a,m} := j(-q^a,q^m)$. The second equality in each line can be found in [15].

Corollary 4.2. *We have*

$$2(q+q^2)\mathcal{W}(q,q^2) = \mu(-q) - \frac{(q)_\infty}{(q^4;q^4)_\infty} \tag{4.6}$$

and

$$(q^2+q^3)\mathcal{W}(q,q^3) = \frac{j(q^2,q^6)}{\overline{J}_{1,4}(q^3)}\phi_-(q) - q\omega(q). \tag{4.7}$$

Proof. For (4.6), we set $(x,q)=(q,q^2)$ in (4.1), use that fact (mentioned in the introduction) that $m(q,q^2,-1)=1/2$, and compare with (4.3). We find that (4.6) is equivalent to the identity

$$\frac{(q)_\infty}{(q^4;q^4)_\infty} + \frac{J_{2,4}(-q)^2}{J_1^3(-q)} = \frac{j(q^3,q^4)}{\overline{J}_{1,4}(q^2)} + \frac{J_1(q^2)^5}{J_2(q^2)^4}\frac{j(q^3,q^4)}{j(-q,q^2)}.$$

But it is easy to see that the first terms on each side are identical, as are the second terms. This gives (4.6). The proof of (4.7) is similar, using the expressions in (4.4) and (4.5) together with the fact that [15]

$$m(x,q,z_1) - m(x,q,z_0) = \frac{z_0J_1^3 j(z_1/z_0,q)j(xz_0z_1,q)}{j(z_0,q)j(z_1,q)j(xz_0,q)j(xz_1,q)}.$$

□

4.3. **The formal dual.** From [21, equation (4.26)] or [2, Entry 5.4.4], we see that the formal dual

$$q^{-1}\mathcal{W}(1,q^{-1}) = \sum_{n\geq 0}\frac{(q;q^2)_n^2q^n}{(-q)_{2n+1}} = \sum_{n\geq 0}(-1)^nq^{n(n+1)},$$

which implies that

$$q^{-3}\mathcal{W}(1,q^{-4}) = \sum_{n\geq 0}\chi(n)q^{n^2},$$

where $\chi(n)$ is the same Dirichlet character mod 4 defined in Section 2.2. By noting that $\mathcal{W}(1,q^{-1})$ is well-defined at odd roots of unity, we can understand $q^{-3}\mathcal{W}(1,q^{-4})$ as a quantum modular form of weight $\frac{1}{2}$ with the same multiplier system ν as in Theorem 2.3. It is surprising that the duals of $\mathcal{W}(1,q)$ and $\mathcal{V}(1,q)$

are related by the substitution of q by q^2. It is unclear whether this is an accident or evidence of a deeper relation between $\mathcal{W}(x,q)$ and $\mathcal{V}(x,q)$.

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School of Liberal Arts, Seoul National University of Science and Technology, 232 Gongneung-ro, Nowon-gu, Seoul 01811, Korea
Email address: bkim4@seoultech.ac.kr

Department of Mathematics Education, Sungkyunkwan University, 25-2, Sungkyunkwan-ro, Jongno-gu, Seoul 03063, Republic of Korea
Email address: subong@skku.edu

CNRS, LIAFA, Université Denis Diderot - Paris 7, Case 7014, 75205 Paris Cedex 13, FRANCE
Email address: lovejoy@math.cnrs.fr