REAL QUADRATIC DOUBLE SUMS

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Abstract. In 1988, Andrews, Dyson and Hickerson initiated the study of q-hypergeometric series whose coefficients are dictated by the arithmetic in real quadratic fields. In this paper, we provide a dozen q-hypergeometric double sums which are generating functions for the number of ideals of a given norm in rings of integers of real quadratic fields and prove some related identities.

1. Introduction

In 1988, Andrews, Dyson and Hickerson [2] initiated the study of q-hypergeometric series whose coefficients are dictated by the arithmetic in real quadratic fields. They considered a q-series from Ramanujan's lost notebook,

$$\sigma(q) := \sum_{n \geq 0} \frac{q^{\binom{n+1}{2}}}{(-q)_n}, \tag{1.1}$$

and proved the Hecke-type identity,

$$\sigma(q) = \sum_{\substack{n \geq 0 \\ -n \leq j \leq n}} (-1)^{n+j} q^{n(3n+1)/2 - j^2} (1 - q^{2n+1}). \tag{1.2}$$

Here and throughout we assume that $|q| < 1$ and use the standard q-hypergeometric notation,

$$(a)_n = (a;q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}),$$

valid for $n \in \mathbb{N} \cup \{\infty\}$. They then used identity (1.2) to relate the coefficients of $\sigma(q)$ to the ring of integers of the real quadratic field $\mathbb{Q}(\sqrt{6})$. As a consequence, they found that these coefficients satisfy an "almost" exact formula, are lacunary and yet, surprisingly, assume all integer values infinitely often.

Other rare and intriguing examples of q-series related to real quadratic fields (predicted to exist by Dyson [6]) have been discovered over the years (see [3], [5], [7] and [8], for example). The key in each of these cases is the use of Bailey pairs to prove a Hecke-type identity resembling (1.2). We recall that a *Bailey pair* relative to a is a pair of sequences $(\alpha_n, \beta_n)_{n \geq 0}$ satisfying

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}}. \tag{1.3}$$

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For example, Bringmann and Kane [3] discovered the following two Bailey pairs. First, (a_n, b_n) is a Bailey pair relative to 1, where

$$a_{2n} = (1 - q^{4n})q^{2n^2-2n} \sum_{j=-n}^{n-1} q^{-2j^2-2j}, \tag{1.4}$$

$$a_{2n+1} = -(1 - q^{4n+2})q^{2n^2} \sum_{j=-n}^{n} q^{-2j^2}, \tag{1.5}$$

and

$$b_n = \frac{(-1)^n (q; q^2)_{n-1}}{(q)_{2n-1}} \chi(n \neq 0). \tag{1.6}$$

Second, (α_n, β_n) is a Bailey pair relative to q, where

$$\alpha_{2n} = \frac{1}{1-q} \left(q^{2n^2+2n} \sum_{j=-n}^{n-1} q^{-2j^2-2j} + q^{2n^2} \sum_{j=-n}^{n} q^{-2j^2} \right), \tag{1.7}$$

$$\alpha_{2n+1} = -\frac{1}{1-q} \left(q^{2n^2+4n+2} \sum_{j=-n}^{n} q^{-2j^2} + q^{2n^2+2n} \sum_{j=-n-1}^{n} q^{-2j^2-2j} \right), \tag{1.8}$$

and

$$\beta_n = \frac{(-1)^n (q; q^2)_n}{(q)_{2n+1}}. \tag{1.9}$$

Recently, we showed that (1.4)–(1.9) are actually special cases of a much more general result (see Theorems 1.1–1.3 in [9]). This led to new Bailey pairs involving indefinite quadratic forms, and we used these new pairs to find many new examples of q-hypergeometric double sums which are mock theta functions [9]. In this paper we use these same pairs to find many new examples of q-hypergeometric double sums which are generating functions for the number of ideals $\mathfrak{a}$ of a given norm $N(\mathfrak{a})$ in the rings of integers $\mathcal{O}_K$ of real quadratic fields K. Our main results are as follows. We use the notation $\sum^*$ to indicate that the sum does not converge in the classical sense, but may be defined as the average of the even and odd partial sums.

Theorem 1.1. *Let $K = \mathbb{Q}(\sqrt{2})$. We have that*

$$L_1(q) := \sum_{n \geq 1} \sum_{n \geq k \geq 1} \frac{(q)_{n-1} (-1)^{n+k} q^{\binom{n+1}{2} + \binom{k+1}{2}}}{(q)_{n-k} (q)_{k-1} (1 - q^{2k-1})}$$

satisfies

$$q^{-17} L_1(q^{32}) = \frac{1}{2} \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ N(\mathfrak{a}) \equiv 15 \pmod{32}}} q^{N(\mathfrak{a})}, \tag{1.10}$$

$$L_2(q) := \sum_{n\geq 0}\sum_{n\geq k\geq 0} \frac{(q)_n(-1)^{n+k}q^{\binom{n+1}{2}+\binom{k+1}{2}}}{(q)_{n-k}(q)_k(1-q^{2k+1})}$$

satisfies

$$q^7L_2(q^{32}) = \frac{1}{2}\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K\\ N(\mathfrak{a})\equiv 7 \pmod{32}}} q^{N(\mathfrak{a})}, \tag{1.11}$$

$$L_3(q) := q\sum_{n\geq 1}\sum_{n\geq k\geq 1} \frac{(q)_{n-1}(-1)^{n+k}q^{\binom{n+1}{2}+\binom{k}{2}}}{(q)_{n-k}(q)_{k-1}(1-q^{2k-1})}$$

