BAILEY PAIRS AND INDEFINITE QUADRATIC FORMS

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Abstract. We construct classes of Bailey pairs where the exponent of q in α_n is an indefinite quadratic form. As an application we obtain families of q-hypergeometric mock theta multisums.

1. Introduction

Recall the q-series notation

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \tag{1.1}$$

valid for $n \in \mathbb{N} \cup \{\infty\}$. A *Bailey pair relative to a* is a pair of sequences $(\alpha_n, \beta_n)_{n\geq 0}$ satisfying

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}}. \tag{1.2}$$

Bailey pairs where the exponent of q in α_n is an indefinite quadratic form play a key role in studies of q-series and their connections to real quadratic fields [7, 11, 14, 19, 20] and mock theta functions [3, 6, 8, 9, 12, 13, 21, 22]. While there are many such studies, they make use of a relatively small number of Bailey pairs. These tend to be either taken from lists of Andrews and Hickerson [3, 8] or produced *ad hoc* as needed. With this in mind, we construct Bailey pairs featuring four classes of indefinite quadratic forms,

$$(K+1)n^2 + (m+1)n - ((2k+1)j^2 + (2\ell+1)j)/2, \tag{1.3}$$

$$(K+1)n^2 + (m+1)n - kj^2 - \ell j, \tag{1.4}$$

$$((2K+1)n^2 + (2m+1)n)/2 - ((2k+1)j^2 + (2\ell+1)j)/2, \tag{1.5}$$

and

$$((2K+1)n^2 + (2m+1)n)/2 - kj^2 - \ell j, \tag{1.6}$$

where $k, K, m,$ and ℓ are intgers with $k, K \geq 1$, $0 \leq m < K$ and $0 \leq \ell < k$. As an application we obtain families of q-hypergeometric mock theta functions.

We state five sets of Bailey pairs. The first four correspond to the four indefinite quadratic forms in (1.3)-(1.6), and the fifth to (1.6) with $\ell = 0$. These will be proven in Section 3 by combining the notion of a dual Bailey pair with two Bailey lattices. The necessary Bailey machinery is reviewed in Section 2.

Theorem 1.1. *Suppose that $k, K \geq 1$, $0 \leq \ell < k$ and $0 \leq m < K$.*

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(i) *The sequences* $(\alpha_n^{(k,K,\ell)}, \beta_n^{(k,K,\ell)})$ *form a Bailey pair relative to* q, *where*

$$\alpha_n^{(k,K,\ell)} = \frac{q^{(K+1)n^2+Kn}(1-q^{2n+1})}{(1-q)} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \tag{1.7}$$

and

$$\beta_n^{(k,K,\ell)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{K-1} n_{k+i}(n_{k+i}+1)+\binom{n_k+1}{2}-\sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i}(-1)^{n_k}}{(q)_{n-n_{k+K-1}}(q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1}(q)_{n_1}}. \tag{1.8}$$

(ii) *The sequences* $(\alpha_n^{(k,K,\ell,m)}, \beta_n^{(k,K,\ell,m)})$ *form a Bailey pair relative to* 1, *where*

$$\begin{aligned} \alpha_n^{(k,K,\ell,m)} &= q^{(K+1)n^2+(m+1)n} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \\ &\quad - \chi(n \neq 0) q^{(K+1)n^2-(m+1)n} \sum_{j=-n+1}^{n-1} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \end{aligned} \tag{1.9}$$

and

$$\beta_n^{(k,K,\ell,m)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{K-1} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i}(-1)^{n_k}}{(q)_{n-n_{k+K-1}}(q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1}(q)_{n_1}}. \tag{1.10}$$

Theorem 1.2. *Suppose that* $k, K \geq 1$, $0 \leq \ell < k$ *and* $0 \leq m < K$.

(i) *The sequences* $(\alpha_n^{(k,K,\ell)}, \beta_n^{(k,K,\ell)})$ *form a Bailey pair relative to* q, *where*

$$\alpha_n^{(k,K,\ell)} = \frac{q^{(K+1)n^2+Kn}(1-q^{2n+1})}{(1-q)} \sum_{j=-n}^{n} (-1)^j q^{-kj^2-\ell j} \tag{1.11}$$

and

$$\beta_n^{(k,K,\ell)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{K-1} n_{k+i}(n_{k+i}+1)+\binom{n_k+1}{2}-\sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i + \binom{n_1+1}{2}}(-1)^{n_k}}{(q)_{n-n_{k+K-1}}(q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1}(q^2;q^2)_{n_1}}. \tag{1.12}$$

(ii) *The sequences* $(\alpha_n^{(k,K,\ell,m)}, \beta_n^{(k,K,\ell,m)})$ *form a Bailey pair relative to* 1, *where*

$$\begin{aligned} \alpha_n^{(k,K,\ell,m)} &= q^{(K+1)n^2+(m+1)n} \sum_{j=-n}^{n} (-1)^j q^{-kj^2-\ell j} \\ &\quad - \chi(n \neq 0) q^{(K+1)n^2-(m+1)n} \sum_{j=-n+1}^{n-1} (-1)^j q^{-kj^2-\ell j} \end{aligned} \tag{1.13}$$

and

$$\beta_n^{(k,K,\ell,m)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{K-1} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i + \binom{n_1+1}{2}} (-1)^{n_k}}{(q)_{n-n_{k+K-1}} (q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1} (q^2;q^2)_{n_1}}. \tag{1.14}$$

