OVERPARTITIONS INTO DISTINCT PARTS WITHOUT SHORT SEQUENCES

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Abstract. In the first part of this paper we introduce overpartitions into distinct parts without \( k \)-sequences. When \( k = 1 \) these are the partitions into parts differing by at least two which occur in the Rogers-Ramanujan identities. For general \( k \) we compute a three-variable double sum \( q \)-hypergeometric generating function and give asymptotic estimates for the number of such overpartitions. When \( k = 2 \) we obtain several more double sum generating functions as well as a combinatorial identity. In the second part of the paper, we establish arithmetic and combinatorial properties of some related \( q \)-hypergeometric double sums.

1. Introduction

For \( k \geq 2 \), a \( k \)-sequence in a partition \( \lambda \) is a string of \( k \) consecutive integers occurring in \( \lambda \). For example, the partition \( \lambda = (13, 12, 10, 9, 9, 7, 6, 5, 5, 3, 1) \) contains the 3-sequence \((7, 6, 5)\) and four 2-sequences, \((13, 12)\), \((10, 9)\), \((7, 6)\), and \((6, 5)\). Let \( p_k(n) \) denote the number of partitions of \( n \) without \( k \)-sequences. The case \( k = 2 \) was first studied by MacMahon [22, Ch. IV], who proved (among other things) that

\[
\sum_{n \geq 0} p_2(n) q^n = 1 + \sum_{n \geq 1} \frac{(q^6; q^6)_n q^n}{(q^3; q^3)_{n-1}(q^2; q^2)_{n-1}(1 - q^n)}.
\]

(1.1)

Here and throughout we use the standard \( q \)-series notation,

\[
(a)_n := (a; q)_n := \prod_{k=1}^n (1 - a q^{k-1}).
\]

For higher \( k \), partitions without \( k \)-sequences first arose in work of Holroyd, Liggett, and Romik [16] in relation to a certain probabilistic model, which led them to asymptotic estimates for \( \log p_k(n) \). Andrews then made an extensive study of \( p_k(n) \) in his inaugural article for the National Academy of Sciences [5]. He established double sum \( q \)-hypergeometric generating functions and made a conjecture on the asymptotics of \( p_k(n) \), which was eventually proven in full by Kane and Rhoades [18]. In proving his own conjecture in the case \( k = 2 \), Andrews made the important observation that the generating function for \( p_2(n) \) is not modular, but the product of a modular form and a mock theta function. Namely, we have

\[
\sum_{n \geq 0} p_2(n) q^n = \frac{(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \chi(q),
\]

(1.3)

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where \( \chi(q) \) is Ramanujan’s third order mock theta function

\[
\chi(q) := \sum_{n \geq 0} q^{n^2} \frac{(-q)_n}{(-q^2;q^3)_n}.
\] (1.4)

This motivated Bringmann and Mahlburg to extend the Hardy-Ramanujan circle method to handle functions of this type [10]. For some of the many other works inspired by Holroyd, Liggett, Romik, and Andrews, we refer to [7, 8, 9, 11, 12, 13].

In the first part of this paper we consider overpartitions into distinct parts without \( k \)-sequences. These generalize partitions into distinct parts without \( k \)-sequences, which were studied in a recent paper of Bringmann, Mahlburg, and Nataraj [13]. Recall that an overpartition is a partition in which the first occurrence of an integer may be overlined, and let \( Q(n) \) denote the number of overpartitions of \( n \) into distinct parts where successive parts differ by at least 2 if the smaller is overlined. For \( k \geq 1 \), we define a \( k \)-sequence in an overpartition \( \lambda \) counted by \( Q(n) \) to be a string of \( k \) consecutive parts of \( \lambda \), if the largest part of the string is not overlined, and a string of \( k + 1 \) consecutive parts if the largest part is overlined. As an illustration, the overpartition into distinct parts \((23, 21, 20, 17, 15, 14, 13, 11, 9, 8, 6, 4, 3, 2)\) contains the 3-sequence \((4, 3, 2)\) and the 2-sequences \((21, 20), (15, 14, 13), (14, 13), (4, 3)\) and \((3, 2)\). Each non-overlined part forms a 1-sequence, as do the strings \((15, 14), (9, 8)\).

Let \( Q_k(n) \) denote the number of overpartitions counted by \( Q(n) \) without \( k \)-sequences, and let \( Q_k(m, n) \) (resp. \( Q_k(\ell, m, n) \)) denote the number of such overpartitions with \( m \) parts (resp. \( m \) parts, \( \ell \) of which are overlined). Our first result is a double sum generating function for

\[
O_k(a, z, q) := \sum_{\ell, m, n \geq 0} Q_k(\ell, m, n) a^\ell z^m q^n.
\]

Theorem 1.1. For a positive integer \( k \),

\[
O_k(a, z, q) = \sum_{j, r \geq 0} (-1)^j (-a)_r z^{kj+r} q^{(kj+r^2)/2+k(j+1)/2} \frac{(q^k; q^k)_j (q)_r}{(q^k; q^k)_j (q)_r}.
\]

