IDENTITIES FOR OVERPARTITIONS WITH EVEN SMALLEST PARTS

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Abstract. We prove several combinatorial identities involving overpartitions whose smallest parts are even. These follow from an infinite product generating function for certain four-colored overpartitions.

1. Statement of results

In a recent study of generalizations of Schur's theorem, the second author gave two identities involving partitions whose smallest parts are even [12, Corollaries 2 and 3]. For example, let $C(n)$ denote the number of partitions of n satisfying the difference conditions

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 5, & \text{if } \lambda_{i+1} \text{ is even or if } \lambda_{i+1} \equiv 5 \pmod 6 \text{ and } \lambda_i \equiv 0,5 \pmod 6, \\ 11, & \text{if } \lambda_{i+1} \equiv 0 \pmod 6 \text{ and } \lambda_i \equiv 0,5 \pmod 6, \end{cases}$$

where, in addition, the smallest s parts are even, where s is the number of parts congruent to 1 or 2 modulo 6. Then $C(n)$ is equal to the number of partitions of n into distinct parts congruent to $0, 2, 3, 4$ modulo 6.

While the connection between partitions with congruence restrictions and partitions with difference conditions is in line with many classical identities (see [1, 2, 6] for some surveys), restricting the parity of the smallest parts according to the number of parts in certain congruence classes is quite different. This restriction arises from the use of a "partial staircase" in place of the usual staircase in the proof of Schur's theorem. For more on partial staircases, see [9, 12].

Motivated by this, our goal in this paper is to prove several identities for *overpartitions* into distinct parts whose smallest parts are even. Recall that an overpartition is a partition in which the final occurrence of an integer may be overlined [7]. Instead of partial staircases, the overpartition identities depend on what we call a "partial generalized staircase". The two simplest results are stated below, using the usual truth function $\chi(A) = 1$ if A is true and 0 otherwise.

Theorem 1. *Let $A_1(n)$ denote the number of overpartitions into parts ≥ 2 satisfying the difference conditions*

$$\lambda_i - \lambda_{i+1} \geq 4 + 2\chi(\lambda_i \text{ is overlined}) + 3\chi(\lambda_{i+1} \text{ is odd}),$$

where, in addition, (i) only parts $\equiv 2, 3 \pmod 4$ may be overlined, (ii) neither $\overline{2}$ nor $\overline{3}$ occurs, and (iii) the smallest s parts are even, where s is the number of non-overlined

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parts $\equiv 2,3 \pmod 4$*. Then* $A_1(n)$ *is equal to the number of partitions into distinct parts* $\equiv 0,2,4,5,6 \pmod 8$*.*

Theorem 2. *Let* $B_1(n)$ *denote the number of overpartitions of* n *satisfying the difference conditions*

$$\lambda_i - \lambda_{i+1} \geq 3 + 2\chi(\lambda_{i+1} \text{ is overlined}) + 5\chi(\lambda_{i+1} \text{ is odd}),$$

where, in addition, (i) *only parts* $\equiv 1,2 \pmod 4$ *may be overlined,* (ii) *if the smallest overlined part is the* k*th smallest part, then it is not* $\overline{4k-2}$ *or* $\overline{4k-3}$*, and* (iii) *the smallest* s *parts are even, where* s *is the number of non-overlined parts* $\equiv 1,2 \pmod 4$*. Then* $B_1(n)$ *is equal to the number of partitions into distinct parts* $\equiv 0,2,3,4,6 \pmod 8$*.*

As an example of Theorem 1, there are 12 partitions of $n = 23$ into distinct parts congruent to $0,2,4,5,$ or 6 modulo 8,

$$(21,2),\ (18,5),\ (16,5,2),\ (14,5,4),\ (13,10),\ (13,8,2),\ (13,6,4),\ (12,6,5),$$

$$(12,5,4,2),\ (10,8,5),\ (10,6,5,2),\ (8,6,5,4),$$

and as predicted, there are exactly 12 overpartitions of $n = 23$ counted by $A_1(23)$,

$$(\overline{23}),\ (21,2),\ (\overline{19},4),\ (19,4),\ (\overline{18},5),\ (17,\overline{6}),\ (17,6),$$

$$(16,\overline{7}),\ (\overline{15},8),\ (15,8),\ (\overline{15},6,2), (13,8,2).$$

