QUADRATIC FORMS AND FOUR PARTITION FUNCTIONS MODULO 3

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ABSTRACT. Recently, Andrews, Hirschhorn and Sellers have proven congruences modulo 3 for four types of partitions using elementary series manipulations. In this paper, we generalize their congruences using arithmetic properties of certain quadratic forms.

1. INTRODUCTION

A partition of a non-negative integer n is a non-increasing sequence whose sum is n. An overpartition of n is a partition of n where we may overline the first occurrence of a part. Let $\overline{p}(n)$ denote the number of overpartitions of n, $\overline{p_o}(n)$ the number of overpartitions of n into odd parts, $ped(n)$ the number of partitions of n without repeated even parts and $pod(n)$ the number of partitions of n without repeated odd parts. The generating functions for these partitions are

$$\sum_{n \geq 0} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}, \tag{1.1}$$

$$\sum_{n \geq 0} \overline{p_o}(n)q^n = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.2}$$

$$\sum_{n \geq 0} ped(n)q^n = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$\sum_{n \geq 0} pod(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}, \tag{1.4}$$

where as usual

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

The infinite products in (1.1)–(1.4) are essentially the four different ways one can specialize the product $(-aq;q)_\infty/(bq;q)_\infty$ to obtain a modular form whose level is relatively prime to 3.

A series of four recent papers examined congruence properties for these partition functions modulo 3 [1, 5, 6, 7]. Among the main theorems in these papers are the following congruences (see Theorem 1.3 in [6], Corollary 3.3 and Theorem 3.5 in [1], Theorem 1.1 in [5] and Theorem 3.2 in [7], respectively). For all $n \geq 0$ and $\alpha \geq 0$ we have

$$\overline{p_o}(3^{2\alpha}(An + B)) \equiv 0 \pmod{3}, \tag{1.5}$$

where $An + B = 9n + 6$ or $27n + 9$,

$$ped\left(3^{2\alpha+3}n + \frac{17 \cdot 3^{2\alpha+2} - 1}{8}\right) \equiv ped\left(3^{2\alpha+2}n + \frac{19 \cdot 3^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod 3, \qquad (1.6)$$

$$\overline{p}(3^{2\alpha}(27n + 18)) \equiv 0 \pmod 3 \qquad (1.7)$$

and

$$pod\left(3^{2\alpha+3} + \frac{23 \cdot 3^{2\alpha+2} + 1}{8}\right) \equiv 0 \pmod 3. \qquad (1.8)$$

We note that congruences modulo 3 for $\overline{p}(n)$, $\overline{p}_o(n)$ and $ped(n)$ are typically valid modulo 6 or 12. The powers of 2 enter trivially (or nearly so), however, so we do not mention them here.

The congruences in (1.5)–(1.8) are proven in [1, 5, 6, 7] using elementary series manipulations. If we allow ourselves some elementary number theory, we find that much more is true.

With our first result we exhibit formulas for $\overline{p}_o(3n)$ and $ped(3n + 1)$ modulo 3 for all $n \geq 0$. These formulas depend on the factorization of n, which we write as

$$n = 2^a 3^b \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j}, \qquad (1.9)$$

where $p_i \equiv 1, 5, 7$ or $11 \pmod{24}$ and $q_j \equiv 13, 17, 19$ or $23 \pmod{24}$. Further, let t denote the number of prime factors of n (counting multiplicity) that are congruent to 5 or 11 (mod 24). Let $R(n, Q)$ denote the number of representations of n by the quadratic form Q.

Theorem 1.1. *For all $n \geq 0$ we have*
$$\overline{p}_o(3n) \equiv f(n)R(n, x^2 + 6y^2) \pmod 3$$
and
$$ped(3n + 1) \equiv (-1)^{n+1}R(8n + 3, 2x^2 + 3y^2) \pmod 3,$$
where $f(n)$ is defined by
$$f(n) = \begin{cases} -1, & n \equiv 1, 6, 9, 10 \pmod{12}, \\ 1, & otherwise. \end{cases}$$
Moreover, we have
$$\overline{p}_o(3n) \equiv f(n)(1 + (-1)^{a+b+t}) \prod_{i=1}^{r}(1 + v_i) \prod_{j=1}^{s}\left(\frac{1 + (-1)^{w_j}}{2}\right) \pmod 3 \qquad (1.10)$$
and
$$(-1)^n ped(3n + 1) \equiv \overline{p}_o(48n + 18) \pmod 3. \qquad (1.11)$$

There are many ways to deduce congruences from Theorem 1.1. For example, calculating the possible residues of $x^2 + 6y^2$ modulo 9 we see that

$$R(3n + 2, x^2 + 6y^2) = R(9n + 3, x^2 + 6y^2) = 0,$$

and then (1.10) implies that $\overline{p}_o(27n) \equiv \overline{p}_o(3n) \pmod 3$. This gives (1.5). The congruences in (1.6) follow from those in (1.5) after replacing $48n + 18$ by $3^{2\alpha}(48(3n+2) + 18)$ and $3^{2\alpha}(48(9n + 6) + 18)$ in (1.11). We record two more corollaries, which also follow readily from Theorem 1.1.

Corollary 1.2. *For all $n \geq 0$ and $\alpha \geq 0$ we have*

$$\overline{p}_o(2^{2\alpha}(An + B)) \equiv 0 \pmod{3},$$

where $An + B = 24n + 9$ or $24n + 15$.