When \( a = 0 \), this is Theorem 1.1 of [13]. When \( k = 1 \), overpartitions into distinct parts without 1-sequences are nothing but partitions where parts differ by at least two, which are the partitions occurring in the Rogers-Ramanujan identities. (See [1] for background on these identities.) Thus we have

\[
O_1(a, z, q) = \sum_{n \geq 0} \frac{a^{n^2} q^{n^2}}{(q)_n},
\] (1.5)

and the Rogers-Ramanujan identities imply that

\[
O_1(1, 1, q) = \sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q, q^4; q^5)_\infty}
\] (1.6)

and

\[
O_1(1, q, q) = \sum_{n \geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}.
\] (1.7)

Here we have extended the notation in (1.2) to

\[
(a_1, a_2, \ldots, a_k; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n.
\] (1.8)

When \( k = 2 \), we shall establish the following generating function.
Theorem 1.2. We have

$$O_2(a, z, q) = \sum_{r,s \geq 0} \frac{a^r z^r q^{r^2 + rs + s^2}}{(q)_r(q)_s}. $$

This is also related to Rogers-Ramanujan type partitions. Namely, if we define $G(z)$ by

$$G(z) := \sum_{n \geq 0} \frac{z^n q^{n^2}}{(q)_n}, $$

then

$$O_2(a, z, q) = \sum_{r \geq 0} \frac{a^r z^r q^{r^2}}{(q)_r} G(zq^r). $$

As a consequence, we obtain a second characterization of the overpartitions counted by $Q_2(\ell, m, n)$.

Corollary 1.3. Let $Q_2(\ell, m, n)$ denote the number of overpartitions of $n$ into $m$ parts, $\ell$ of which are overlined, satisfying

1. Each natural number $< \ell$ appears as a non-overlined part exactly once,
2. There is no non-overlined part of size $\ell$,
3. Non-overlined parts $> \ell$ have gap $\geq 2$.

If $m' = m + \ell - \chi(\ell > 0)$, then

$$Q_2(\ell, m, n) = \overline{Q}_2(\ell, m', n), $$

where $\chi(P)$ is the characteristic function, which is equal to 1 when $P$ is true and 0 if $P$ is false.

For example, $Q_2(2, 4, 14) = 5$ from the partitions $(7, 4, 2, 1), (7, 4, 3, 1), (6, 4, 3, 1), (6, 4, 3, 1)$, and $(6, 3, 3, 2, 1)$. We see that $Q_2(2, 5, 14)$ also equals 5 from the partitions $(7, 3, 2, 1, 1), (6, 4, 2, 1, 1), (6, 3, 3, 1, 1, 1), (5, 3, 3, 2, 1)$, and $(5, 3, 3, 2, 1)$.

While the generating function for $Q_1(n)$ is modular by (1.6), we do not have any information on the modular transformations of the generating function for $Q_2(n)$. Nevertheless we are able to use expressions for $G(z)$ in Ramanujan’s lost notebook to obtain the following elegant expressions involving positive definite binary quadratic forms and Gaussian polynomials, the latter defined by

$$\left[ \begin{array}{c} m \\ n \end{array} \right] = \frac{(q)_m}{(q)_n(q)_{m-n}}, \text{ if } 0 \leq n \leq m, \\
0, \text{ otherwise.}$$

Theorem 1.4. The following identities are true.

$$O_2(1, 1, q) = \frac{1}{(q)^\infty} \sum_{k,n \geq 0} (-1)^n \left[ \begin{array}{c} k \\ n \end{array} \right] q^{k^2 + n(3n-1)/2}(1 - q^{3n+3k+3}) $$

$$= \frac{1}{(q)^\infty} \sum_{k,n \geq 0} (-1)^n \left[ \begin{array}{c} k + n \\ n \end{array} \right] q^{k^2 + n(5n+1)/2+2kn}(1 - q^{4n+2k+2}). $$

Other identities in the literature involving $G(z)$ could be used to give further expressions for $O_2(1, 1, q)$. We record one more straightforward example later in Prop. 3.1, by directly substituting results of Andrews [2] and Garrett, Ismail and Stanton [14] into (1.10).

For $k \geq 2$ we study the asymptotics of $Q_k(n)$, following the methods of [8] and [13]. We first define two functions

$$g_k(u) := -2\pi^2 u^2 + \text{Li}_2(e^{2\pi i u}) - \frac{1}{k} \text{Li}_2(e^{2\pi i ku}) - \text{Li}_2(-e^{2\pi i u})$$
and

\[ h_k(x) := x^{k+2} + x^{k+1} - x^2 - 2x + 1, \]

where \( \text{Li}_2(x) \) is the dilogarithm function defined by

\[ \text{Li}_2(x) = \sum_{n \geq 0} \frac{x^n}{n^2}. \]

By adopting the same argument in [13, Prop 2.1], we can easily see that

1. \( h_k(x) \) has the unique root \( w_k \in (0, 1) \) for \( k \geq 2 \).
2. On the positive real axis, \( g_k(u) \) has the unique critical point at \( v_k \), where \( w_k = e^{2\pi i v_k} \).