Theorem 1.3. *Suppose that $k, K \geq 1$, $0 \leq \ell < k$ and $0 \leq m < K$.*

(i) *The sequences $(\alpha_n^{(k,K,\ell)}, \beta_n^{(k,K,\ell)})$ form a Bailey pair relative to q, where*

$$\alpha_n^{(k,K,\ell)} = \frac{q^{((2K+1)n^2+(2K-1)n)/2}(1-q^{2n+1})}{(1-q)} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \tag{1.15}$$

and

$$\beta_n^{(k,K,\ell)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{K-1} n_{k+i}(n_{k+i}+1) - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i} (-1)^{n_k} (-q)_{n_k}}{(q)_{n-n_{k+K-1}} (q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1} (q)_{n_1} (-q)_{n_{k+1}}}. \tag{1.16}$$

(ii) *The sequences $(\alpha_n^{(k,K,\ell,m)}, \beta_n^{(k,K,\ell,m)})$ form a Bailey pair relative to 1, where*

$$\begin{aligned} \alpha_n^{(k,K,\ell,m)} &= q^{((2K+1)n^2+(2m+1)n)/2} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \\ &\quad - \chi(n \neq 0) q^{((2K+1)n^2-(2m+1)n)/2} \sum_{j=-n+1}^{n-1} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \end{aligned} \tag{1.17}$$

and

$$\beta_n^{(k,K,\ell,m)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{K-1} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i} (-1)^{n_k} (-q)_{n_k}}{(q)_{n-n_{k+K-1}} (q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1} (q)_{n_1} (-q)_{n_{k+1}}}. \tag{1.18}$$

Theorem 1.4. *Suppose that $k, K \geq 1$, $0 \leq \ell < k$ and $0 \leq m < K$.*

(i) *The sequences $(\alpha_n^{(k,K,\ell)}, \beta_n^{(k,K,\ell)})$ form a Bailey pair relative to q, where*

$$\alpha_n^{(k,K,\ell)} = \frac{q^{((2K+1)n^2+(2K-1)n)/2}(1-q^{2n+1})}{(1-q)} \sum_{j=-n}^{n} (-1)^j q^{-kj^2-\ell j} \tag{1.19}$$

and

$$\beta_n^{(k,K,\ell)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{K-1} n_{k+i}(n_{k+i}+1) - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i + \binom{n_1+1}{2}} (-1)^{n_k} (-q)_{n_k}}{(q)_{n-n_{k+K-1}} (q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1} (q^2;q^2)_{n_1} (-q)_{n_{k+1}}}. \tag{1.20}$$

(ii) The sequences $(\alpha_n^{(k,K,\ell,m)}, \beta_n^{(k,K,\ell,m)})$ *form a Bailey pair relative to* 1*, where*

$$\begin{aligned} \alpha_n^{(k,K,\ell,m)} &= q^{((2K+1)n^2+(2m+1)n)/2} \sum_{j=-n}^{n} (-1)^j q^{-kj^2-\ell j} \\ &\quad - \chi(n \neq 0) q^{((2K+1)n^2-(2m+1)n)/2} \sum_{j=-n+1}^{n-1} (-1)^j q^{-kj^2-\ell j} \end{aligned} \tag{1.21}$$

and

$$\beta_n^{(k,K,\ell,m)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{K-1} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i + \binom{n_1+1}{2}} (-1)^{n_k} (-q)_{n_k}}{(q)_{n-n_{k+K-1}} (q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1} (q^2;q^2)_{n_1} (-q)_{n_{k+1}}}. \tag{1.22}$$

Theorem 1.5. *Suppose that* $k, K \geq 1$ *and* $0 \leq m < K$*.*

(i) The sequences $(\alpha_n^{(k,K)}, \beta_n^{(k,K)})$ *form a Bailey pair relative to* q*, where*

$$\alpha_n^{(k,K)} = \frac{q^{((2K+1)n^2+(2K-1)n)/2}(1-q^{2n+1})}{(1-q)} \sum_{j=-n}^{n} (-1)^j q^{-kj^2} \tag{1.23}$$

and

$$\beta_n^{(k,K)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{2q^{\sum_{i=1}^{K-1} n_{k+i}(n_{k+i}+1) + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1}} (-1)^{n_k}}{(1+q^{n_k})(q)_{n-n_{k+K-1}} (q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}}. \tag{1.24}$$

(ii) The sequences $(\alpha_n^{(k,K,m)}, \beta_n^{(k,K,m)})$ *form a Bailey pair relative to* 1*, where*

$$\begin{aligned} \alpha_n^{(k,K,m)} &= q^{((2K+1)n^2+(2m+1)n)/2} \sum_{j=-n}^{n} (-1)^j q^{-kj^2} \\ &\quad - \chi(n \neq 0) q^{((2K+1)n^2-(2m+1)n)/2} \sum_{j=-n+1}^{n-1} (-1)^j q^{-kj^2} \end{aligned} \tag{1.25}$$

and

$$\beta_n^{(k,K,m)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{2q^{\sum_{i=1}^{K-1} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1}} (-1)^{n_k}}{(1+q^{n_k})(q)_{n-n_{k+K-1}} (q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}}. \tag{1.26}$$

Recall that if (α_n, β_n) is a Bailey pair relative to a, then [4]

$$\sum_{n \geq 0} (\rho_1)_n (\rho_2)_n (aq/\rho_1\rho_2)^n \beta_n = \frac{(aq/\rho_1)_\infty (aq/\rho_2)_\infty}{(aq)_\infty (aq/\rho_1\rho_2)_\infty} \sum_{n \geq 0} \frac{(\rho_1)_n (\rho_2)_n (aq/\rho_1\rho_2)^n}{(aq/\rho_1)_n (aq/\rho_2)_n} \alpha_n. \tag{1.27}$$

Using the Bailey pairs from Theorems 1.1 - 1.5 in specializations of (1.27) will give many identities between q-hypergeometric multisums and indefinite theta series $f_{a,b,c}(x,y,q)$, where

$$f_{a,b,c}(x,y,q) := \left(\sum_{r,s \geq 0} - \sum_{r,s < 0} \right) (-1)^{r+s} x^r y^s q^{a\binom{r}{2} + brs + c\binom{s}{2}}. \tag{1.28}$$

We limit ourselves to a few examples, where the multisums are *mock theta functions*. These identities are proven in Section 4, and the connection between indefinite theta series and mock theta functions is discusse in Section 5.

