RANK AND CONJUGATION FOR A SECOND FROBENIUS REPRESENTATION OF AN OVERPARTITION

JEREMY LOVEJOY

Abstract. The notions of rank and conjugation are developed in the context of a second Frobenius representation of an overpartition. Some identities for overpartitions that are invariant under this conjugation are presented.

1. Introduction

An overpartition of n is a partition of n in which the first occurrence of a number may be overlined. Two different notions of "rank" have proven important in the study of overpartitions. The first of these, called just the rank, was essential to understanding the combinatorics of certain q-series identities like the ${}_1\psi_1$ and q-Gauss summations [5, 6]. The rank of an overpartition λ is equal to one less than the largest part $\ell(\lambda)$ minus the number of overlined parts less than $\ell(\lambda)$. The second statistic, called the D-rank, plays an important role in the combinatorics of multiple-series identities that are overpartition analogues of the Andrews-Gordon identities [7, 8]. The D-rank of an overpartition λ is the largest part $\ell(\lambda)$ minus the number of parts $n(\lambda)$, a straight application of Dyson's partition rank [9] to overpartitions.

In this paper we shall introduce and study a third rank, which we call the M_2-rank. Using the notation from the previous paragraph in addition to the symbol λ_o for the partition consisting of the non-overlined odd parts of an overpartition λ, we have the following definition for the M_2-rank.

Definition 1.1. *The M_2-rank of an overpartition λ is*

$$M_2\text{-}rank(\lambda) = \left\lceil \frac{\ell(\lambda)}{2} \right\rceil - n(\lambda) + n(\lambda_o) - \chi(\lambda),$$

where $\chi(\lambda) = 1$ if the largest part of λ is odd and non-overlined and $\chi(\lambda) = 0$ otherwise.

For comparison, we record that the overpartition $(11, 11, \overline{10}, \overline{8}, 8, 7, 5, \overline{3}, 2, 1, 1)$ has rank 7, D-rank 0, and M_2-rank 0, while the overpartition $(\overline{10}, \overline{9}, 7, 7, 7, \overline{5}, 1)$ has rank 7, D-rank 3, and M_2-rank 2. When the overpartition has neither non-overlined odd parts nor overlined even parts, then the M_2-rank becomes the M_2-rank for partitions with no repeated odd parts studied in [2].

The first principal object of this paper is to establish some generating functions for the M_2-rank. We assume that the empty overpartition of 0 has rank 0. In addition to the notations above, we use $e(\lambda)$ and $o(\lambda)$ to denote the number of even parts and the number of odd parts

Date: March 18, 2008.
2000 *Mathematics Subject Classification.* 11P81, 05A17.
The author was partially supported by an ACI "Jeunes Chercheurs et Jeunes Chercheuses".

of λ. Overlining any function indicates that we are only considering the overlined parts. A subscript $<$ means that the largest part is excluded.

Theorem 1.2. *Employing the usual q-series notation* [10], *the coefficient of $a^r b^s c^t z^u q^N$ in*

$$\sum_{n\geq 0} \frac{(-1/a, -q/b; q^2)_n (abcq)^n}{(zq^2, cq^2/z; q^2)_n} \tag{1.1}$$

is equal to the number of overpartitions λ of N such that $r = \overline{e}(\lambda) + n(\lambda_o)$, $s = o(\lambda)$, $t = n(\lambda)$, and $u = M_2$-rank(λ).

Corollary 1.3. *Let $p_2(m,n)$ denote the number of overpartitions of n with M_2-rank m. For any $m \in \mathbb{Z}$,*

$$\sum_{n\geq 1} p_2(m,n)q^n = 2\frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 1} \frac{(-1)^{n+1} q^{n^2+2|m|n}(1-q^{2n})}{(1+q^{2n})}. \tag{1.2}$$

It will be convenient to view (1.1) as the generating function for certain generalized Frobenius partitions, where the summation variable n counts the number of columns. We can then employ a bijection of [5] to transform these Frobenius partitions into overpartitions and establish Theorem 1.2. This is much like the situation in [13], but here the Frobenius partitions are different. This second representation is detailed in Section 2.

The invariance of the generating function (1.1) under the mapping $z \to cz^{-1}$ implies that there should be a nice combinatorial involution on overpartitions that reverses the sign of the M_2-rank. The second main goal of this paper is to define this involution, which will be called $2F$-conjugation. The $2F$-conjugation will be most easily viewed via the second Frobenius representation. It is distinct from the classical Ferrers diagram conjugation of an overpartition as well as the F-conjugation defined in [13]. This conjugation is described in Section 4.

The final goal of the paper is to give some examples of how to use q-series identities to derive identities for overpartitions which are $2F$-self-conjugate. Appearing will be the usual suspects, such as theta functions and divisor functions. The interested reader may turn to Section 5 for a preview.

2. Overpartition basics

In this section we recall some basic facts about overpartitions and detail the second Frobenius representation. The fundamental generating function for overpartitions is the following proposition. We invite the reader to recall the definition of the rank of an overpartition from the introduction.

Proposition 2.1. *The coefficient of $a^r z^s q^N$ in*

$$\frac{(-a;q)_n q^n}{(zq;q)_n} \tag{2.1}$$

is equal to the number of overpartitions λ of N with n parts, r overlined parts and rank s

Proof. The function $q^k/(zq;q)_k$ generates a partition λ into k positive (non-overlined) parts, where the exponent of z records the largest part minus 1. Note that since there are not yet any overlined parts, this is the same as the rank. Now $(-a;q)_k$ generates a partition $\mu = \mu_1 + \cdots + \mu_j$ into distinct non-negative parts less than k, with the exponent of a tracking the number of parts.