Moreover, \( g_k''(v_k) < 0 \).

Using the constant term method and the saddle point method as in [8] and [13], we can prove the following asymptotic formula.

**Theorem 1.5.** Let \( k \geq 2 \). Using the notation above, as \( n \to \infty \),

\[ \bar{Q}_k(n) \sim \frac{\sqrt{\pi} g_k(v_k)^{1/4}}{\sqrt{-g_k''(v_k)}} e^{2\sqrt{g_k(v_k)n}}. \]

In the second part of the paper we leave overpartitions without \( k \)-sequences and turn to double \( q \)-hypergeometric sums resembling the one in Theorem 1.2. We study instances of \( D_\alpha(q) \) and \( E_k(q) \), defined for integers \( \alpha \geq -1 \) and \( k > 0 \) by

\[ D_\alpha(q) := \sum_{n,m \geq 0} \frac{q^{n^2+\alpha nm+m^2}}{(q)_n(q)_m}, \]

and

\[ E_k(z,q) := \sum_{n,m \geq 0} \frac{z^m q^{kn^2-(2k-1)nm+km^2}}{(q)_n(q)_m}. \]

These lead to some new partition functions with interesting combinatorial and arithmetic properties.

To keep the introduction from running on, we postpone a discussion of this until Sections 5 and 6.

The remainder of paper up to Section 5 is organized as follows. In Section 2, we prove the generating function for \( O_k(a,z,q) \) in Theorem 1.1. In Section 3, we focus on the case \( k = 2 \), obtaining the simpler generating function for \( O_2(a,z,q) \) in Theorem 1.2 and its partition theoretic implication in Corollary 1.3, as well as the generating functions for \( O_2(1,1,q) \) in Theorem 1.4. In Section 4, we prove the asymptotic formula for \( Q_k(n) \).

2. **The two-variable generating function**

We begin by proving Theorem 1.1. We define \( \overline{D}_k \) to be the set of overpartitions counted by \( \bar{Q}_k(n) \) for some \( n \). For an overpartition \( \lambda \in \overline{D}_k \), we denote its weight by \( |\lambda| \), the number of its overlined parts by \( o(\lambda) \), and the number of its parts by \( \ell(\lambda) \).

Note that

\[ O_k(a,z,q) = \sum_{\lambda \in \overline{D}_k} a^{o(\lambda)} z^{\ell(\lambda)} q^{|\lambda|}. \]

For a given partition \( \lambda \in \overline{D}_k \), we remove the “staircase” by subtracting \( \ell(\lambda) - 1 \) from the largest part, \( \ell(\lambda) - 2 \) from the second largest part, and so on down to 1 from the second smallest part and 0 from the smallest part. Denote the resulting partition by \( \lambda' \). By the definition of \( \overline{D}_k \), in \( \lambda' \) we have
that each non-overlined part can be repeated at most $k - 1$ times and there could be an overlined part of the same size. Therefore, we see that
\[ \sum_{\lambda \in D_k} a^o(\lambda) z^{\ell(\lambda')} q^{l(\lambda')} = \frac{(-azq)^{\infty}}{(zq)^{\infty}} (z^k q^k; q^k)_\infty. \]

Using the $q$-binomial theorem
\[ \sum_{n \geq 0} \frac{(-a)_n}{(q)_n} (zq)^n = \frac{(-azq)^{\infty}}{(zq)^{\infty}} \]
and the case $(z, q) = (z^k, q^k)$ of the $q$-binomial series
\[ (zq)^{\infty} = \sum_{n \geq 0} (-1)^n \frac{z^n q^{n+1}}{(q)_n}, \] (2.1)
we can express this product as a double sum,
\[ \sum_{\lambda \in D_k} a^o(\lambda) z^{\ell(\lambda')} q^{l(\lambda')} = \sum_{j, r \geq 0} \frac{(-1)^j z^k q^{j(j+1)/2}}{(q)_j} (a^r z^r q^r) \frac{(-a)^r z^r q^r}{(q)_r}. \] From the fact that $\ell(\lambda) = \ell(\lambda')$ and $|\lambda| = |\lambda'| + \ell(\lambda)(\ell(\lambda) - 1)/2$, we deduce Theorem 1.1. \qed

3. The case $k = 2$

In this section we prove the results on $O_2(a, z, q)$ in Theorems 1.2 and 1.4 and Corollary 1.3. We begin with Theorem 1.2.