satisfies

$$q^{-33}L_3(q^{32}) = \frac{1}{2}\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K\\ N(\mathfrak{a})\equiv 31 \pmod{32}}} q^{N(\mathfrak{a})}, \tag{1.12}$$

and

$$L_4(q) := -1 + \sum_{n\geq 0}\sum_{n\geq k\geq 0} \frac{(q)_n(-1)^{n+k}q^{\binom{n+1}{2}+\binom{k}{2}}}{(q)_{n-k}(q)_k(1-q^{2k+1})}$$

satisfies

$$q^{-9}L_4(q^{32}) = \frac{1}{2}\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K\\ N(\mathfrak{a})\equiv 23 \pmod{32}}} q^{N(\mathfrak{a})}. \tag{1.13}$$

Theorem 1.2. *Let $L = \mathbb{Q}(\sqrt{3})$. We have that*

$$L_5(q) := q\sum_{n\geq 1}\sum_{n\geq k\geq 1} \frac{(-1)_n(q)_{n-1}(-1)^{n+k}q^{n+k^2-k}}{(q)_{n-k}(q^2;q^2)_{k-1}(1-q^{2k-1})}$$

satisfies

$$q^{-2}L_5(q^2) = 2\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_L\\ N(\mathfrak{a})\equiv 0 \pmod{2}\\ \mathfrak{a}=(x),N(x)<0}} q^{N(\mathfrak{a})}, \tag{1.14}$$

$$L_6(q) := \sum_{n\geq 1}\sum_{n\geq k\geq 1} \frac{(-1)_n(q)_{n-1}(-1)^{n+k}q^{n+k^2}}{(q)_{n-k}(q^2;q^2)_{k-1}(1-q^{2k-1})}$$

satisfies

$$q^{-1}L_6(q^2) = 2\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_L\\ N(\mathfrak{a})\equiv 1 \pmod{2}\\ \mathfrak{a}=(x),N(x)<0}} q^{N(\mathfrak{a})}, \tag{1.15}$$

$$L_7(q) := 2{\sum_{n\geq 0}}^{*} \sum_{n\geq k\geq 0} \frac{(q^2;q^2)_n(-1)^{n+k}q^{k^2+k}}{(q)_{n-k}(q^2;q^2)_k(1-q^{2k+1})}$$

satisfies

$$qL_7(q^6) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_L \\ N(\mathfrak{a})\equiv 1 \pmod 6}} q^{N(\mathfrak{a})}, \tag{1.16}$$

and

$$L_8(q) := -1 + 2{\sum_{n\geq 0}}^{*} \sum_{n\geq k\geq 0} \frac{(q^2;q^2)_n(-1)^{n+k}q^{k^2}}{(q)_{n-k}(q^2;q^2)_k(1-q^{2k+1})}$$

satisfies

$$q^{-2}L_8(q^6) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_L \\ N(\mathfrak{a})\equiv 4 \pmod 6}} q^{N(\mathfrak{a})}. \tag{1.17}$$

Theorem 1.3. *Let $M = \mathbb{Q}(\sqrt{6})$. We have that*

$$L_9(q) := \sum_{n\geq 1}\sum_{n\geq k\geq 1} \frac{(-1)_n(q)_{n-1}(-1)^{n+k}q^{n+\binom{k+1}{2}}}{(q)_{n-k}(q)_{k-1}(1-q^{2k-1})}$$

satisfies

$$q^{-9}L_9(q^{16}) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_M \\ N(\mathfrak{a})\equiv 7 \pmod{16}}} q^{N(\mathfrak{a})}, \tag{1.18}$$

$$L_{10}(q) := q\sum_{n\geq 1}\sum_{n\geq k\geq 1} \frac{(-1)_n(q)_{n-1}(-1)^{n+k}q^{n+\binom{k}{2}}}{(q)_{n-k}(q)_{k-1}(1-q^{2k-1})}$$

satisfies

$$q^{-17}L_{10}(q^{16}) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_M \\ N(\mathfrak{a})\equiv 15 \pmod{16}}} q^{N(\mathfrak{a})}, \tag{1.19}$$

$$L_{11}(q) := 2{\sum_{n\geq 0}}^{*} \sum_{n\geq k\geq 0} \frac{(q^2;q^2)_n(-1)^{n+k}q^{\binom{k+1}{2}}}{(q)_{n-k}(q)_k(1-q^{2k+1})}$$

satisfies

$$q^5L_{11}(q^{48}) = \frac{1}{2}\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_M \\ N(\mathfrak{a})\equiv 5 \pmod{48}}} q^{N(\mathfrak{a})}, \tag{1.20}$$

and

$$L_{12}(q) := -2 + 2\sum_{n\geq 0}^{*} \sum_{n\geq k\geq 0} \frac{(q^2;q^2)_n(-1)^{n+k}q^{\binom{k}{2}}}{(q)_{n-k}(q)_k(1-q^{2k+1})}$$

satisfies

$$q^{-19}L_{12}(q^{48}) = \frac{1}{2} \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_M \\ N(\mathfrak{a})\equiv 29 \pmod{48}}} q^{N(\mathfrak{a})}. \tag{1.21}$$

We note that it follows from the above Hecke-type identities and Theorem 1 in [10] that all of the series $L_i(q)$ are lacunary. Exact formulas for the number of elements/ideals in $\mathcal{O}_K$, $\mathcal{O}_L$ and $\mathcal{O}_M$ with prime power norm then imply that their coefficients assume all eligible integer values infinitely often.