To illustrate Theorem 2, note that there are 15 partitions of $n = 22$ into distinct parts congruent to $0,2,3,4,$ or 6 modulo 8,

$$(22),\ (20,2),\ (19,3),\ (18,4),\ (16,6),\ (16,4,2),\ (14,8),\ (14,6,2),\ (12,10),\ (12,8,2),$$

$$(12,6,4),\ (11,8,3),\ (11,6,3,2),\ (10,8,4),\ (10,6,4,2),$$

and as predicted, there are also 15 overpartitions counted by $B_1(22)$,

$$(\overline{22}),\ (22),\ (20,2),\ (19,3),\ (\overline{18},4),\ (18,4),\ (\overline{17},\overline{5}),\ (16,\overline{6}),\ (16,6),$$

$$(15,7),\ (\overline{14},8),\ (14,8),\ (\overline{14},6,2),\ (14,6,2),\ (12,8,2).$$

Our results depend on combinatorial interpretations of the infinite product

$$(-aq;q^2)_\infty(-bq)_\infty(-q)_\infty,$$

which we prove in the next section. Here we use the usual notation

$$(a)_n = (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}),$$

valid for $n \in \mathbb{N} \cup \{0,\infty\}$. These combinatorial interpretations are in terms of certain four-colored overpartitions. Theorems 1 and 2 then follow after dilating by $q = q^4$ and making appropriate substitutions for a and b. This approach goes back to Alladi and Gordon's treatment of Schur's partition theorem. See [4, 5] for Alladi and Gordon's original work and [2, 3, 8] for more on their idea and its applications. Two more identities resembling Theorems 1 and 2 are given in Theorems 5 and 6 in Section 3.

2. Identities for four-colored overpartitions

In this section we consider overpartitions into distinct parts colored by u, au, b, and ab such that (i) only parts colored by b or ab may be overlined, and (ii) the s smallest parts do not have a in their color, where s is the number of non-overlined parts labeled by b or ab. Denote this set of overpartitions by $\mathcal{O}$. Define $A(i,j,n)$ to be the number of four-colored overpartitions of n in $\mathcal{O}$ with i parts having a in their color and j parts having b in their color, with no part equal to $\overline{1}$, and such that the minimal difference between adjacent parts is given by the matrix

$$M_A = \begin{array}{c} \\ \overline{b} \\ b \\ u \\ \overline{ab} \\ ab \\ au \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} \overline{b} & b & u & \overline{ab} & ab & au \end{array} \\ \begin{pmatrix} 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix} \end{array}.$$

By this we mean that the minimal difference between λ_i of color x and λ_{i+1} of color y is given by the (x,y) entry of M. We have the following generating function for $A(i,j,n)$.

Theorem 3. *We have*

$$\sum_{i,j,n\geq 0} A(i,j,n)a^i b^j q^n = \left(-aq;q^2\right)_\infty (-bq)_\infty(-q)_\infty. \tag{2.1}$$

Proof: We first consider three partitions - one ordinary partition with "uncolored" parts labeled by u, another ordinary partition with parts labeled by b, and a partition into distinct parts ≥ 2 labeled by b. We overline the parts in the third partition and then put the three partitions together with the order

$$\overline{x}_b < x_b < x_u, \tag{2.2}$$

obtaining a colored overpartition such that only parts labeled by b may be overlined, with no part $\overline{1}_b$, and satisfying the difference conditions in the matrix

$$\begin{array}{c} \\ \overline{b} \\ b \\ u \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \overline{b} & b & u \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}. \tag{2.3}$$