Theorem 1.6. *We have the following identities.*

(1) *For $k \geq 1$, $0 \leq \ell < k$, and $0 \leq m \leq k$,*

$$\sum_{n_{2k} \geq n_{2k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{k} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i} (-1)^{n_k}}{(q)_{n_{2k}-n_{2k-1}} (q)_{n_{2k-1}-n_{2k-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}} = \frac{1}{(q)_\infty} \Big(f_{3,4k+5,3}(q^{m+2-\ell}, q^{m+3+\ell}, q) + q^{k+m+3} f_{3,4k+5,3}(q^{2k+m+6-\ell}, q^{2k+m+7+\ell}, q) \Big). \tag{1.29}$$

(2) *For $k \geq 1$, $0 \leq \ell < k$, and $0 \leq m \leq k$,*

$$\begin{aligned} \sum_{n_{2k} \geq n_{2k-1} \geq \cdots \geq n_1 \geq 0} & \frac{(q;q^2)_{n_{2k}} q^{n_{2k}^2 + 2\sum_{i=1}^{k-1} n_{k+i}^2 + 2\sum_{i=1}^{m} n_{k+i} + n_k(n_k+1) - 2\sum_{i=1}^{k-1} n_i n_{i+1} - 2\sum_{i=1}^{\ell} n_i} (-1)^{n_k+n_{2k}}}{(q^2;q^2)_{n_{2k}-n_{2k-1}} (q^2;q^2)_{n_{2k-1}-n_{2k-2}} \cdots (q^2;q^2)_{n_2-n_1} (q^2;q^2)_{n_1}} \\ &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \Big(f_{1,2k+2,1}(-q^{3-2\ell+2m}, -q^{5+2\ell+2m}, q^4) - q^{2k+5+2m} f_{1,2k+2,1}(-q^{4k+9-2\ell+2m}, -q^{4k+11+2\ell+2m}, q^4) \Big) \\ &= \frac{(q;q^2)_\infty}{2(q^2;q^2)_\infty} \Big(f_{1,2k+2,1}(-q^{m-\ell+1}, q^{m+\ell+2}, q) + f_{1,2k+2,1}(q^{m-\ell+1}, -q^{m+\ell+2}, q) \Big). \end{aligned} \tag{1.30}$$

(3) *For $k \geq 1$, $0 \leq \ell < k$, and $0 \leq m \leq k$,*

$$\sum_{n_{2k} \geq n_{2k-1} \geq \cdots \geq n_1 \geq 0} \frac{(q;q^2)_{n_{2k}} q^{\sum_{i=1}^{k-1} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i} (-1)^{n_{2k}+n_k}}{(q)_{n_{2k}-n_{2k-1}} (q)_{n_{2k-1}-n_{2k-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}} = \frac{(q;q^2)_\infty}{2(q^2;q^2)_\infty} \Big(f_{1,4k+3,1}(-q^{m+1-\ell}, -q^{m+2+\ell}, q) - q^{k+2+m} f_{1,4k+3,1}(-q^{2k+3-\ell+m}, -q^{2k+4+\ell+m}, q) \Big). \tag{1.31}$$

(4) *For $k \geq 1$ and $0 \leq \ell < k$,*

$$\sum_{n_{2k} \geq n_{2k-1} \geq \cdots \geq n_1 \geq 0} \frac{(-q)_{n_{2k}} q^{\binom{n_{2k}+1}{2} + \sum_{i=1}^{k-1} n_{k+i}(n_{k+i}+1) + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i} (-1)^{n_k}}{(q)_{n_{2k}-n_{2k-1}} (q)_{n_{2k-1}-n_{2k-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}} = \frac{(-q)_\infty}{(q)_\infty} \left(f_{1,2k+2,1}(q^{k+1-\ell}, q^{k+2+\ell}, q^2) + q^{2k+2} f_{1,2k+2,1}(q^{3k+4-\ell}, q^{3k+5+\ell}, q^2) \right). \tag{1.32}$$

(5) *For $k \geq 1$ and $0 \leq m \leq k$,*

$$\sum_{n_{2k} \geq n_{2k-1} \geq \cdots \geq n_1 \geq 0} \frac{2q^{\sum_{i=1}^{k} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1}} (-1)^{n_k}}{(1+q^{n_k})(q)_{n_{2k}-n_{2k-1}}(q)_{n_{2k-1}-n_{2k-2}} \cdots (q)_{n_2-n_1}(q)_{n_1}}$$
$$= \frac{1}{(q)_\infty}\Big(f_{3,4k+3,3}(q^{m+2}, q^{m+2}, q) + q^{k+m+2} f_{3,4k+3,3}(q^{2k+m+5}, q^{2k+m+5}, q)\Big)$$
$$= \frac{1}{(q)_\infty} f_{3,4k+3,3}(q^{(4m+5)/8}, -q^{(4m+5)/8}, -q^{1/4}). \tag{1.33}$$

(6) *For $k \geq 1$ and $0 \leq m \leq k$,*

$$\sum_{n_{2k} \geq n_{2k-1} \geq \cdots \geq n_1 \geq 0} \frac{2(q;q^2)_{n_{2k}} q^{n_{2k}^2 + 2\sum_{i=1}^{k-1} n_{k+i}^2 + 2\sum_{i=1}^{m} n_{k+i} + n_k(n_k+1) - 2\sum_{i=1}^{k-1} n_i n_{i+1}} (-1)^{n_k+n_{2k}}}{(1+q^{2n_k})(q^2;q^2)_{n_{2k}-n_{2k-1}}(q^2;q^2)_{n_{2k-1}-n_{2k-2}} \cdots (q^2;q^2)_{n_2-n_1}(q^2;q^2)_{n_1}}$$
$$= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\Big(f_{1,2k+1,1}(-q^{2m+3}, -q^{2m+3}, q^4) - q^{2k+2m+3} f_{1,2k+1,1}(-q^{4k+2m+7}, -q^{4k+2m+7}, q^4)\Big)$$
$$= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} f_{1,2k+1,1}(-q^{m+1}, q^{m+1}, q). \tag{1.34}$$