With the following corollary we establish identities between some of the real quadratic double sums appearing in Theorems 1.1 and 1.2 and those which feature prominently in previous related works. Recall the following q-series (see (1.2) in [5], (1.9) in [7] and Theorems 1.6 and 1.7 in [3]):

$$\mathcal{Z}_2(q) := \sum_{n\geq 1} \frac{q^n(-q^2;q^2)_{n-1}}{(-q;q^2)_n},$$

$$\mathcal{Z}_3(q) := \sum_{n\geq 1} \frac{(-1)^n q^{n^2+n}(q^2;q^2)_{n-1}}{(-q)_{2n}},$$

$$\mathcal{Z}_4(q) := \sum_{n\geq 0} \frac{(-1)^n q^{n^2+n}(q^2;q^2)_{n}}{(-q)_{2n+1}},$$

$$\mathcal{Z}_5(q) := \sum_{n\geq 1} \frac{(-1)^n q^{n}(q^2;q^2)_{n}}{(q^n)_{n}}.$$

Corollary 1.4. *We have the following identities:*

$$\mathcal{Z}_2(q) = q^{-2}L_1(q^4) + qL_2(q^4) + q^{-4}L_3(q^4) + q^{-1}L_4(q^4), \tag{1.22}$$

$$2\mathcal{Z}_3(q) = -q^{-2}L_5(q^2) + q^{-1}L_6(q^2), \tag{1.23}$$

$$\mathcal{Z}_4(-q) = L_7(q^2) + q^{-1}L_8(q^2), \tag{1.24}$$

$$-2\mathcal{Z}_5(q^2) = L_6(q). \tag{1.25}$$

The paper is organized as follows. In Section 2, we first recall some preliminaries on Bailey pairs and key results from [2] and [9]. In Section 3, we prove Theorems 1.1–1.3 and Corollary 1.4. In Section 4, we mention some questions for further study.

2. Preliminaries

Before proceeding to the proofs of Theorems 1.1–1.3, we briefly discuss some preliminaries. First, the Bailey lemma (see Chapter 3 in [1]) says that if (α_n, β_n) is a Bailey pair relative to a, then so is (α_n', β_n'), where

$$\alpha_n' = \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1)_n(aq/\rho_2)_n}\alpha_n \tag{2.1}$$

and

$$\beta_n' = \sum_{k=0}^{n} \frac{(\rho_1)_k(\rho_2)_k(aq/\rho_1\rho_2)_{n-k}(aq/\rho_1\rho_2)^k}{(aq/\rho_1)_n(aq/\rho_2)_n(q)_{n-k}}\beta_k. \tag{2.2}$$

The limiting form of the Bailey lemma is found by putting (2.1) and (2.2) into (1.3) and letting $n \to \infty$, giving

$$\sum_{n\geq 0}(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\beta_n = \frac{(aq/\rho_1)_\infty(aq/\rho_2)_\infty}{(aq)_\infty(aq/\rho_1\rho_2)_\infty}\sum_{n\geq 0}\frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1)_n(aq/\rho_2)_n}\alpha_n. \tag{2.3}$$

Next, we recall six key Bailey pairs which were established in [9].

Proposition 2.1. *We have the following two Bailey pairs. First, the sequences (a_n, b_n) form a Bailey pair relative to 1, where*

$$a_{2n} = (1-q^{4n})q^{2n^2-2n+1}\sum_{j=-n}^{n-1} q^{-2j^2}, \tag{2.4}$$

$$a_{2n+1} = -(1-q^{4n+2})q^{2n^2}\sum_{j=-n}^{n} q^{-2j^2-2j}, \tag{2.5}$$

and

$$b_n = \begin{cases} 0, & \textit{if } n=0, \\ \frac{(-1)^n q^{-n+1}}{(q^2;q^2)_{n-1}(1-q^{2n-1})}, & \textit{otherwise.} \end{cases} \tag{2.6}$$

Second, the sequences (α_n, β_n) form a Bailey pair relative to q, where

$$\alpha_{2n} = \frac{1}{1-q}\left(q^{2n^2}\sum_{j=-n}^{n} q^{-2j^2-2j} + q^{2n^2+2n+1}\sum_{j=-n}^{n-1} q^{-2j^2}\right), \tag{2.7}$$

$$\alpha_{2n+1} = -\frac{1}{1-q}\left(q^{2n^2+2n+1}\sum_{j=-n-1}^{n} q^{-2j^2} + q^{2n^2+4n+2}\sum_{j=-n}^{n} q^{-2j^2-2j}\right), \tag{2.8}$$

and

$$\beta_n = \frac{(-1)^n q^{-n}}{(q^2;q^2)_n(1-q^{2n+1})}. \tag{2.9}$$

Proposition 2.2. *We have the following two Bailey pairs. First, the sequences (a_n, b_n) form a Bailey pair relative to 1, where*

$$a_{2n} = (1-q^{4n})q^{2n^2-2n}\sum_{j=-n}^{n-1} q^{-4j^2-3j}, \tag{2.10}$$

$$a_{2n+1} = -(1-q^{4n+2})q^{2n^2}\sum_{j=-n}^{n} q^{-4j^2-j}, \tag{2.11}$$

and

$$b_n = \begin{cases} 0, & \text{if } n=0, \\ \frac{(-1)^n q^{-\binom{n}{2}}}{(q)_{n-1}(1-q^{2n-1})}, & \text{otherwise.} \end{cases} \tag{2.12}$$