Proof of Theorem 1.2. From the proof in the previous section we have that
\[ O_k(a, z, q) = [y^0] (-ayz)^{\infty} (yz)^{\infty} \sum_{n \in \mathbb{Z}} y^{-n} q^{n+1/2}. \] (3.1)

Using this with $k = 2$ we have
\[ O_2(a, z, q) = [y^0] (-ayz)^{\infty} (-yz)^{\infty} \sum_{n \in \mathbb{Z}} y^{-n} q^{n+1/2} \]
\[ = [y^0] \sum_{r \geq 0} \frac{a^r y^r z^r q^{r(r+1)}}{(q)_r} \sum_{s \geq 0} \frac{y^s z^s q^{s(s+1)}}{(q)_s} \sum_{n \in \mathbb{Z}} y^{-n} q^{n+1/2} \]
\[ = \sum_{r, s \geq 0} \frac{a^r z^{r+s} q^{r^2+rs+s^2}}{(q)_r (q)_s}. \]

We remark that a similar argument can be used to deduce (1.5) from Theorem 1.1. Namely, using (3.1) with $k = 1$ we have
\[ O_1(a, z, q) = [y^0] (-ayz)^{\infty} \sum_{n \in \mathbb{Z}} y^{-n} q^{n+1/2} \]
\[ = [y^0] \sum_{n \geq 0} \frac{a^n y^n z^n q^{n(n+1)}}{(q)_n} \sum_{n \in \mathbb{Z}} y^{-n} q^{n+1/2} \]
\[ = \sum_{n \geq 0} \frac{a^n z^n q^{2n}}{(q)_n}. \]
Next, we establish Corollary 1.3.

**Proof of Corollary 1.3.** We begin by writing the generating function for $Q_2(\ell,m,n)$ as

$$
O_2(a, z, q) = \sum_{r \geq 0} a^rz^r q^{(r+1)/2} \frac{q^{(r-1)/2} G(z q^r)}{(q)_r}.
$$

(3.2)

Recalling (1.9), we note that $q^{(r-1)/2} G(z q^r)$ is the generating function for partitions $\lambda$ such that

(i) the natural numbers $< r$ appear as parts exactly once, 
(ii) $r$ does not appear as a part, and 
(iii) the parts $> r$ have gap $\geq 2$, where the exponent of $z$ counts the number of parts $> r$. Next, we have that $a^r z^r q^{(r+1)/2}/(q)_r$ is the generating function for partitions $\mu$ into $r$ distinct parts. Overlining the parts of $\mu$ and adding in the parts of $\lambda$ non-overlined, we obtain the overpartitions described in the statement of the theorem. This completes the proof of Corollary 1.3. □

We turn now to the identities in Theorem 1.4.

**Proof of Theorem 1.4.** We recall the following entry from Ramanujan’s lost notebook [6, Entry 4.2.1]:

$$
(aq)_n \sum_{n \geq 0} a^n q^{n^2} (q)_n = \sum_{n \geq 0} (-1)^n(aq)_n (1 - aq^{2n}) a^{2n} q^{n(5n-1)/2}
$$

(3.3)

$$
= \sum_{n \geq 0} (-1)^n(aq)_n (1 - a^2 q^{4n}) a^{2n} q^{n(5n+1)/2}.
$$

(3.4)

Using the case $a = q^j$ of (3.3), we have

$$
O_2(1, 1, q) = \sum_{j \geq 0} \frac{q^{j^2}}{(q)_j} \sum_{n \geq 0} q^{n(n+j)} (q)_n
$$

$$
= \sum_{j \geq 0} \frac{q^{j^2}}{(q)_j (q^{j+1})} \sum_{n \geq 0} \frac{(1 - q^{2n+j})(-1)^n q^{n(5n-1)/2+2nj}(q^n j)_n}{(1 - q^j)(q)_n}
$$

$$
= \frac{1}{(q)_\infty} \sum_{j, n \geq 0} \frac{(1 - q^{2n+j})(-1)^n q^{j^2+2nj}(5n-1)/2+2nj(q^n j)_{n+j-1}}{(q)_j(q)_n}
$$

Since

$$
\frac{(1 - q^{2n+j})(q)_{n+j-1}}{(q)_j(q)_n} = \frac{1 - q^{n+j} q^n (1 - q^n)}{(q)_j(q)_n} = \frac{(q^{n+j} (q)_{n+j-1}}{(q)_j(q)_{n-1}},
$$

we obtain

$$
O_2(1, 1, q)
$$

$$
= \frac{1}{(q)_\infty} \left\{ \sum_{j, n \geq 0} \frac{(-1)^n q^{j^2+n(5n-1)/2+2nnj}(q^n j)_{n+j}}{(q)_j(q)_n} + \sum_{j \geq 0, n \geq 1} \frac{(-1)^n q^{j^2+2n(5n+1)/2+2nj}+3j^3(q^n j)_{n+j-1}}{(q)_j(q)_{n-1}} \right\}
$$

$$
= \frac{1}{(q)_\infty} \left\{ \sum_{j, n \geq 0} \frac{(-1)^n q^{j^2+n(5n-1)/2+2nnj}(q^n j)_{n+j}}{(q)_j(q)_n} - \sum_{j, n \geq 0} \frac{(-1)^n q^{j^2+2n(5n+1)/2+2nj+3j+3}(q^n j)_{n+j}}{(q)_j(q)_n} \right\}.
$$
We have Proposition 3.1, which proves (1.13).

For (1.12), we do not give all of the details. First we define the quantum dilogarithm

\[ \text{Li}_2(x; q) := -\log(x; q) = \sum_{n \geq 1} \frac{x^n}{n(1 - q^n)} \]

which combines a result of Garrett, Ismail, and Stanton [14] and a result of Andrews [2].