(7) *For $k \geq 1$ and $0 \leq m \leq k$,*

$$\sum_{n_{2k} \geq n_{2k-1} \geq \cdots \geq n_1 \geq 0} \frac{2(q;q^2)_{n_{2k}} q^{\sum_{i=1}^{k-1} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1}} (-1)^{n_k+n_{2k}}}{(1+q^{n_k})(q)_{n_{2k}-n_{2k-1}}(q)_{n_{2k-1}-n_{2k-2}} \cdots (q)_{n_2-n_1}(q)_{n_1}}$$
$$= \frac{(q;q^2)_\infty}{2(q^2;q^2)_\infty}\Big(f_{1,4k+1,1}(-q^{m+1}, -q^{m+1}, q) - q^{k+m+1} f_{1,4k+1,1}(-q^{2k+m+2}, -q^{2k+m+2}, q)\Big) \tag{1.35}$$
$$= \frac{(q;q^2)_\infty}{2(q^2;q^2)_\infty} f_{1,4k+1,1}(q^{(4m+3)/8}, -q^{(4m+3)/8}, q^{1/4}).$$

(8) *For $k \geq 1$,*

$$\sum_{n_{2k} \geq n_{2k-1} \geq \cdots \geq n_1 \geq 0} \frac{2(-q)_{n_{2k}} q^{\binom{n_{2k}+1}{2} + \sum_{i=1}^{k-1} n_{k+i}(n_{k+i}+1) + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1}} (-1)^{n_k}}{(1+q^{n_k})(q)_{n_{2k}-n_{2k-1}}(q)_{n_{2k-1}-n_{2k-2}} \cdots (q)_{n_2-n_1}(q)_{n_1}}$$
$$= \frac{(-q)_\infty}{(q)_\infty}\Big(f_{1,2k+1,1}(q^{k+1}, q^{k+1}, q^2) + q^{2k+1} f_{1,2k+1,1}(q^{3k+3}, q^{3k+3}, q^2)\Big) \tag{1.36}$$
$$= \frac{(-q)_\infty}{(q)_\infty}\Big(f_{1,2k+1,1}(q^{(2k+1)/4}, -q^{(2k+1)/4}, -q^{1/2})\Big).$$

We make two remarks before continuing. First, if $m < k$ in (1.31) or (1.35) then the covergence of the multisum is an issue. This is rectified by defining the sum to be the average of the limits of the even and the odd partial sums (with respect to n_{2k}).

Second, there is further simplification of the $f_{a,b,c}(x,y,q)$ in some special cases. For example, if $m = k$ in (1.31) we have that for $k \geq 1$ and $0 \leq \ell < k$,

$$\begin{aligned}\frac{(q;q^2)_\infty}{2(q^2;q^2)_\infty}\Big(f_{1,4k+3,1}(-q^{k+1-\ell},-q^{k+2+\ell},q) - q^{2k+2}f_{1,4k+3,1}(-q^{3k+3-\ell},-q^{3k+4+\ell},q)\Big)\\ = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}f_{1,4k+3,1}(-q^{k+1-\ell},-q^{k+2+\ell},q).\end{aligned} \tag{1.37}$$

Similarly, if $m = k$ in (1.35) we have that for $k \geq 1$,

$$\begin{aligned}\frac{(q;q^2)_\infty}{2(q^2;q^2)_\infty}\Big(f_{1,4k+1,1}(-q^{k+1},-q^{k+1},q) - q^{2k+1}f_{1,4k+1,1}(-q^{3k+2},-q^{3k+2},q)\Big)\\ = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}f_{1,4k+1,1}(-q^{k+1},-q^{k+1},q).\end{aligned} \tag{1.38}$$

Equations (1.37) and (1.38) follow from the fact that [17, Prop. 5.1]

$$f_{a,b,c}(x,y,q) = -\frac{q^{a+b+c}}{xy}f_{a,b,c}(q^{2a+b}/x,q^{2c+b}/y,q). \tag{1.39}$$

2. The Bailey machinery

In this section we collect results on Bailey pairs needed for the proof of Theorems 1.1 - 1.5. For more background and further details, see [1, 2, 4, 5, 10, 18, 25]. The first result is the Bailey lemma.

Lemma 2.1. [2, p. 270] *If (α_n, β_n) is a Bailey pair relative to a, then so is (α_n', β_n'), where*

$$\alpha_n' = \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1)_n(aq/\rho_2)_n}\alpha_n \tag{2.1}$$

and

$$\beta_n' = \sum_{k=0}^{n}\frac{(\rho_1)_k(\rho_2)_k(aq/\rho_1\rho_2)_{n-k}(aq/\rho_1\rho_2)^k}{(aq/\rho_1)_n(aq/\rho_2)_n(q)_{n-k}}\beta_k. \tag{2.2}$$

Iterating Lemma 2.1 produces a sequence of Bailey pairs relative to a called the Bailey chain. Combining the Bailey chain with the following result, which changes a to a/q, gives a Bailey lattice.