Second, the sequences (α_n, β_n) form a Bailey pair relative to q, where

$$\alpha_{2n} = \frac{1}{1-q}\left(q^{2n^2}\sum_{j=-n}^{n} q^{-4j^2-j} + q^{2n^2+2n}\sum_{j=-n}^{n-1} q^{-4j^2-3j}\right), \tag{2.13}$$

$$\alpha_{2n+1} = -\frac{1}{1-q}\left(q^{2n^2+2n}\sum_{j=-n-1}^{n} q^{-4j^2-3j} + q^{2n^2+4n+2}\sum_{j=-n}^{n} q^{-4j^2-j}\right), \tag{2.14}$$

and

$$\beta_n = \frac{(-1)^n q^{-\binom{n+1}{2}}}{(q)_n(1-q^{2n+1})}. \tag{2.15}$$

Proposition 2.3. *We have the following two Bailey pairs. First, the sequences (a_n, b_n) form a Bailey pair relative to 1, where*

$$a_{2n} = (1-q^{4n})q^{2n^2-2n+1}\sum_{j=-n}^{n-1} q^{-4j^2-j}, \tag{2.16}$$

$$a_{2n+1} = -(1-q^{4n+2})q^{2n^2}\sum_{j=-n}^{n} q^{-4j^2-3j}, \tag{2.17}$$

and

$$b_n = \begin{cases} 0, & \text{if } n=0, \\ \frac{(-1)^n q^{-\binom{n+1}{2}+1}}{(q)_{n-1}(1-q^{2n-1})}, & \text{otherwise.} \end{cases} \tag{2.18}$$

Second, the sequences (α_n, β_n) form a Bailey pair relative to q, where

$$\alpha_{2n} = \frac{1}{1-q}\left(q^{2n^2}\sum_{j=-n}^{n} q^{-4j^2-3j} + q^{2n^2+2n+1}\sum_{j=-n}^{n-1} q^{-4j^2-j}\right), \tag{2.19}$$

$$\alpha_{2n+1} = -\frac{1}{1-q}\left(q^{2n^2+2n+1}\sum_{j=-n-1}^{n} q^{-4j^2-j} + q^{2n^2+4n+2}\sum_{j=-n}^{n} q^{-4j^2-3j}\right), \tag{2.20}$$

and

$$\beta_n = \frac{(-1)^n q^{-n(n+3)/2}}{(q)_n(1-q^{2n+1})}. \tag{2.21}$$

Finally, we record a useful lemma for rewriting Hecke-type sums in terms of rings of integers of real quadratic fields.

Lemma 2.4. [2, Lemma 3] *Let (x_1, y_1) be the fundamental solution of $x^2 - Dy^2 = 1$, i.e., the solution in which x_1 and y_1 are minimal positive. If $m > 0$, then each equivalence class of solutions of $u^2 - Dv^2 = m$ contains a unique (u, v) with $u > 0$ and*

$$-\frac{y_1}{x_1+1}u < v \leq \frac{y_1}{x_1+1}u.$$

If $m < 0$, the corresponding conditions are $v > 0$ and

$$-\frac{Dy_1}{x_1+1}v < u \leq \frac{Dy_1}{x_1+1}v.$$

3. Proofs of Theorems 1.1–1.3 and Corollary 1.4

We briefly discuss the strategy for proving Theorems 1.1–1.3. The first step is to make substitutions for ρ_1 and ρ_2 such that the product on the right-hand side of (2.3) is eliminated. For example, for Bailey pairs (α_n, β_n) relative to $a = 1$, we can let $\rho_1 \to \infty$, divide both sides by $1 - \rho_2$, then let $\rho_2 \to 1$ in (2.3) to obtain

$$\sum_{n\geq 1}(-1)^n (q)_{n-1} q^{\frac{n(n+1)}{2}}\beta_n = \sum_{n\geq 1}\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1-q^n}\alpha_n. \tag{3.1}$$

Alternatively, we can take $\rho_1 = -1$ and divide both sides by $1 - \rho_2$, then let $\rho_2 \to 1$ in (2.3) to get

$$\sum_{n\geq 1}(-1)_n (q)_{n-1}(-q)^n\beta_n = 2\sum_{n\geq 1}\frac{(-q)^n}{1-q^{2n}}\alpha_n. \tag{3.2}$$

For Bailey pairs (α_n, β_n) relative to $a = q$, we can let $b \to \infty$ and $c = q$ in (2.3) to obtain

$$\sum_{n\geq 0}(-1)^n (q)_n q^{\frac{n(n+1)}{2}}\beta_n = (1-q)\sum_{n\geq 0}(-1)^n q^{\frac{n(n+1)}{2}}\alpha_n \tag{3.3}$$

or take $\rho_1 = q$ and $\rho_2 = -q$ in (2.3) to obtain

$$\sum_{n\geq 0}(q^2;q^2)_n(-1)^n\beta_n = \frac{1-q}{2}\sum_{n\geq 0}(-1)^n\alpha_n. \tag{3.4}$$

We then employ the Bailey pairs in Propositions 2.1–2.3 and the Bailey lemma in (2.1) and (2.2) to obtain a new Bailey pair. Finally, we insert this new pair into one of (3.1)–(3.4), express the "α_n" side in terms of indefinite quadratic forms and appeal to Lemma 2.4. We now prove Theorems 1.1–1.3.