The generating function for such overpartitions is

$$\sum_{r,s,t\geq 0} \frac{q^r}{(q)_r}\frac{b^s q^s}{(q)_s}\frac{b^t q^{t+\binom{t+1}{2}}}{(q)_t}. \tag{2.4}$$

Next we add a *staircase* to this overpartition, meaning that we add 0 to the smallest part, 1 to the next smallest part, and so on. This multiplies the generating function by the term $q^{\binom{r+s+t}{2}}$ and increases the minimal difference between parts by 1, resulting in the difference conditions

$$\begin{array}{c} \\ \overline{b} \\ b \\ u \end{array}\begin{array}{c} \begin{array}{ccc} \overline{b} & b & u \end{array} \\ \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \end{array}.$$

Finally, we multiply by the term $(-a)_{r+t}$, which generates a partition into distinct parts between 0 and $r+t-1$ and in which the exponent of a counts the number of parts. For each part m in this partition, we add 1 to each of the m largest parts of the overpartition and then add the label a to the $m+1$st part. Note that if all of the parts between 0 and $r+t-1$ occur, then we have an ordinary staircase of size $r+t$. For this reason, such a partition into distinct parts is often called a *generalized staircase*. See [9, 11, 12] for more on the use of generalized staircases. Here, since the parts in the generalized staircase do not go all the way up to the length of the overpartition to which it is added, we call it a *partial generalized staircase*. The largest possible part in this partial generalized staircase is $r+t-1$, while our overpartition has $r+s+t$ parts. Thus there is no label a in the s smallest parts of the resulting overpartition and we obtain an overpartition counted by $A(i,j,n)$. We have the triple sum generating function,

$$\sum_{i,j,n\geq 0} A(i,j,n)a^i b^j q^n = \sum_{r,s,t\geq 0} \frac{q^r}{(q)_r}\frac{b^s q^s}{(q)_s}\frac{b^t q^{t+\binom{t+1}{2}}}{(q)_t}(-a)_{r+t}q^{\binom{r+s+t}{2}}. \tag{2.5}$$

To complete the proof of the theorem, we will show that the triple sum above is equal to the product on the right-hand side of (2.1). To this end, we recall a few well-known q-series identities [10]: the q-Chu-Vandermonde summation

$$\sum_{k=0}^{n} \frac{(a)_k\,(q^{-n})_k\,q^k}{(q)_k(c)_k} = \frac{(c/a)_n a^n}{(c)_n}, \tag{2.6}$$

the q-binomial identity

$$\sum_{n\geq 0} \frac{q^{\binom{n+1}{2}} z^n}{(q)_n} = (-zq)_\infty\,, \tag{2.7}$$

and Lebesgue's identity,

$$\sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}(-z)_n}{(q)_n} = \left(-zq;q^2\right)_\infty(-q)_\infty, \tag{2.8}$$

noting the special case $c=0$ and $a=q^{-m}$ of (2.6),

$$\sum_{k=0}^{n} \frac{(q^{-n})_k\,(q^{-m})_k\,q^k}{(q)_k} = q^{-mn}. \tag{2.9}$$