Lemma 2.2. [1, p. 59] *If (α_n, β_n) is a Bailey pair relative to a, then (α_n', β_n') is a Bailey pair relative to a/q, where $\alpha_0' = \alpha_0$ and for $n \geq 1$,*

$$\alpha_n' = (1-a)\left(\frac{a}{\rho\sigma}\right)^n\frac{(\rho)_n(\sigma)_n}{(a/\rho)_n(a/\sigma)_n}\left(\frac{\alpha_n}{1-aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}}\right) \tag{2.3}$$

and

$$\beta_n' = \sum_{k=0}^{n}\frac{(\rho)_k(\sigma)_k(a/\rho\sigma)_{n-k}(a/\rho\sigma)^k}{(a/\rho)_n(a/\sigma)_n(q)_{n-k}}\beta_k \tag{2.4}$$

We shall also make use of the so-called Bailey lattice replacement.

Lemma 2.3. [10, Prop 4.1] *If (α_n, β_n) is a Bailey pair relative to 1, where*

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{An^2}(q^{(A-1)n} + q^{-(A-1)n}), & \text{otherwise,} \end{cases} \tag{2.5}$$

then so is (α'_n, β'_n), where

$$\alpha'_n = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{An^2}(q^{An} + q^{-An}), & \text{otherwise,} \end{cases} \tag{2.6}$$

and

$$\beta'_n = q^n \beta_n. \tag{2.7}$$

The following gives a different Bailey lattice.

Lemma 2.4. [18, p. 1510] *If (α_n, β_n) is a Bailey pair relative to a, then (α'_n, β'_n) is a Bailey pair relative to aq, where*

$$\alpha'_n = \frac{(1-aq^{2n+1})(aq/b)_n(-b)^n q^{n(n-1)/2}}{(1-aq)(bq)_n} \sum_{r=0}^{n} \frac{(b)_r}{(aq/b)_r}(-b)^{-r} q^{-r(r-1)/2}\alpha_r \tag{2.8}$$

and

$$\beta'_n = \frac{(b)_n}{(bq)_n}\beta_n. \tag{2.9}$$

Finally we recall the dual Bailey pair.

Lemma 2.5. [2, pp. 278-279] *If $(\alpha_n, \beta_n) = (\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair relative to a, then so is (α'_n, β'_n), where*

$$\alpha'_n = a^n q^{n^2} \alpha_n(a^{-1}, q^{-1}) \tag{2.10}$$

and

$$\beta'_n = a^{-n} q^{-n^2-n} \beta_n(a^{-1}, q^{-1}). \tag{2.11}$$

3. Proof of Theorems 1.1 - 1.5

We are now prepared to prove the main theorems.

Proof of Theorem 1.1.
Step 1. Begin with the "unit" Bailey pair relative to 1 [2, p. 272],

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{\binom{n}{2}}(1+q^n), & \text{if } n > 0, \end{cases} \tag{3.1}$$

and

$$\beta_n = \beta_{n_0} = \chi(n = 0). \tag{3.2}$$

Step 2. Note that if $\rho_1, \rho_2 \to \infty$ in Lemma 2.1 we have

$$\alpha'_n = a^n q^{n^2} \alpha_n \tag{3.3}$$

and

$$\beta'_n = \sum_{k=0}^{n} \frac{a^k q^{k^2}}{(q)_{n-k}} \beta_k. \tag{3.4}$$

Alternately apply this (with $a = 1$) and Lemma 2.3 ℓ times. The result is

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n \left(q^{((2\ell+1)n^2+(2\ell+1)n)/2} + q^{((2\ell+1)n^2-(2\ell+1)n)/2} \right), & \text{if } n > 0, \end{cases} \tag{3.5}$$

and

$$\beta_n = \beta_{n_\ell} = \sum_{n_\ell \geq n_{\ell-1} \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{\ell-1} n_i^2 + \sum_{i=1}^{\ell} n_i}}{(q)_{n_\ell - n_{\ell-1}} (q)_{n_{\ell-1}-n_{\ell-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}}. \tag{3.6}$$

Step 3. Apply (3.3) and (3.4) (with $a = 1$) $k - \ell$ times. This gives

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n \left(q^{((2k+1)n^2+(2\ell+1)n)/2} + q^{((2k+1)n^2-(2\ell+1)n)/2} \right), & \text{if } n > 0, \end{cases} \tag{3.7}$$

and

$$\beta_n = \beta_{n_k} = \sum_{n_k \geq n_{k-1} \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{k-1} n_i^2 + \sum_{i=1}^{\ell} n_i}}{(q)_{n_k - n_{k-1}} (q)_{n_{k-1}-n_{k-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}}. \tag{3.8}$$

Step 4. Compute the dual of the above Bailey pair using Lemma 2.5. Use the fact that $(q^{-1}; q^{-1})_n q^{\binom{n+1}{2}} (-1)^n = (q)_n$ to find

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n \left(q^{(-(2k-1)n^2-(2\ell+1)n)/2} + q^{(-(2k-1)n^2+(2\ell+1)n)/2} \right), & \text{if } n > 0, \end{cases} \tag{3.9}$$

and

$$\beta_n = \beta_{n_k} = (-1)^{n_k} q^{-\binom{n_k+1}{2}} \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{-\sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i}}{(q)_{n_k - n_{k-1}} \cdots (q)_{n_2-n_1} (q)_{n_1}}. \tag{3.10}$$

Step 5. Note that Lemma 2.4 with $a = 1$ and $b \to 0$ gives a Bailey pair relative to q,

$$\alpha'_n = \frac{q^{n^2}(1 - q^{2n+1})}{1 - q} \sum_{j=0}^{n} q^{-j^2} \alpha_j \tag{3.11}$$

and

$$\beta'_n = \beta_n. \tag{3.12}$$

Apply this to the result of Step 4 to obtain the Bailey pair relative to q,

$$\alpha_n = \frac{q^{n^2}(1 - q^{2n+1})}{1 - q} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \tag{3.13}$$

and

$$\beta_n = \beta_{n_k} = (-1)^{n_k} q^{-\binom{n_k+1}{2}} \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{-\sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i}}{(q)_{n_k - n_{k-1}} \cdots (q)_{n_2-n_1} (q)_{n_1}}. \tag{3.14}$$