Corollary 1.3. *If $\ell \equiv 1, 5, 7$ or $11 \pmod{24}$ is prime, then for all n with $\ell \nmid n$ we have*

$$\overline{p}_o(3\ell^2 n) \equiv 0 \pmod{3}. \tag{1.12}$$

For the functions $\overline{p}(3n)$ and $pod(3n + 2)$ we have relations not to binary quadratic forms but to $r_5(n)$, the number of representations of n as the sum of five squares. Our second result is the following.

Theorem 1.4. *For all $n \geq 0$ we have*

$$\overline{p}(3n) \equiv (-1)^n r_5(n) \pmod{3}$$

and

$$pod(3n + 2) \equiv (-1)^n r_5(8n + 5) \pmod{3}.$$

Moreover, for all odd primes ℓ and $n \geq 0$, we have

$$\overline{p}(3\ell^2 n) \equiv \left(\ell - \ell\left(\frac{n}{\ell}\right) + 1\right) \overline{p}(3n) - \ell\overline{p}(3n/\ell^2) \pmod{3} \tag{1.13}$$

and

$$(-1)^{n+1} pod(3n + 2) \equiv \overline{p}(24n + 15) \pmod{3}, \tag{1.14}$$

where $\left(\frac{\bullet}{\ell}\right)$ denotes the Legendre symbol.

Here we have taken $\overline{p}(3n/\ell^2)$ to be 0 unless $\ell^2 \mid 3n$. Again there are many ways to deduce congruences. For example, (1.7) follows readily upon combining (1.13) in the case $\ell = 3$ with the fact that

$$r_5(9n + 6) \equiv 0 \pmod{3},$$

which is a consequence of the fact that $R(9n + 6, x^2 + y^2 + 3z^2) = 0$. One can check that (1.8) follows similarly. For another example, we may apply (1.13) with n replaced by $n\ell$ for $\ell \equiv 2 \pmod{3}$ to obtain

Corollary 1.5. *If $\ell \equiv 2 \pmod{3}$ is prime and $\ell \nmid n$, then*

$$\overline{p}(3\ell^3 n) \equiv 0 \pmod{3}.$$

2. Proofs of Theorems 1.1 and 1.4

Proof of Theorem 1.1. On page 364 of [6] we find the identity

$$\sum_{n \geq 0} \overline{p}_o(3n)q^n = \frac{D(q^3)D(q^6)}{D(q)^2},$$

where

$$D(q) := \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}.$$

Reducing modulo 3, this implies that

$$\sum_{n \geq 0} \overline{p_o}(3n)q^n \equiv \sum_{x,y \in \mathbb{Z}} (-1)^{x+y} q^{x^2+6y^2} \pmod{3}$$

$$\equiv \sum_{n \geq 0} f(n) R(n, x^2 + 6y^2) q^n \pmod{3}.$$

Now it is known (see Corollary 4.2 of [3], for example) that if n has the factorization in (1.9), then

$$R(n, x^2 + 6y^2) = (1 + (-1)^{a+b+t}) \prod_{i=1}^{r} (1 + v_i) \prod_{j=1}^{s} \left(\frac{1 + (-1)^{w_j}}{2} \right). \tag{2.1}$$

This gives (1.10). Next, from [1] we find the identity

$$\sum_{n \geq 0} ped(3n+1)q^n = \frac{D(q^3)\psi(-q^3)}{D(q)^2},$$

where

$$\psi(q) := \sum_{n \geq 0} q^{n(n+1)/2}.$$

Reducing modulo 3, replacing q by $-q^8$ and multiplying by q^3 gives

$$\sum_{n \geq 0} (-1)^{n+1} ped(3n+1)q^{8n+3} \equiv \sum_{n \geq 0} R(8n+3, 2x^2 + 3y^2)q^{8n+3} \pmod{3}.$$

It is known (see Corollary 4.3 of [3], for example) that if n has the factorization given in (1.9), then

$$R(n, 2x^2 + 3y^2) = (1 - (-1)^{a+b+t}) \prod_{i=1}^{r} (1 + v_i) \prod_{j=1}^{s} \left(\frac{1 + (-1)^{w_j}}{2} \right).$$

Comparing with (2.1) finishes the proof of (1.11). $\qquad \square$

Proof of Theorem 1.4. On page 3 of [5] we find the identity

$$\sum_{n \geq 0} \overline{p}(3n)q^n \equiv \frac{D(q^3)^2}{D(q)} \pmod{3}.$$

Reducing modulo 3 and replacing q by $-q$ yields

$$\sum_{n \geq 0} (-1)^n \overline{p}(3n)q^n \equiv \sum_{n \geq 0} r_5(n)q^n \pmod{3}.$$

It is known (see Lemma 1 in [4], for example) that for any odd prime ℓ we have

$$r_5(\ell^2 n) = \left(\ell^3 - \ell \left(\frac{n}{\ell} \right) + 1 \right) r_5(n) - \ell^3 r_5(n/\ell^2).$$

Here $r_5(n/\ell^2) = 0$ unless $\ell^2 \mid n$. Replacing $r_5(n)$ by $(-1)^n \overline{p}(3n)$ throughout gives (1.13). Now equation (1) of [7] reads

$$\sum_{n \geq 0} (-1)^n pod(3n+2)q^n = \frac{\psi(q^3)^3}{\psi(q)^4}.$$

Reducing modulo 3 we have

$$\sum_{n \geq 0} (-1)^n pod(3n+2)q^n \equiv \psi(q)^5 \pmod{3}$$

$$\equiv \sum_{n \geq 0} r_5(8n+5)q^n \pmod{3}$$

$$\equiv -\sum_{n \geq 0} \overline{p}(24n+15)q^n \pmod{3},$$

where the second congruence follows from Theorem 1.1 in [2]. This implies (1.14) and thus the proof of Theorem 1.4 is complete. $\qquad\square$

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