Proof of Theorem 1.1. To prove (1.10), we insert (2.10)–(2.12) into (2.1) and (2.2) with $(a, \rho_1, \rho_2) = (1, \infty, \infty)$, then apply (3.1) to get

$$L_1(q) = \sum_{\substack{n\geq 1\\ -n\leq j\leq n-1}} q^{8n^2-n-4j^2-3j} + q^{8n^2+n-4j^2-3j} + \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{8n^2+7n+2-4j^2-j} + q^{8n^2+9n+3-4j^2-j} \tag{3.5}$$

and so

$$q^{-17}L_1(q^{32}) = \sum_{\substack{n\geq 1\\ -n\leq j\leq n-1}} q^{(16n-1)^2-2(8j+3)^2} + q^{(16n+1)^2-2(8j+3)^2} + \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(16n+7)^2-2(8j+1)^2} + q^{(16n+9)^2-2(8j+1)^2}. \tag{3.6}$$

Thus,

$$\begin{aligned} 2q^{-17}L_1(q^{32}) = & \sum_{\substack{n\geq 1\\ -n\leq j\leq n-1}} q^{(16n-1)^2-2(8j+3)^2} + q^{(16n+1)^2-2(8j+3)^2} \\ & + \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(16n+7)^2-2(8j+1)^2} + q^{(16n+9)^2-2(8j+1)^2} \\ & + \sum_{\substack{n\geq 1\\ -n+1\leq j\leq n}} q^{(16n-1)^2-2(8j-3)^2} + q^{(16n+1)^2-2(8j-3)^2} \\ & + \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(16n+7)^2-2(8j-1)^2} + q^{(16n+9)^2-2(8j-1)^2} \end{aligned} \tag{3.7}$$

where we have let $j \to -j$ in the second copy of (3.6) to obtain the third and fourth sum in (3.7). By Lemma 2.4 and unique factorization in $\mathcal{O}_K$, each ideal $\mathfrak{a}$ can be uniquely written as $\mathfrak{a} = (u + v\sqrt{2})$ with $u > 0$ and $-\frac{1}{2}u < v \leq \frac{1}{2}u$. This representation combined with the condition $N(\mathfrak{a}) \equiv 15 \pmod{32}$ is equivalent to either $u \equiv \pm 1 \pmod{16}$, $v \equiv \pm 3 \pmod 8$ or $u \equiv \pm 7 \pmod{16}$, $v \equiv \pm 1 \pmod 8$. Comparing this with (3.7) implies (1.10).

To prove (1.11), we insert (2.13)–(2.15) into (2.1) and (2.2) with $(a, \rho_1, \rho_2) = (q, \infty, \infty)$, then apply (3.3) to get

$$L_2(q) = \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{8n^2+3n-4j^2-j} + q^{8n^2+13n+5-4j^2-j} + \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{8n^2+11n+3-4j^2-3j} + q^{8n^2+21n+13-4j^2-3j} \tag{3.8}$$

and so

$$q^7L_2(q^{32}) = \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(16n+3)^2-2(8j+1)^2} + q^{(16n+13)^2-2(8j+1)^2} + \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{(16n+21)^2-2(8j+3)^2} + q^{(16n+11)^2-2(8j+3)^2}. \tag{3.9}$$

Thus,

$$\begin{aligned} 2q^7L_2(q^{32}) = &\sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(16n+3)^2-2(8j+1)^2} + q^{(16n+13)^2-2(8j+1)^2} \\ &+ \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{(16n+21)^2-2(8j+3)^2} + q^{(16n+11)^2-2(8j+3)^2} \\ &+ \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(16n+3)^2-2(8j-1)^2} + q^{(16n+13)^2-2(8j-1)^2} \\ &+ \sum_{\substack{n\geq 0\\ -n\leq j\leq n+1}} q^{(16n+21)^2-2(8j-3)^2} + q^{(16n+11)^2-2(8j-3)^2}. \end{aligned} \tag{3.10}$$

Again, Lemma 2.4, unique factorization and the condition $N(\mathfrak{a}) \equiv 7 \pmod{32}$ imply (1.11) after comparing with (3.10).

For (1.12), we insert (2.16)–(2.18) into (2.1) and (2.2) with $(a, \rho_1, \rho_2) = (1, \infty, \infty)$, then apply (3.1) to obtain

$$L_3(q) = \sum_{\substack{n\geq 1\\ -n\leq j\leq n-1}} q^{8n^2-n+1-4j^2-j} + q^{8n^2+n+1-4j^2-j} + \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{8n^2+7n+2-4j^2-3j} + q^{8n^2+9n+3-4j^2-3j} \tag{3.11}$$

and so

$$q^{-33}L_3(q^{32}) = \sum_{\substack{n\geq 1\\ -n\leq j\leq n-1}} q^{(16n-1)^2-2(8j+1)^2} + q^{(16n+1)^2-2(8j+1)^2} + \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(16n+7)^2-2(8j+3)^2} + q^{(16n+9)^2-2(8j+3)^2}. \tag{3.12}$$

Thus,

$$\begin{aligned}2q^{-33}L_3(q^{32}) = &\sum_{\substack{n\geq 1\\ -n\leq j\leq n-1}} q^{(16n-1)^2-2(8j+1)^2} + q^{(16n+1)^2-2(8j+1)^2}\\ &+ \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(16n+7)^2-2(8j+3)^2} + q^{(16n+9)^2-2(8j+3)^2}\\ &+ \sum_{\substack{n\geq 1\\ -n+1\leq j\leq n}} q^{(16n-1)^2-2(8j-1)^2} + q^{(16n+1)^2-2(8j-1)^2}\\ &+ \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(16n+7)^2-2(8j-3)^2} + q^{(16n+9)^2-2(8j-3)^2}.\end{aligned} \tag{3.13}$$

Arguing as above gives (1.12).