Letting \( n + j \to k \) and \( n \to \ell \), we get

\[
O_2(1, 1, q) = \frac{1}{(q)_\infty} \left\{ \sum_{k, \ell \geq 0} \frac{(-1)^{\ell} q^{k^2 + \ell(3\ell - 1)/2}}{(q)_{k-\ell}(q)_\ell} \right\} - \sum_{k, \ell \geq 0} \frac{(-1)^{\ell} q^{k^2 + 3k + 3\ell(3\ell + 5)/2}}{(q)_{k-\ell}(q)_\ell} \left\{ \sum_{k \geq 0} q^{k^2} \sum_{\ell \geq 0} (-1)^{\ell} q^{\ell(3\ell - 1)/2} \right\} \left\{ \sum_{k \geq 0} q^{k^2 + 3k + 3\ell(3\ell + 5)/2} \right\},
\]

which establishes (1.12).

For (1.13), we set \( a = q^j \) in (3.4). This gives

\[
O_2(1, 1, q) = \sum_{j=0}^{\infty} q^{j^2} \frac{q^{n(n+j)}}{(q)_j (q)_{n0}} \left( \sum_{n \geq 0} \frac{(-1)^{n} q^{n+1} (1 - q^{4n+2j+2})}{(q)_n} \right) q^{n(5n+1)/2+2jn} = \frac{1}{(q)_\infty} \sum_{j \geq 0} q^{j^2} \left( \sum_{n \geq 0} (-1)^{n} \binom{j+n}{n} q^{n(5n+1)/2+2jn} (1 - q^{4n+2j+2}) \right),
\]

which proves (1.13).

We end this section by recording a formula to which we alluded in the introduction.

**Proposition 3.1.** We have

\[
O_2(1, 1, q) = \sum_{n \geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(q)_n} \left( \frac{A_n(q)}{(q, q^4; q^5)_\infty} - \frac{B_n(q)}{(q^2, q^3; q^5)_\infty} \right),
\]

where \( A_0 = 1 \), and \( B_0 = 0 \), and for \( n > 0 \)

\[
A_n(q) = \sum_{k \geq 0} q^{k^2 + k \binom{n+k-2}{k}} = \sum_{k \in \mathbb{Z}} (-1)^k q^{k(5k-3)/2} \left\lceil \frac{n-1}{2} \right\rceil_{n-1-5k} ,
\]

\[
B_n(q) = \sum_{k \geq 0} q^{k} \binom{n+k-1}{k} = \sum_{k \in \mathbb{Z}} (-1)^k q^{k(5k+1)/2} \left\lceil \frac{n-1}{2} \right\rceil_{n-1-5k} .
\]

**Proof.** This follows immediately from the fact that

\[
G(q^n) = (-1)^n q^{-n(n-1)/2} \left( \frac{A_n(q)}{(q, q^4; q^5)_\infty} - \frac{B_n(q)}{(q^2, q^3; q^5)_\infty} \right) ,
\]

which combines a result of Garrett, Ismail, and Stanton [14] and a result of Andrews [2].

\[ \square \]

### 4. Asymptotic Formulas

Here we prove asymptotic formula for \( \overline{Q}_k(n) \) for \( k \geq 3 \). As our proof is similar with that of [13, Thm 1.2], we do not give all of the details. First we define the quantum dilogarithm

\[
\text{Li}_2(x; q) := -\log(x; q) = \sum_{n \geq 1} \frac{x^n}{n(1 - q^n)} .
\]
Recall that uniformly in $x$ as $\epsilon \to 0^+$,
\[
\text{Li}_2(x; e^{-\epsilon}) = \frac{1}{\epsilon} \text{Li}_2(x) - \frac{1}{2} \log(1 - x) + O(\epsilon).
\]

**Proof of Theorem 1.5.** We first investigate the asymptotic behavior of $\mathcal{O}_k(1, 1, q) =: \mathcal{O}_k(q)$ with $q = e^{-\epsilon}$ as $\epsilon \to 0^+$. From (3.1), we see that
\[
\mathcal{O}_k(q) = [y^0]q(y^{-1}q^{1/2}; q^{1/2}) \exp \left( -\text{Li}_2(y^k; q^k) - \text{Li}_2(-y; q) + \text{Li}_2(y; q) \right),
\]
where $\theta(y; q) = \sum_{n \in \mathbb{Z}} y^n q^{n^2}$. Thus, by Cauchy’s formula, we derive that
\[
\mathcal{O}_k(q) = \int_{[0,1]+ic} \theta(y^{-1}q^{1/2}; q^{1/2}) \exp \left( -\text{Li}_2(y^k; q^k) - \text{Li}_2(-y; q) + \text{Li}_2(y; q) \right) du,
\]
where $c = \frac{1}{2\pi} \log w_k^{-1}$ is a positive constant and $y = \exp(2\pi i u)$. By employing the transformation formula for the theta function (for example, see [13, eqn. 2.5]), we see that
\[
\mathcal{O}_k(q) = \sqrt{\frac{2\pi}{\epsilon}} \int_{\mathbb{R}+ic} \exp \left( -\frac{2\pi^2}{\epsilon} \left( u - \frac{ic}{2\pi} \right)^2 - \text{Li}_2(y^k; q^k) - \text{Li}_2(-y; q) + \text{Li}_2(y; q) \right) du.
\]