We also note that

$$(q)_{n-k} = \frac{(q)_n}{(q^{-n})_k}(-1)^k q^{\binom{k}{2}-nk}. \tag{2.10}$$

Now we rewrite the triple sum in (2.5) as

$$\begin{aligned}
\sum_{i,j,n\geq 0} & A(i,j,n)a^i b^j q^n \\
&= \sum_{r,s,t\geq 0} \frac{q^{\binom{r+s+t}{2}+r+s+t+\binom{t+1}{2}}(-a)_{r+t}b^{s+t}}{(q)_r(q)_s(q)_t} \\
&= \sum_{\substack{r,s,t\geq 0\\ t\leq \min\{r,s\}}} \frac{q^{\binom{r+s-t}{2}+r+s+\binom{t}{2}}(-a)_r b^s}{(q)_{r-t}(q)_{s-t}(q)_t} \qquad \text{(by setting } (r,s)\mapsto(r-t,s-t)) \\
&= \sum_{\substack{r,s,t\geq 0\\ t\leq \min\{r,s\}}} \frac{q^{\binom{r+1}{2}+\binom{s+1}{2}+rs+t}\,(q^{-r})_t\,(q^{-s})_t\,(-a)_r b^s}{(q)_r(q)_s(q)_t} \qquad \text{(by (2.10))} \\
&= \sum_{r,s\geq 0} \frac{q^{\binom{r+1}{2}+\binom{s+1}{2}}(-a)_r b^s}{(q)_r(q)_s} \qquad \text{(by (2.9))} \\
&= \frac{(-aq;q^2)_\infty\,(-bq)_\infty}{(q;q^2)_\infty} \qquad \text{(by (2.7) and (2.8))},
\end{aligned}$$

which gives the desired result. □

We may now deduce Thereom 1.

Proof of Theorem 1: With the dilations

$$q \to q^4, \qquad a \to aq, \qquad b \to bq^{-2} \tag{2.11}$$

in Theorem 3, the infinite product becomes

$$(-aq^5;q^8)_\infty(-bq^2;q^4)_\infty(-q^4;q^4)_\infty,$$

while the four-colored integers are transformed by

$$x_u \to 4x_u, \quad x_{au} \to (4x+1)_{au}, \quad x_b \to (4x-2)_b, \quad x_{ab} \to (4x-1)_{ab}.$$

This gives the full set of integers ≥ 2, where only parts which are congruent to 2 or 3 modulo 4 may be overlined. The absence of a $\overline{1}$ before dilating means there is no $\overline{2}$ or $\overline{3}$ after dilating. The condition on the s smallest parts on overpartitions in $\mathcal{O}$ becomes the condition in the statement of the theorem. Finally, the matrix of difference conditions M_A becomes

$$M_{A_1} = \begin{array}{c} \\ \overline{b} \\ b \\ u \\ \overline{ab} \\ ab \\ au \end{array}\begin{array}{c} \begin{array}{cccccc} \overline{b} & b & u & \overline{ab} & ab & au \end{array} \\ \begin{pmatrix} 8 & 8 & 6 & 11 & 11 & 9 \\ 4 & 4 & 6 & 7 & 7 & 9 \\ 6 & 6 & 4 & 9 & 9 & 7 \\ 9 & 9 & 7 & 12 & 12 & 10 \\ 5 & 5 & 7 & 8 & 8 & 10 \\ 7 & 7 & 5 & 10 & 10 & 8 \end{pmatrix} \end{array},$$

and we obtain

$$\sum_{i,j,n\geq 0} A_1(i,j,n)a^i b^j q^n = \left(-aq^5;q^8\right)_\infty \left(-bq^2;q^4\right)_\infty \left(-q^4;q^4\right)_\infty,$$

where $A_1(i,j,n)$ is equal to the number of overpartitions of n counted by $A_1(n)$ having i (resp. j) parts with a (resp. b) in their color. This is a refinement of the statement of Theorem 1 and the proof is complete.

□

To arrive at Theorem 2 we will show that a slightly different set of overpartitions in $\mathcal{O}$ also has the generating function on the right-hand side of (2.1). Namely, let $B(i,j,n)$ denote the number of overpartitions of n in $\mathcal{O}$ with i parts having a in their color and j parts having b in their color, where the kth smallest part is not $\overline{k}$, and such that the minimal difference between adjacent parts is given by the matrix

$$M_B := \begin{array}{c} \\ \overline{b} \\ b \\ u \\ \overline{ab} \\ ab \\ au \end{array}\begin{array}{c} \begin{array}{cccccc} \overline{b} & b & u & \overline{ab} & ab & au \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 2 & 3 & 2 & 3 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 2 & 3 & 2 & 3 \\ 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix} \end{array}.$$

Theorem 4. *We have*

$$\sum_{i,j,n\geq 0} B(i,j,n)a^i b^j q^n = \left(-aq;q^2\right)_\infty (-bq)_\infty (-q)_\infty. \tag{2.12}$$