For (1.13), we insert (2.19)–(2.21) into (2.1) and (2.2) with $(a,\rho_1,\rho_2) = (q,\infty,\infty)$, then apply (3.3) to obtain

$$L_4(q) = -1 + \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{8n^2+3n-4j^2-3j} + q^{8n^2+13n+5-4j^2-3j} + \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{8n^2+11n+4-4j^2-j} + q^{8n^2+21n+14-4j^2-j} \tag{3.14}$$

and so

$$q^{-9}L_4(q^{32}) = -q^{-9} + \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(16n+3)^2-2(8j+3)^2} + q^{(16n+13)^2-2(8j+3)^2} + \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{(16n+21)^2-2(8j+1)^2} + q^{(16n+11)^2-2(8j+1)^2}. \tag{3.15}$$

Slightly modifying the summation limits we obtain

$$\begin{aligned} q^{-9}L_4(q^{32}) = & \sum_{\substack{n\geq 0 \\ -n\leq j\leq n-1}} q^{(16n+3)^2-2(8j+3)^2} + \sum_{\substack{n\geq 0 \\ -n-1\leq j\leq n}} q^{(16n+13)^2-2(8j+3)^2} \\ & + \sum_{\substack{n\geq -1 \\ -n-1\leq j\leq n+1}} q^{(16n+21)^2-2(8j+1)^2} + \sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} q^{(16n+11)^2-2(8j+1)^2}. \end{aligned} \tag{3.16}$$

Thus,

$$\begin{aligned} 2q^{-9}L_4(q^{32}) = & \sum_{\substack{n\geq 0 \\ -n\leq j\leq n-1}} q^{(16n+3)^2-2(8j+3)^2} + \sum_{\substack{n\geq 0 \\ -n-1\leq j\leq n}} q^{(16n+13)^2-2(8j+3)^2} \\ & + \sum_{\substack{n\geq -1 \\ -n-1\leq j\leq n+1}} q^{(16n+21)^2-2(8j+1)^2} + \sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} q^{(16n+11)^2-2(8j+1)^2} \\ & + \sum_{\substack{n\geq 0 \\ -n+1\leq j\leq n}} q^{(16n+3)^2-2(8j-3)^2} + \sum_{\substack{n\geq 0 \\ -n\leq j\leq n+1}} q^{(16n+13)^2-2(8j-3)^2} \\ & + \sum_{\substack{n\geq -1 \\ -n-1\leq j\leq n+1}} q^{(16n+21)^2-2(8j-1)^2} + \sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} q^{(16n+11)^2-2(8j-1)^2}. \end{aligned} \tag{3.17}$$

Arguing as before gives (1.13). □

Proof of Theorem 1.2. For (1.14), insert (2.4)–(2.6) into (2.1) and (2.2) with $(a, \rho_1, \rho_2) = (1, \infty, \infty)$, then apply (3.2) to obtain

$$L_5(q) = 2 \sum_{\substack{n\geq 1 \\ -n\leq j\leq n-1}} q^{6n^2+1-2j^2} + 2 \sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} q^{6n^2+6n+2-2j^2-2j} \tag{3.18}$$

and so

$$q^{-2}L_5(q^2) = 2 \sum_{\substack{n\geq 1 \\ -n\leq j\leq n-1}} q^{3(2n)^2-(2j)^2} + 2 \sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} q^{3(2n+1)^2-(2j+1)^2}. \tag{3.19}$$

By Lemma 2.4 and unique factorization in $\mathcal{O}_L$, each (principal) ideal $\mathfrak{a}$ generated by an element of negative norm can be uniquely written as $\mathfrak{a} = (u + v\sqrt{3})$ with $v > 0$ and $-v < u \leq v$. This representation combined with the condition $N(\mathfrak{a}) \equiv 0 \pmod 2$ is equivalent to either $u \equiv 0 \pmod 2$, $v \equiv 0 \pmod 2$ or $u \equiv 1 \pmod 2$, $v \equiv 1 \pmod 2$. Comparing this with (3.19) implies (1.14).

For (1.15), insert (1.4)–(1.6) into (2.1) and (2.2) with $(a, \rho_1, \rho_2) = (q, \infty, \infty)$, then apply (3.2) to get

$$L_6(q) = 2 \sum_{\substack{n\geq 1 \\ -n\leq j\leq n-1}} q^{6n^2-2j^2-2j} + 2 \sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} q^{6n^2+6n+1-2j^2} \tag{3.20}$$

and so

$$q^{-1}L_6(q^2) = 2 \sum_{\substack{n\geq 1\\ -n\leq j\leq n-1}} q^{3(2n)^2-(2j+1)^2} + 2 \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{3(2n+1)^2-(2j)^2}. \tag{3.21}$$

By Lemma 2.4 and unique factorization in $\mathcal{O}_L$, each principal ideal $\mathfrak{a}$ generated by an element of negative norm can be uniquely written as $\mathfrak{a} = (u + v\sqrt{3})$ with $v > 0$, $-v < u \leq v$. Arguing as usual gives (1.15).