Note that the integrand can be rewritten as
\[
\exp \left( \frac{g_k(u)}{\epsilon} + \pi iu + \frac{1}{2} \log(1 - y^k) + \frac{1}{2} \log(1 + y) - \frac{1}{2} \log(1 - y) + O(\epsilon) \right),
\]
therefore by expanding around $u = v_k$ and replacing $u = v_k + \sqrt{\epsilon}z$, we obtain that the integrand equals
\[
\sqrt{\frac{w_k(1 - w_k^k)(1 + w_k)}{1 - w_k}} \exp \left( \frac{g_k(v_k)}{\epsilon} + \frac{g_k''(v_k)}{2} z^2 + O(\epsilon^{1/2}) \right).
\]

From the definition of $w_k$, we see that $\frac{w_k(1 - w_k^k)(1 + w_k)}{1 - w_k} = 1$, which implies that
\[
\mathcal{O}_k(q) = \frac{2\pi}{\sqrt{-g_k''(v_k)}} \left( 1 + O(\epsilon^{1/2}) \right) \exp \left( \frac{g_k(v_k)}{\epsilon} \right)
\]
by the Gaussian integral evaluation. Since $\overline{Q}_k(n + 1) \geq \overline{Q}_k(n)$ (this is because for a given overpartition counted by $\overline{Q}_k(n)$ we obtain a partition counted by $\overline{Q}_k(n + 1)$ by increasing the largest part by 1), we can apply Ingham’s Tauberian theorem [13, Theorem 4.2] (c.f. [17]), which proves the theorem. \hfill \square

### 5. The functions $D_\alpha(q)$

Recall the definition of $D_\alpha(q)$ in (1.14). In his proof of the Lusztig-Macdonald-Wall conjectures, Andrews [3] established the modularity of $D_{-1}(q)$ by proving the identity
\[
D_{-1}(q) = \frac{(-q; q^2)_\infty^2}{(q; q^2)_\infty^2}.
\]

This identity subsequently played a role in several works [15, 21, 26]. When $\alpha = 0$ we again have a modular function (simply the square of (1.6)), and when $\alpha = 1$ we have the generating function for overpartitions without 2-sequences studied in the first part of this paper. In this section we make some remarks on the next case, $\alpha = 2$. We define $a(n)$ by
\[
\sum_{n\geq0} a(n)q^n = D_2(q) = 1 + 2q + 2q^2 + 2q^3 + 5q^4 + 6q^5 + 9q^6 + 10q^7 + 13q^8 + \cdots. 
\]
Moreover, let \( \mathcal{P}_n \) denote the set of all partitions of \( n \), and for a partition \( \lambda \), let \( d(\lambda) \) denote the size of the Durfee square, \( \lambda_r \) the partition to the right of the Durfee square, and \( h(\lambda) \) the first hook-length, i.e., the largest part plus the number of parts minus 1 (and \(-1\) for the empty partition). Then we have the following:

**Theorem 5.1.** We have

(i) \( a(2n + 1) \equiv 0 \pmod{2} \),

(ii) \( a(n) = \sum_{\lambda \in \mathcal{P}_n, h(\lambda_r) < d(\lambda)} (d(\lambda) - h(\lambda_r)) \).

For example, among 7 partitions of 5, there are 3 partitions with \( d(\lambda) > h(\lambda_r) \). The partition \( 1 + 1 + 1 + 1 + 1 \) has \( d(\lambda) = 1 \) and \( h(\lambda_r) = -1 \), the partition \( 3 + 2 \) has \( d(\lambda) = 2 \) and \( h(\lambda_r) = 1 \), and the partition \( 2 + 2 + 1 \) has \( d(\lambda) = 2 \) and \( h(\lambda_r) = -1 \). Thus we have \( a(5) = 6 \), as expected.

**Proof.** The first part follows easily from the symmetry in the double series definition of \( D_2(q) \). In (1.14), we consider the cases \( m = n \) and \( m \neq n \) separately, which gives

\[
\sum_{n \geq 0} a(n) q^n = \sum_{m,n \geq 0} \frac{q^{n+m}^2}{(q)_n(q)_m} = \sum_{n \geq 0} \frac{q^{4n^2}}{(q^2)_n^2} + 2 \sum_{n \geq n \geq 0} \frac{q^{n+m}^2}{(q)_n(q)_m} \equiv \sum_{n \geq 0} \frac{q^{4n^2}}{(q^2; q^2)_n^2} \pmod{2}.
\]

We note in passing that \( q \)-expansion is the generating function for the number of partitions with gap \( \geq 4 \) and the smallest part \( > 1 \), after replacing \( q^2 \) by \( q \).