Step 6. Apply (3.3) and (3.4) (with $a = q$) $m + 1$ times. The result is

$$\alpha_n = \frac{q^{(m+2)n^2+(m+1)n}(1 - q^{2n+1})}{1 - q} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \tag{3.15}$$

and

$$\beta_n = \beta_{n_{k+m+1}} = \sum_{n_{k+m+1} \geq n_{k+m} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{m} n_{k+i}^2 + n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i} (-1)^{n_k}}{(q)_{n_{k+m+1}-n_{k+m}} (q)_{n_{k+m}-n_{k+m-1}} \cdots (q)_{n_2-n_1} (q)_{n_1}}. \tag{3.16}$$

If $m = K-1$ then this gives part (i) of Theorem 1.1.
Step 7. Note that if $a = q$ and $\rho = \sigma = \sqrt{q}$ in Lemma 2.2 then we have $\alpha_0' = \alpha_0$ and for $n \geq 1$,

$$\alpha_n' = \frac{1-q}{1-q^{2n+1}} \alpha_n - \frac{q^{2n-1}(1-q)}{1-q^{2n-1}} \alpha_{n-1} \tag{3.17}$$

and

$$\beta_n' = \beta_n. \tag{3.18}$$

Apply this to the pair obtained in Step 6 to obtain the Bailey pair relative to 1,

$$\begin{aligned} \alpha_n = {} & q^{(m+2)n^2+(m+1)n} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \\ & - \chi(n \neq 0) q^{(m+2)n^2-(m+1)n} \sum_{j=-n+1}^{n-1} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \end{aligned} \tag{3.19}$$

and

$$\beta_n = \beta_{n_{k+m+1}} = \sum_{n_{k+m+1} \geq n_{k+m} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{m} n_{k+i}^2 + n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i} (-1)^{n_k}}{(q)_{n_{k+m+1}-n_{k+m}} (q)_{n_{k+m}-n_{k+m-1}} \cdots (q)_{n_2-n_1} (q)_{n_1}}. \tag{3.20}$$

Step 8. Finally apply (3.3) and (3.4) (with $a = 1$) $K-1-m$ times to get

$$\begin{aligned} \alpha_n = \alpha_n^{(k,K,\ell,m)} = {} & q^{(K+1)n^2+(m+1)n} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \\ & - \chi(n \neq 0) q^{(K+1)n^2-(m+1)n} \sum_{j=-n+1}^{n-1} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2} \end{aligned} \tag{3.21}$$

and

$$\beta_n = \beta_n^{(k,K,\ell,m)} = \sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{K-1} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i} (-1)^{n_k}}{(q)_{n-n_{k+K-1}} (q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}}, \tag{3.22}$$

which is part (ii) of the theorem. This finishes the proof. □

Theorems 1.2 - 1.5 are proven in the same way, with some minor variations as indicated below.

Proof of Theorem 1.2. We proceed as in the proof of Theorem 1.1, except that in Step 1, instead of beginning with the unit Bailey pair in (3.1) and (3.2), we begin with the pair [24, p.468]

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ 2(-1)^n, & \text{if } n > 0, \end{cases} \tag{3.23}$$

and

$$\beta_n = \frac{(-1)^n}{(q^2;q^2)_n}. \tag{3.24}$$

At the first application of (3.3) and (3.4) in Step 2 we keep in mind the identity

$$\sum_{k=0}^{n} \frac{(-1)^k q^{k^2}}{(q)_{n-k}(q^2;q^2)_k} = \frac{1}{(q^2;q^2)_n}, \tag{3.25}$$

which is the case $c = -q$ and $a \to \infty$ of the q-Chu-Vandermonde identity [16, p. 354, Eq. (II.7)],

$$\sum_{k=0}^{n} \frac{(a)_k(q^{-n})_k}{(q)_k(c)_k}(cq^n/a)^k = \frac{(c/a)_n}{(c)_n}. \tag{3.26}$$

The rest of the proof is exactly the same. □

Proof of Theorem 1.3. Here we break from the proof of Theorem 1.1 at Step 6. Instead of applying (3.3) and (3.4) $m+1$ times, we apply Lemma 2.1 once with $\rho_1 = -q$ and $\rho_2 \to \infty$ and then we apply (3.3) and (3.4) m times. The rest of the proof is the same. □

Proof of Theorem 1.4. Here we follow the proof of Theorem 1.2 with the same variation in Step 6 as in the proof of Theorem 1.3. □

Proof of Theorem 1.5. We proceed as in the proof of Theorem 1.1 with $\ell = 0$. At Step 5 we apply Lemma 2.4 with $b = -1$ instead of $b \to \infty$. The rest of the proof is the same. □

4. Proof of Theorem 1.6

Proof of Theorem 1.6. If we take the case $K = k$ of the Bailey pair relative to 1 in Theorem 1.1 (i.e. equations (1.9) and (1.10)) and substitute into (1.27) with $\rho_1, \rho_2 \to \infty$, we obtain

$$\begin{aligned} \sum_{n_{2k} \geq n_{2k-1} \geq \cdots \geq n_1 \geq 0} & \frac{q^{\sum_{i=1}^{k} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i}(-1)^{n_k}}{(q)_{n_{2k}-n_{2k-1}}(q)_{n_{2k-1}-n_{2k-2}} \cdots (q)_{n_2-n_1}(q)_{n_1}} \\ &= \frac{1}{(q)_\infty}\Bigg(\sum_{\substack{n \geq 0 \\ |j| \leq n}} q^{(k+2)n^2+(m+1)n-((2k+1)j^2+(2\ell+1)j)/2}(-1)^j \\ &\quad - \sum_{\substack{n \geq 1 \\ |j| \leq n-1}} q^{(k+2)n^2-(m+1)n-((2k+1)j^2+(2\ell+1)j)/2}(-1)^j \Bigg), \end{aligned} \tag{4.1}$$