Proof: The proof is the almost the same as the proof of Theorem 3, except that when we start with parts colored by b and u generated by (2.4) we use the order

$$x_b < \overline{x}_b < x_u \tag{2.13}$$

instead of the order (2.2). This gives the difference conditions

$$\begin{array}{c} \\ \overline{b} \\ b \\ u \end{array}\begin{array}{c} \begin{array}{ccc} \overline{b} & b & u \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

in place of (2.3), and so after adding the staircase and the partial generalized staircase corresponding to $(-a)_{r+t}$ we obtain the difference conditions in M_B above. The only subtle point is that since $\overline{1}_b$ does not occur originally, after adding the staircase we have that the kth smallest part is not $\overline{k}_b$, and after adding the partition corresponding to $(-a)_{r+t}$ the kth smallest part cannot be $\overline{k}_{ab}$, either. $\square$

We may now deduce Theorem 2.

Proof of Theorem 2: With the dilations

$$q \to q^4, \qquad a \to aq^{-1}, \qquad b \to bq^{-2}, \tag{2.14}$$

the infinite product in (2.12) becomes

$$(-aq^3;q^8)_\infty(-bq^2;q^4)_\infty(-q^4;q^4)_\infty,$$

while the four-colored integers are transformed by

$$x_u \to 4x_u, \quad x_{au} \to (4x-1)_{au}, \quad x_b \to (4x-2)_b, \quad x_{ab} \to (4x-3)_{ab}.$$

The gives the full set of integers, where only parts $\equiv 1$ or $2 \pmod 4$ may be overlined. The difference condition matrix M_B becomes

$$M_{B_1} = \begin{array}{c} \\ \overline{b} \\ b \\ u \\ \overline{ab} \\ ab \\ au \end{array} \begin{array}{c} \begin{array}{cccccc} \overline{b} & b & u & \overline{ab} & ab & au \end{array} \\ \begin{pmatrix} 8 & 4 & 6 & 13 & 9 & 11 \\ 8 & 4 & 6 & 13 & 9 & 11 \\ 6 & 6 & 4 & 11 & 11 & 9 \\ 7 & 3 & 5 & 12 & 8 & 10 \\ 7 & 3 & 5 & 12 & 8 & 10 \\ 5 & 5 & 3 & 10 & 10 & 8 \end{pmatrix} \end{array},$$

and we obtain

$$\sum_{i,j,n\geq 0} B_1(i,j,n)a^ib^jq^n = (-aq^3;q^8)_\infty(-bq^2;q^4)_\infty(-q^4;q^4)_\infty,$$

where $B_1(i,j,n)$ is equal to the number of overpartitions counted by $B_1(n)$ having i (resp. j) parts with a (resp. b) in their color. This is a refinement of the statement of Theorem 2 and the proof is complete. $\square$

Before continuing, we make two remarks. First, the condition on the s smallest parts in the definition of the overpartitions in $\mathcal{O}$ is necessary in order for $A(i,j,n)$ and $B(i,j,n)$ to have a nice infinite product generating function. Indeed, if this condition is dropped and we use a complete generalized staircase, then the generating function is the triple sum

$$\sum_{r,s,t\geq 0} \frac{q^{\binom{r+s+t}{2}+r+s+t+\binom{t+1}{2}}(-a)_{r+s+t}b^{s+t}}{(q)_r(q)_s(q)_t},$$

and it is easy to check that this is not even a simple infinite product for $a=b=1$. Second, we used the different dilations (2.11) and (2.14) in Theorems 3 and 4 in order to have a little variety in the infinite products, but we could have also used (2.11) in Theorem 4 or (2.14) in Theorem 3 and obtained similar identities. In fact, there are many dilations which could be used in these theorems, each giving a slightly different result.