For (1.16), insert (1.7)–(1.9) into (2.1) and (2.2) with $(a, \rho_1, \rho_2) = (q, \infty, \infty)$, then apply (3.4) to get

$$\begin{aligned} L_7(q) = &\sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{6n^2+16n+10-2j^2-2j} + q^{6n^2+8n+2-2j^2-2j} \\ &+ \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{6n^2+2n-2j^2} + q^{6n^2+10n+4-2j^2} \end{aligned} \tag{3.22}$$

and so

$$\begin{aligned} qL_7(q^6) = &\sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{(6n+8)^2-3(2j+1)^2} + q^{(6n+4)^2-3(2j+1)^2} \\ &+ \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(6n+1)^2-3(2j)^2} + q^{(6n+5)^2-3(2j)^2}. \end{aligned} \tag{3.23}$$

Arguing as usual gives (1.16).

For (1.17), insert (2.7)–(2.9) into (2.1) and (2.2) with $(a, \rho_1, \rho_2) = (q, \infty, \infty)$, then apply (3.4) to obtain

$$\begin{aligned} L_8(q) = -1 + &\sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{6n^2+2n-2j^2-2j} + q^{6n^2+10n+4-2j^2-2j} \\ &+ \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{6n^2+16n+11-2j^2} + q^{6n^2+8n+3-2j^2} \end{aligned} \tag{3.24}$$

and so

$$\begin{aligned} q^{-2}L_8(q^6) = q^{-2} + &\sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(6n+1)^2-3(2j+1)^2} + q^{(6n+5)^2-3(2j+1)^2} \\ &+ \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{(6n+8)^2-3(2j)^2} + q^{(6n+4)^2-3(2j)^2}. \end{aligned}$$

Slightly modifying the summation bounds and simplifying gives

$$\begin{aligned} q^{-2}L_8(q^6) = &\sum_{\substack{n\geq 0 \\ -n\leq j\leq n-1}} q^{(6n+1)^2-3(2j+1)^2} + \sum_{\substack{n\geq 0 \\ -n-1\leq j\leq n}} q^{(6n+5)^2-3(2j+1)^2} \\ &+ \sum_{\substack{n\geq -1 \\ -n-1\leq j\leq n+1}} q^{(6n+8)^2-3(2j)^2} + \sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} q^{(6n+4)^2-3(2j)^2}. \end{aligned}$$

Arguing as usual gives (1.17). □

Proof of Theorem 1.3. For (1.18), we insert (2.10)–(2.12) into (2.1) and (2.2) with $(a, \rho_1, \rho_2) = (1, \infty, \infty)$, then apply (3.2) to get

$$L_9(q) = 2\sum_{\substack{n\geq 1 \\ -n\leq j\leq n-1}} q^{6n^2-4j^2-3j} + 2\sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} q^{6n^2+6n+2-4j^2-j} \tag{3.25}$$

and so

$$\begin{aligned} q^{-9}L_9(q^{16}) = &\sum_{\substack{n\geq 1 \\ -n\leq j\leq n-1}} q^{6(4n)^2-(8j+3)^2} + \sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} q^{6(4n+2)^2-(8j+1)^2} \\ &+ \sum_{\substack{n\geq 1 \\ -n+1\leq j\leq n}} q^{6(4n)^2-(8j-3)^2} + \sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} q^{6(4n+2)^2-(8j-1)^2}. \end{aligned} \tag{3.26}$$

By Lemma 2.4 and unique factorization in $\mathcal{O}_M$, each ideal $\mathfrak{a}$ with $N(\mathfrak{a}) \equiv 7 \pmod{16}$ can be uniquely written as $\mathfrak{a} = (u + v\sqrt{6})$ with $v > 0$ and $-2v < u \leq 2v$. Arguing as usual gives (1.18).

For (1.19), we insert (2.16)–(2.18) into (2.1) and (2.2) with $(a, \rho_1, \rho_2) = (1, \infty, \infty)$, then apply (3.2) to get

$$L_{10}(q) = 2\sum_{\substack{n\geq 1 \\ -n\leq j\leq n-1}} q^{6n^2+1-4j^2-j} + 2\sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} q^{6n^2+6n+2-4j^2-3j} \tag{3.27}$$

and so

$$\begin{aligned} q^{-17}L_{10}(q^{16}) = &\sum_{\substack{n\geq 1 \\ -n\leq j\leq n-1}} q^{6(4n)^2-(8j+1)^2} + \sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} q^{6(4n+2)^2-(8j+3)^2} \\ &+ \sum_{\substack{n\geq 1 \\ -n+1\leq j\leq n}} q^{6(4n)^2-(8j-1)^2} + \sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} q^{6(4n+2)^2-(8j-3)^2}. \end{aligned} \tag{3.28}$$

Arguing as usual gives (1.19).

For (1.20), we insert (2.13)–(2.15) into (2.1) and (2.2) with $(a, \rho_1, \rho_2) = (q, \infty, \infty)$, then apply (3.4) to obtain

$$L_{11}(q) = \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{6n^2+2n-4j^2-j} + q^{6n^2+10n+4-4j^2-j} + \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{6n^2+16n+10-4j^2-3j} + q^{6n^2+8n+2-4j^2-3j} \tag{3.29}$$

and so

$$q^{10}L_{11}(q^{96}) = \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(24n+4)^2-6(8j+1)^2} + q^{(24n+20)^2-6(8j+1)^2} + \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{(24n+32)^2-6(8j+3)^2} + q^{(24n+16)^2-6(8j+3)^2}. \tag{3.30}$$