valid for $k \geq 1$, $0 \leq \ell < k$, and $0 \leq m < k$. On the right-hand side we let $n = (r+s)/2$ and $j = (r-s)/2$ in the first sum and $n = -(r+s)/2$ and $j = (r-s)/2$ in the second sum to obtain

$$\begin{aligned} \sum_{n_{2k} \geq n_{2k-1} \geq \cdots \geq n_1 \geq 0} & \frac{q^{\sum_{i=1}^{k} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i}(-1)^{n_k}}{(q)_{n_{2k}-n_{2k-1}}(q)_{n_{2k-1}-n_{2k-2}} \cdots (q)_{n_2-n_1}(q)_{n_1}} \\ = \frac{1}{(q)_\infty} & \Bigg(\sum_{\substack{r,s \geq 0 \\ r \equiv s \pmod 2}} - \sum_{\substack{r,s < 0 \\ r \equiv s \pmod 2}} \Bigg) (-1)^{\frac{r-s}{2}} q^{\frac{3}{8}r^2 + \frac{3}{8}s^2 + (k+\frac{5}{4})rs + (\frac{m}{2} - \frac{\ell}{2} + \frac{1}{4})r + (\frac{m}{2} + \frac{\ell}{2} + \frac{3}{4})s}. \end{aligned} \tag{4.2}$$

Replacing (r,s) by $(2r,2s)$ and $(2r+1,2s+1)$ gives

$$
\begin{aligned}
&\frac{1}{(q)_\infty}\left(\sum_{r,s\geq 0}-\sum_{r,s<0}\right)(-1)^{r+s}q^{\frac{3}{2}r^2+\frac{3}{2}s^2+(4k+5)rs+(m-\ell+\frac{1}{2})r+(m+\ell+\frac{3}{2})s}\\
&+\frac{q^{k+m+3}}{(q)_\infty}\left(\sum_{r,s\geq 0}-\sum_{r,s<0}\right)(-1)^{r+s}q^{\frac{3}{2}r^2+\frac{3}{2}s^2+(4k+5)rs+(m-\ell+2k+\frac{9}{2})r+(m+\ell+2k+\frac{11}{2})s} \qquad (4.3)\\
&=\frac{1}{(q)_\infty}\Big(f_{3,4k+5,3}(q^{m+2-\ell},q^{m+3+\ell},q)+q^{k+m+3}f_{3,4k+5,3}(q^{2k+m+6-\ell},q^{2k+m+7+\ell},q)\Big),
\end{aligned}
$$

which is identity (1.29) for $0\leq m<k$. The identity for $m=k$ follows similarly after inserting the Bailey pair relative to q in Theorem 1.1 into (1.27) with $\rho_1,\rho_2\to\infty$.

The remaining identities in Theorem 1.6 follow in the same way, so we merely indicate the appropriate Bailey pair and specialization of (1.27). For parts (2) - (4) we use Theorem 1.1. For (2), use $q=q^2$, $\rho_1=q$, and $\rho_2\to\infty$ in (1.27). For part (3) use $q=q^2$, $\rho_1=-\rho_2=\sqrt{q}$ and for part (4) use only (1.7) and (1.8) with $\rho=-q$ and $\rho_2\to\infty$.

For parts (5) - (8) we use Theorem 1.5. For (5), use $\rho_1,\rho_2\to\infty$ in (1.27), for (6) use $q=q^2$, $\rho_1=q$, and $\rho_2\to\infty$, for (7) use $q=q^2$, $\rho_1=-\rho_2=\sqrt{q}$, and for (8) use only (1.15) and (1.16) with $\rho=-q$ and $\rho_2\to\infty$.

The second equality in each of equations (1.33) - (1.36) as well as in (1.30) follows from an application of [17, Prop 5.2]

$$
\begin{aligned}
f_{a,b,c}(x,y,q)=&f_{a,b,c}(-x^2q^a,-y^2q^c,q^4)-xf_{a,b,c}(-x^2q^{3a},-y^2q^{c+2b},q^4)\\
&-yf_{a,b,c}(-x^2q^{a+2b},-y^2q^{3c},q^4)+xyq^bf_{a,b,c}(-x^2q^{3a+2b},-y^2q^{3c+2b},q^4). \qquad (4.4)
\end{aligned}
$$

□

5. Appell-Lerch series, indefinite theta series and mock theta functions

We begin this section by stating a result of Hickerson and Mortenson which expresses the indefinite theta series (1.28) in terms of Appell-Lerch series. The Appell-Lerch series is defined by

$$m(x,q,z):=\frac{1}{j(z,q)}\sum_{r\in\mathbb{Z}}\frac{(-1)^rq^{\binom{r}{2}}z^r}{1-q^{r-1}xz}, \qquad (5.1)$$

where x, $z\in\mathbb{C}^*:=\mathbb{C}\setminus\{0\}$ with neither z nor xz an integral power of q, and

$$j(x,q):=(x)_\infty(q/x)_\infty(q)_\infty.$$

Define

$$g_{a,b,c}(x,y,q,z_1,z_0) := \sum_{t=0}^{a-1}(-y)^t q^{c\binom{t}{2}} j(q^{bt}x,q^a) m\left(-q^{a\binom{b+1}{2}-c\binom{a+1}{2}-t(b^2-ac)}\frac{(-y)^a}{(-x)^b}, q^{a(b^2-ac)}, z_0\right)$$
$$+\sum_{t=0}^{c-1}(-x)^t q^{a\binom{t}{2}} j(q^{bt}y,q^c) m\left(-q^{c\binom{b+1}{2}-a\binom{c+1}{2}-t(b^2-ac)}\frac{(-x)^c}{(-y)^b}, q^{c(b^2-ac)}, z_1\right) \tag{5.2}$$