3. Further results

In this section we consider different orders among the parts x_u, $\overline{x}_b$ and x_b instead of (2.2) and (2.13) in the proof of Theorems 3 and 4 respectively, so that we obtain similar results to Theorems 1 and 2. We will be brief with the details. First we use the order

$$x_u < x_b < \overline{x}_b$$

instead of (2.2). After adding the staircase and the partial generalized staircase, we have the difference conditions

$$M_D := \begin{array}{c} \\ \overline{b} \\ b \\ u \\ \overline{ab} \\ ab \\ au \end{array} \begin{array}{c} \begin{array}{cccccc} \overline{b} & b & u & \overline{ab} & ab & au \end{array} \\ \begin{pmatrix} 2 & 1 & 1 & 3 & 2 & 2 \\ 2 & 1 & 1 & 3 & 2 & 2 \\ 2 & 2 & 1 & 3 & 3 & 2 \\ 2 & 1 & 1 & 3 & 2 & 2 \\ 2 & 1 & 1 & 3 & 2 & 2 \\ 2 & 2 & 1 & 3 & 3 & 2 \end{pmatrix} \end{array}.$$

The fact that there was orginally no $\overline{1}_b$ means that in the end if the smallest overlined part is the kth smallest part, then it is not $\overline{(k+\ell)}_b$ or $\overline{(k+\ell)}_{ab}$, where ℓ is the number of parts following it which have a in their color. With the dilations

$$q \to q^4, \qquad a \to aq, \qquad b \to bq^2,$$

the difference condition matrix M_D becomes

$$M_{D_1} = \begin{array}{c} \\ \overline{b} \\ b \\ u \\ \overline{ab} \\ ab \\ au \end{array} \begin{array}{c} \begin{array}{cccccc} \overline{b} & b & u & \overline{ab} & ab & au \end{array} \\ \begin{pmatrix} 8 & 4 & 6 & 11 & 7 & 9 \\ 8 & 4 & 6 & 11 & 7 & 9 \\ 6 & 6 & 4 & 9 & 9 & 7 \\ 9 & 5 & 7 & 12 & 8 & 10 \\ 9 & 5 & 7 & 12 & 8 & 10 \\ 7 & 7 & 5 & 10 & 10 & 8 \end{pmatrix} \end{array},$$

while the infinite product $(-aq;q^2)_\infty(-bq)_\infty(-q)_\infty$ becomes

$$\left(-aq^5;q^8\right)_\infty \left(-bq^6;q^4\right)_\infty \left(-q^4;q^4\right)_\infty.$$

Thus we have the following theorem.

Theorem 5. *Let $D_1(n)$ denote the number of overpartitions into parts ≥ 4 satisfying the difference conditions*

$$\lambda_i - \lambda_{i+1} \geq 4 + 2\chi(\lambda_{i+1} \text{ is overlined}) + 3\chi(\lambda_{i+1} \text{ is odd}),$$

where, in addition, (i) only parts $\equiv 2,3 \pmod 4$ may be overlined, (ii) if the smallest overlined part is the kth smallest part, then it is neither $\overline{4(k+\ell)+2}$ nor $\overline{4(k+\ell)+3}$, *where ℓ is the number of odd parts coming after it, and* (iii) *the smallest s parts are even, where s is the number of non-overlined parts* $\equiv 2,3 \pmod 4$*. Then $D_1(n)$ is equal to the number of partitions into distinct parts > 2 which are congruent to* $0, 2, 4, 5,$ *or* 6 *modulo* 8*.*

Now we apply the order

$$x_u < \overline{x}_b < x_b$$

instead of (2.13). After adding the staircase and the partial generalized staircase, we have the difference conditions given by the matrix

$$M_E := \begin{array}{c} \\ \overline{b} \\ b \\ u \\ \overline{ab} \\ ab \\ au \end{array} \begin{array}{c} \begin{array}{cccccc} \overline{b} & b & u & \overline{ab} & ab & au \end{array} \\ \begin{pmatrix} 2 & 2 & 1 & 3 & 3 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 3 & 3 & 2 \\ 2 & 2 & 1 & 3 & 3 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 3 & 3 & 2 \end{pmatrix} \end{array}.$$