For the second part, we begin by writing

\[
\sum_{m,n \geq 0} \frac{q^{n+m}^2}{(q)_m(q)_n} = \sum_{m \geq 0} \frac{q^{m^2}}{(q)_m} \sum_{k=0}^{\lceil m \rceil} \binom{m}{k}.
\]

The right-hand side can be regarded as a weighted count of partitions \( \lambda \) as follows. First recall that if \( P(M, N) \) is the set of partitions of whose Ferrers diagram fits inside an \( M \times N \) rectangle, then it is well-known that \( \binom{m}{k} \) is the generating function for the partitions in \( P(m - k, k) \) (or equivalently \( P(k, m - k) \)). Now, the index of summation \( m = d(\lambda) \) is the size of the Durfee square, the term \( 1/(q)_m \) corresponds to parts under the Durfee square, and the term \( \sum_{k=0}^{\lceil m \rceil} \binom{m}{k} \) corresponds to the partition to the right of the Durfee square, \( \lambda_r \). Observe that \( \lambda_r \) generated by \( \binom{d(\lambda)}{k} \) satisfies \( h(\lambda_r) < d(\lambda) \), and such a \( \lambda_r \) is contained in the set \( P(k, d(\lambda) - k) \) for precisely \( d(\lambda) - h(\lambda_r) \) \( k \)'s.

Therefore, we have proven that

\[
a(n) = \sum_{\lambda \in \mathcal{P}_n, h(\lambda_r) < d(\lambda)} (d(\lambda) - h(\lambda_r)),
\]

as claimed. \( \square \)
The functions $E_k(q)$

Recall the definition of $E_k(z, q)$ in equation (1.15). In the same paper where he established the modularity of $D_{-1}(q)$ in (5.1) (note that $D_{-1}(q) = E_1(1, q)$), Andrews [3] observed (without giving a proof) that the general $E_k(1, q)$ can be expressed as infinite product. In this section we give a short proof of Andrews’ observation (for general $z$) and then study the combinatorics of these series, using the notion of a Rogers-Ramanujan subpartition following ideas of Kolitsch [20] and Kim and Kim [19]. We begin with the infinite product representation.

**Theorem 6.1.** For $k > 0$ we have

$$E_k(z, q) = \frac{1}{(q)_{\infty}} \sum_{n \in \mathbb{Z}} z^n q^{kn^2}$$

$$= \frac{(-zq^k, -q^k/z, q^{2k}; q^{2k})_{\infty}}{(q)_{\infty}}.$$  \hfill (6.2)

**Proof.** The case $k = 1$ was established by the third author [21, p.304], and his simple proof readily generalizes as follows. Employing the $q$-binomial theorem in (2.1) and the Jacobi triple product identity,

$$\sum_{n \in \mathbb{Z}} z^n q^{\frac{n(n+1)}{2}} = (-1/z, -zq)_{\infty},$$

we calculate

$$\sum_{n,m \geq 0} \frac{z^{m-n} q^{kn^2 - (2k-1)nm + km^2}}{(q)_n(q)_m} = [x^0] \sum_{\ell \in \mathbb{Z}} x^{-\ell} q^{(2k-1)\ell} \sum_{m \geq 0} (zq^{-k+1})^m x^{-m} q^{m}_2 (q)_m \sum_{n \geq 0} (zq^{-k+1})^{-n} x^n q^{\frac{n(n+1)}{2}} (q)_n$$

$$= [x^0] \sum_{\ell \in \mathbb{Z}} x^{-\ell} q^{(2k-1)\ell} (-zq^{-k+1}/x)_{\infty} (-xq^k/z)_{\infty}$$

$$= \frac{1}{(q)_{\infty}} [x^0] \sum_{\ell \in \mathbb{Z}} x^{-\ell} q^{(2k-1)\ell} (xq^{k-1}/z)^n q^{\frac{n(n+1)}{2}}$$

$$= \frac{1}{(q)_{\infty}} \sum_{n \in \mathbb{Z}} z^{-n} q^{kn^2}$$

$$= \frac{(-zq^k, -q^k/z, q^{2k}; q^{2k})_{\infty}}{(q)_{\infty}}.$$

\hfill $\blacksquare$

Let $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$ be an ordinary partition. In a recent paper [19], the second author and E. Kim defined a subpartition of an ordinary partition as follows. Let us fix positive integers $d$ and $t$. Then, for a given partition, the subpartition with gap $d$ and tail condition $t$ is defined as the longest sequence satisfying $\alpha_1 > \alpha_2 > \cdots > \alpha_s$ and $\alpha_s > \alpha_{s+1}$, where $\alpha_i - \alpha_j \geq d$ for all $i < j \leq s$ and $\alpha_s - \alpha_{s+1} \geq t$. Here $\alpha_{s+1}$ must be understood to be zero if it comes after the final part. This is a generalization of L. Kolitsch’s Rogers-Ramanujan subpartition [20], which is the case $d = 2$ and $t = 1$. We define the length of the subpartition with gap $d$ as the number of parts in the subpartition.