and

$$\theta_{n,p}(x,y,q) := \frac{1}{\overline{J}_{0,np(2n+p)}}\sum_{r^*=0}^{p-1}\sum_{s^*=0}^{p-1} q^{n\binom{r-(n-1)/2}{2}+(n+p)(r-(n-1)/2)(s+(n+1)/2)+n\binom{s+(n+1)/2}{2}}$$
$$\times \frac{(-x)^{r-(n-1)/2}(-y)^{s+(n+1)/2}J^3_{p^2(2n+p)}j(-q^{np(s-r)}x^n/y^n,q^{np^2})j(q^{p(2n+p)(r+s)+p(n+p)}x^py^p,q^{p^2(2n+p)})}{j(q^{p(2n+p)r+p(n+p)/2}(-y)^{n+p}/(-x)^n,q^{p^2(2n+p)})j(q^{p(2n+p)s+p(n+p)/2}(-x)^{n+p}/(-y)^n,q^{p^2(2n+p)})},$$

where $r := r^* + \{(n-1)/2\}$ and $s := s^* + \{(n-1)/2\}$ with $0 \le \{\alpha\} < 1$ denoting the fractional part of α. Also, $J_m := J_{m,3m}$ with $J_{a,m} := j(q^a, q^m)$, and $\overline{J}_{a,m} := j(-q^a, q^m)$. Following [17], we use the term "generic" to mean that the parameters do not cause poles in the Appell-Lerch sums or in the quotients of theta functions.

Theorem 5.1. [17, Theorem 0.3] *Let n and p be positive integers with $(n,\, p) = 1$. For generic $x,\, y \in \mathbb{C}^*$*

$$f_{n,n+p,n}(x,y,q) = g_{n,n+p,n}(x,y,q,-1,-1) + \theta_{n,p}(x,y,q). \tag{5.3}$$

The fact that specializations of Appell-Lerch series are mock theta functions (or, in rare instances, modular forms) is now well-known and follows from work of Zwegers [26], [27, Ch. 1]. By Theorem 5.1 the indefinite theta series $f_{n,n+p,n}(x,y,q)$ can be expressed as $\sum_{i=1}^{2n} f_i g_i + F$, where the f_i and F are modular forms and the g_i are Appell-Lerch series. When it happens that all nonzero f_i are identical to one f, then dividing the indefinite theta series by f yields a sum of Appell-Lerch series and a modular form. The result is a mock theta function. This is precisely what happens with our choice of $k = K$ for the examples in Theorem 1.6. We only display the simplest case, which is (1.34). Here Theorem 5.1 turns the right-hand side into

$$2q^{-\binom{m+1}{2}} m\left(q^{2k^2+(1-2m)k}, q^{4k^2+4k}, -1\right) + \theta_{1,2k}(-q^{m+1}, q^{m+1}, q). \tag{5.4}$$

The other cases are similar, giving us the following:

Theorem 5.2. *For generic parameters, the series in Theorem 1.6 are mock theta functions.*

We close with some remarks. First, there are instances of (1.29)-(1.36) where the mock theta functions are in fact modular forms. For example, when Theorem 5.1 is applied to the second line of (1.36), the terms $j(q^{bt}x, q^a)$ and $j(q^{bt}y, q^c)$ in (5.2) are identically 0 when k is odd.

Second, for small p there are alternatives to Theorem 5.1 which yield simpler formulas [17, Section 0]. For example, an application of [17, Theorem 0.11] with $n = 1$ to the case $k = 2$ of

(1.34) gives

$$\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} f_{1,5,1}(-q^{m+1}, q^{m+1}, q)$$
$$= 2q^{-\binom{m+1}{2}} m(q^{10-4m}, q^{24}, -1) - q^{1-2m} \frac{J_{2+4m,24}\overline{J}_{7+2m,12}J_{12}J_{4,16}}{\overline{J}_{14+4m,24}J_{48}\overline{J}_{1,4}}.$$

This may be compared with the case $k = 2$ of (5.4), where $\theta_{1,4}$ is *a priori* the sum of nine modular forms.

Third, the mock theta functions in Theorem 1.6 may be compared with those in [21]. For example, there it was shown that

$$\sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 0} \frac{q^{\binom{n_k+1}{2}+\binom{n_{k-1}+1}{2}+\sum_{i=1}^{k-2} 2^{k-2-i} n_i}(-1)^{n_1}(-q)_{n_{k-1}} \prod_{i=1}^{k-2}(-q^{2^{k-2-i}}; q^{2^{k-2-i}})_{2n_i}}{(q)_{n_k-n_{k-1}}(q^2;q^2)_{n_{k-1}-n_{k-2}} \cdots (q^{2^{k-2}};q^{2^{k-2}})_{n_2-n_1}(q^{2^{k-1}};q^{2^{k-1}})_{n_1}}$$
$$= \frac{(-q)_\infty}{(q)_\infty}\Big(f_{1,2^{k-1}+1,1}(q^{2^{k-2}+1}, q^{2^{k-2}+1}, q^2) + q^{2^{k-1}+1} f_{1,2^{k-1}+1,1}(q^{3(2^{k-2}+1)}, q^{3(2^{k-2}+1)}, q^2)\Big) \tag{5.5}$$

is a mock theta function when $k \geq 3$ [21, Equation (1.14)].

Finally, we expect numerous relations between the multisums in equations (1.29)-(1.36) and classical mock theta functions, as well as among the multisums. Any such relation can be reduced to a finite computation as in [21]-[23] or [15].

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