With the dilations

$$q \to q^4, \qquad a \to aq^{-1}, \qquad b \to bq^2,$$

the difference condition matrix M_E becomes

$$M_{E_1} = \begin{array}{c} \\ \overline{b} \\ b \\ u \\ \overline{ab} \\ ab \\ au \end{array} \begin{array}{c} \begin{array}{cccccc} \overline{b} & b & u & \overline{ab} & ab & au \end{array} \\ \begin{pmatrix} 8 & 8 & 6 & 13 & 13 & 11 \\ 4 & 4 & 6 & 9 & 9 & 11 \\ 6 & 6 & 4 & 11 & 11 & 9 \\ 7 & 7 & 5 & 12 & 12 & 10 \\ 3 & 3 & 5 & 8 & 8 & 10 \\ 5 & 5 & 3 & 10 & 10 & 8 \end{pmatrix} \end{array},$$

and the infinite product becomes

$$\left(-aq^3; q^8\right)_\infty \left(-bq^6; q^4\right)_\infty \left(-q^4; q^4\right)_\infty .$$

Therefore we have another similar result.

Theorem 6. *Let $E_1(n)$ denote the number of overpartitions into parts ≥ 3 satisfying the difference conditions*

$$\lambda_i - \lambda_{i+1} \geq 3 + 2\chi(\lambda_i \text{ is overlined}) + 5\chi(\lambda_{i+1} \text{ is odd}),$$

where, in addition, (i) *only parts* $\equiv 1, 2 \pmod 4$ *may be overlined,* (ii) *if the smallest overlined part is the kth smallest part, then it is not* $\overline{4(k+\ell)+1}$ *or* $\overline{4(k+\ell)+2}$*, where ℓ is the number of odd parts coming after it, and* (iii) *the smallest s parts are even, where s is the number of non-overlined parts* $\equiv 1, 2 \pmod 4$*. Then $E_1(n)$ is equal to the number of partitions into distinct parts > 2 which are congruent to* $0, 2, 3, 4,$ *or* 6 *modulo* 8*.*

We close by illustrating Theorems 5 and 6 for $n = 29$. There are 13 partitions of 29 into distinct parts congruent to $0, 2, 4, 5$, or 6 modulo 8,

$$(29),\ (24,5),\ (21,8),\ (20,5,4),\ (18,6,5),\ (16,13),\ (16,8,5),\ (14,10,5),$$

$$(14,6,5,4),\ (13,12,4),\ (13,10,6),\ (12,8,5,4),\ (10,8,6,5),$$

as well as 13 overpartitions counted by $D_1(29)$,

$$(29),\ (25,4),\ (24,5),\ (\overline{23},6),\ (21,8),\ (20,9),\ (\overline{19},\overline{10}),\ (\overline{19},10),\ (19,\overline{10}),$$

$$(17,12),\ (17,8,4),\ (16,9,4),\ (15,10,4).$$

Similarly, there are 17 partitions of 29 into distinct parts congruent to $0, 2, 3, 4$, or 6 modulo 8,

$$(26,3),\ (22,4,3),\ (20,6,3),\ (19,10),\ (19,6,4),\ (18,11),\ (18,8,3),\ (16,10,3),$$

$$(16,6,4,3),\ (14,12,3),\ (14,11,4),\ (14,8,4,3),\ (12,11,6),$$

$$(12,10,4,3),\ (12,8,6,3),\ (11,10,8),\ (11,8,6,4),$$

as well as 17 overpartitions counted by $E_1(29)$,

$$(\overline{29}),\ (\overline{26},3),\ (\overline{25},4),\ (25,4),\ (23,6),\ (\overline{22},7),\ (\overline{21},8),\ (21,8),\ (20,\overline{9}),\ (19,\overline{10}),$$

$$(19,10),\ (18,7,4),\ (\overline{17},12),\ (17,12),\ (\overline{17},8,4),\ (17,8,4),\ (15,10,4).$$

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