Thus,

$$\begin{aligned}2q^{10}L_{11}(q^{96}) = &\sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(24n+4)^2-6(8j+1)^2} + q^{(24n+20)^2-6(8j+1)^2} \\ &+ \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{(24n+32)^2-6(8j+3)^2} + q^{(24n+16)^2-6(8j+3)^2} \\ &+ \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(24n+4)^2-6(8j-1)^2} + q^{(24n+20)^2-6(8j-1)^2} \\ &+ \sum_{\substack{n\geq 0\\ -n\leq j\leq n+1}} q^{(24n+32)^2-6(8j-3)^2} + q^{(24n+16)^2-6(8j-3)^2}.\end{aligned} \tag{3.31}$$

Arguing as usual gives

$$q^{10}L_{11}(q^{96}) = \frac{1}{2} \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_M\\ N(\mathfrak{a})\equiv 10 \pmod{96}}} q^{N(\mathfrak{a})}, \tag{3.32}$$

and dividing by the unique ideal $(2+\sqrt{6})$ in $\mathcal{O}_M$ of norm 2 gives (1.20).

For (1.21), we insert (2.19)–(2.21) into (2.1) and (2.2) with $(a, \rho_1, \rho_2) = (q, \infty, \infty)$, then apply (3.4) to obtain

$$L_{12}(q) = -2 + \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{6n^2+2n-4j^2-3j} + q^{6n^2+10n+4-4j^2-3j} + \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{6n^2+16n+11-4j^2-j} + q^{6n^2+8n+3-4j^2-j}. \tag{3.33}$$

Thus,

$$q^{-38}L_{12}(q^{96}) = -2q^{-38} + \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(24n+4)^2-6(8j+3)^2} + q^{(24n+20)^2-6(8j+3)^2} + \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{(24n+32)^2-6(8j+1)^2} + q^{(24n+16)^2-6(8j+1)^2}.$$

Slightly modifying the summation bounds and simplifying gives

$$\begin{aligned} q^{-38}L_{12}(q^{96}) = &\sum_{\substack{n\geq 1\\ -n\leq j\leq n-1}} q^{(24n+4)^2-6(8j+3)^2} + \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{(24n+20)^2-6(8j+3)^2} \\ &+ \sum_{\substack{n\geq -1\\ -n-1\leq j\leq n+1}} q^{(24n+32)^2-6(8j+1)^2} + \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(24n+16)^2-6(8j+1)^2}. \end{aligned} \tag{3.34}$$

Thus,

$$\begin{aligned} 2q^{-38}L_{12}(q^{96}) = &\sum_{\substack{n\geq 1\\ -n\leq j\leq n-1}} q^{(24n+4)^2-6(8j+3)^2} + \sum_{\substack{n\geq 0\\ -n-1\leq j\leq n}} q^{(24n+20)^2-6(8j+3)^2} \\ &+ \sum_{\substack{n\geq 1\\ -n+1\leq j\leq n}} q^{(24n+4)^2-6(8j-3)^2} + \sum_{\substack{n\geq 0\\ -n\leq j\leq n+1}} q^{(24n+20)^2-6(8j-3)^2} \\ &+ \sum_{\substack{n\geq -1\\ -n-1\leq j\leq n+1}} q^{(24n+32)^2-6(8j+1)^2} + \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(24n+16)^2-6(8j+1)^2} \\ &+ \sum_{\substack{n\geq -1\\ -n-1\leq j\leq n+1}} q^{(24n+32)^2-6(8j-1)^2} + \sum_{\substack{n\geq 0\\ -n\leq j\leq n}} q^{(24n+16)^2-6(8j-1)^2}. \end{aligned} \tag{3.35}$$

Arguing as usual gives (1.21). ☐

Proof of Corollary 1.4. By Theorem 3.3 of [5] we have that

$$q^{-1}\mathcal{Z}_2(q^8) = \frac{1}{2}\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K\\ N(\mathfrak{a})\equiv 7 \pmod 8}} q^{N(\mathfrak{a})}. \tag{3.36}$$

Comparing (3.36) with equations (1.10)–(1.13) gives (1.22). Next, Theorem 1.2 of [7] is equivalent to

$$\mathcal{Z}_3(q) = -\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_L\\ \mathfrak{a}=(x), N(x)<0}} (-1)^{N(\mathfrak{a})}q^{N(\mathfrak{a})}. \tag{3.37}$$

One compares (3.37) with equations (1.14) and (1.15) to obtain (1.23). In Theorem 1.7 of [3] it is shown that

$$q\mathcal{Z}_4(q^3) = -\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_L \\ N(\mathfrak{a})\equiv 1 \pmod 3}} (-1)^{N(\mathfrak{a})} q^{N(\mathfrak{a})}. \tag{3.38}$$

Comparing (3.38) with equations (1.16) and (1.17) gives (1.24). Finally, in Theorem 1.6 of [3] it is shown that

$$q^{-1}\mathcal{Z}_5(q^4) = -\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_L \\ N(\mathfrak{a})\equiv 3 \pmod 4}} q^{N(\mathfrak{a})}. \tag{3.39}$$

The sum on the right-hand side of (3.39) is identical to the sum on the right-hand side of (1.15), giving (1.25). □

4. Questions for further study

The series $\sigma(q)$ has been related to Maass waveforms by Cohen [4] and to quantum modular forms by Zagier [11]. The relation of the series $L_i(q)$ to Maass waveforms could be made precise using work of Zwegers [12], but it is unclear whether there is a relation to quantum modular forms. This is worth investigating. The combinatorics of these series is also worth pursuing. Do they have an elegant partition-theoretic interpretation? Is there a natural explanation for the positivity of their coefficients?

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