For $\ell \geq 0$ let $A_k(\ell, n)$ denote the number of partitions of $n$ with subpartitions with gap $2k$ and tail condition $k$, where the length of the subpartition is at least $\ell$. (When $\ell = 0$ then we are simply counting the number of partitions of $n$.) Then, following the combinatorial argument in [19], we have
Proposition 6.2.

\[ \sum_{n \geq 0} \sum_{\ell \in \mathbb{Z}} A_k(|\ell|, n) z^\ell q^n = \frac{1}{(q)_\infty} \sum_{n \in \mathbb{Z}} z^n q^{kn^2}. \]  

(6.4)

We consider the cases \( z = 1 \) and \( -1 \) in (6.4). We define

\[ A_k(n) := \sum_{\ell \geq 1} A_k(\ell, n) = \sum_{\ell \geq 1} \ell p(n, 2k, k, \ell) \]

\[ B_k(n) := \sum_{\ell \geq 1} (-1)^\ell A_k(\ell, n) = - \sum_{\ell \geq 1} \ell \equiv 1 \pmod{2} p(n, 2k, k, \ell) \]

where \( p(n, d, t, \ell) \) be the number of partitions of \( n \) having length \( \ell \) subpartitions with gap \( d \) and tail condition \( t \). For example, since 4 has the subpartition 4, 3 + 1 has the subpartition 3 + 1 of length 2, and 2 + 1 + 1 has the subpartition 2, we have \( A_1(4) = 1 + 2 + 1 = 4 \) and \( B_1(4) = -2 \). Proposition 6.2 together with (6.3) imply that the generating functions for \( A_1(n) \) and \( B_1(n) \) are differences of infinite products:

Corollary 6.3. We have

\[ 2 \sum_{n \geq 1} A_1(n) q^n = \frac{(-q; q^2)^2}{(q; q^2)_\infty} - \frac{1}{(q)_\infty}, \]

\[ 2 \sum_{n \geq 1} B_1(n) q^n = (q; q^2)_\infty - \frac{1}{(q)_\infty}. \]

We close with some simple congruences for \( A_1(n) \). Note that

\[ \sum_{n \geq 1} A_1(n) q^n = \frac{1}{(q)_\infty} \sum_{n \geq 1} q^{n^2}. \]  

(6.5)

In seminal work [4], Andrews considered (among many other things) the number of 2-colored Frobenius partitions, of \( n \), denoted \( c\phi_2(n) \). Its generating function is given by

\[ \sum_{n \geq 0} c\phi_2(n) q^n = \frac{1}{(q)^2 \infty} \sum_{n \in \mathbb{Z}} q^{n^2}, \]  

(6.6)

which resembles (6.5). Motivated by congruence properties for \( c\phi_2(n) \) \[24\], we prove the following.

Proposition 6.4. For all nonnegative integers \( n \),

\[ A_1(25n + 9) \equiv A_1(25n + 14) \equiv A_1(25n + 24) \equiv 0 \pmod{5}. \]

Proof. From the famous partition function congruence \( p(5n + 4) \equiv 0 \pmod{5} \), it suffices to prove \( c(25n + \ell) \equiv 0 \pmod{5} \) for \( \ell = 9, 14, 24 \), where \( c(n) \) is defined by \( \sum_{n \geq 0} c(n) q^n = \frac{1}{(q)_\infty} \theta(q) \). Since \( (q^5; q^5)^5 \equiv (q^{25}, q^{25})_\infty \pmod{5} \), this is equivalent to showing that the coefficients of \( q^{25n+\ell+1} \) of

\[ q^{(q^5, q^5^5)_\infty} \theta(q) \theta(q^{25}) =; F(q) = \sum_{n \geq 1} a_F(n) q^n \]

are divisible by 5. As we are going to use the standard facts from the theory of modular forms [23, 25], we do not give all of the details. It is not hard to see that \( F(q) \) is a holomorphic modular...
form of weight 3 and level 100 for some odd character $\chi$, and thus so is $G := \sum_{n \geq 1} a_F(5n)q^n$. Note that

$$G - G \otimes \left( \frac{\cdot}{5} \right) = \sum_{n \equiv 0 \pmod{5}} a_F(5n)q^n + 2 \sum_{n \equiv 2, 3 \pmod{5}} a_F(5n)q^n$$

is again a modular form of weight 3 and level 100 with the same character $\chi$, and we can prove that $G - G \otimes \left( \frac{\cdot}{5} \right) \equiv 0 \pmod{5}$ by checking the first 45 coefficients [25, Corollary 9.20]. This completes the proof of the proposition. □

We remark that it appears from numerical computation that Proposition 6.4 can be extended to a family of congruences modulo powers of 5. For example, it is likely true that

$$A_1(125n + 74) \equiv A_1(125n + 124) \equiv 0 \pmod{25}$$

and

$$A_1(3125n + 1849) \equiv A_1(3125n + 3099) \equiv 0 \pmod{125}.$$  

Proving such a family of congruences would amount to proving a family of congruences mod powers of 5 for $\theta(q)/(q)_{\infty}$, as in Paule and Radu’s proof of congruences for $c\phi_2(n)$. We leave this open.

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REFERENCES


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