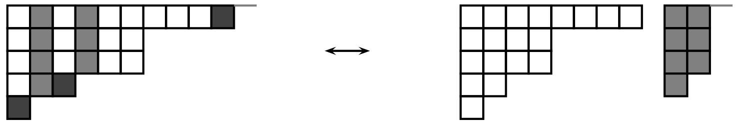

Figure 1. An example of the Joichi-Stanton map

For each of these μ_i beginning with the largest, we add 1 to the first μ_i parts of λ, and then overline the (μ_i+1)th part of λ. Here the parts of λ are written in non-increasing order. This operation leaves the rank invariant and counts one overlined part for each part of μ. For example, if $k=5$, $\lambda = 8+4+4+2+1$, and $\mu = 4+3+0$, then we have

$$\begin{aligned}(8+4+4+2+1, 4+3+0) &\iff (9+5+5+3+\overline{1}, 3+0)\\ &\iff (10+6+6+\overline{3}+\overline{1}, 0)\\ &\iff (\overline{10}+6+6+\overline{3}+\overline{1})\end{aligned}$$

The result is obviously an overpartition and the process is easily inverted. A graphical version of this example is presented in Figure 1, where the parts of μ are the lightly shaded columns and the parts of λ are the white rows. □

In the sequel we will refer to the bijection in the proof of Proposition 2.1 as the "Joichi-Stanton map", since to our knowledge it was in [11] that the mapping was first described (with $z=1$). Actually, we will use several corollaries of Proposition 2.1, each of which corresponds to a slight variation of the Joichi-Stanton map. For example, the proposition implies that $(-q;q^2)_n/(q^2;q^2)_n$ is the generating function for partitions into exactly n non-negative parts where the odd parts may not repeat. In the corresponding Joichi-Stanton map, we proceed as in the proof above, except the partition λ becomes a partition into n non-negative even parts, the partition μ becomes a partition into distinct odd parts less than $2n$, and a part $\mu_i = 2x-1$ is applied to λ by adding two to the $x-1$ largest parts of λ and one to the xth largest part. The reader should have no trouble dealing with these minor variations of the Joichi-Stanton map.

It has often been convenient to regard an overpartition as a generalized Frobenius partition. A generalized Frobenius partition, or, more succinctly, an F-partition of n [1] is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix} \tag{2.2}$$

where $\sum a_i$ is a partition taken from a set A, $\sum b_i$ is a partition taken from a set B, and $\sum(a_i+b_i)=n$. The number of such Frobenius partitions of n is denoted by $p_{A,B}(n)$. If Q denotes the set of partitions into distinct parts and if $\mathcal{O}$ denotes the set of overpartitions into non-negative parts, then $p_{Q,\mathcal{O}}(n)$ is equal to the number of overpartitions of n. This fact is equivalent to the q-Chu-Vandermonde identity [6].

There is another representation of an overpartition as an F-partition, the existence of which is equivalent to the q-Gauss summation, which we may write

$$\sum_{n\geq 0} \frac{(-1/a, -q/b; q^2)_n (abcq)^n}{(cq^2, q^2; q^2)_n} = \frac{(-acq^2, -bcq; q^2)_\infty}{(cq^2, abcq; q^2)_\infty}. \tag{2.3}$$

This second Frobenius representation first arose in the context of multiple q-series identities of the Rogers-Ramanujan type [12]. It will perhaps be observed that in Proposition 2.2 below we actually describe an F-partition that is slightly different from the one in [12]. However, these are equivalent and easily identified, so we shall refer to them both as the second Frobenius representation of an overpartition.

Proposition 2.2. *Let $\mathcal{D}$ denote the set of overpartitions into odd parts. Let $\mathcal{E}$ denote the set of partitions into non-negative integers where odd parts may not repeat. Then there is a one-to-one correspondence between overpartitions λ of N and generalized Frobenius partitions ν of N counted by $p_{\mathcal{D},\mathcal{E}}(N)$ such that (i) $n(\nu_1) - \overline{n}(\nu_1) = \overline{e}(\lambda) + n(\lambda_o)$, (ii) $e(\nu_2) = o(\lambda)$, and (iii) $\frac{\ell(\nu_1)-1}{2} - \overline{n}_<(\nu_1) + n(\nu_2) = n(\lambda)$.*

Proof. We begin with a combinatorial interpretation of the sum on the left hand side of (2.3). By Proposition 2.1, we may deduce that $a^n q^n(-1/a;q^2)_n/(cq^2;q^2)_n$ is the generating function for overpartitions $\nu_1 \in \mathcal{D}$ having exactly n parts, where the exponent of a counts the number of non-overlined parts $(n(\nu_1) - \overline{n}(\nu_1))$ and the exponent of c counts half of one less than the largest part minus the number of overlined parts that are less than the largest part (i.e., $\frac{\ell(\nu_1)-1}{2} - \overline{n}_<(\nu_1)$). Again by Proposition 2.1, $b^n c^n(-q/b;q^2)_n/(q^2;q^2)_n$ is the generating function for partitions $\nu_2 \in \mathcal{E}$ having exactly n parts, where the exponent of b counts the number of even parts $(e(\nu_2))$ and the exponent of c counts the number of parts $(n(\nu_2))$. In other words,

$$\frac{(-1/a,-q/b;q^2)_n(abcq)^n}{(cq^2,q^2;q^2)_n}$$

is the generating function for those generalized Frobenius partitions ν counted by $p_{\mathcal{D},\mathcal{E}}(N)$ having n columns, and where the exponent of a is $n(\nu_1) - \overline{n}(\nu_1)$, the exponent of b is $e(\nu_2)$, and the exponent of c is $\frac{\ell(\nu_1)-1}{2} - \overline{n}_<(\nu_1) + n(\nu_2)$.

Now, of course, in light of (2.3), the proposition is established. However, it will be essential to have a simple bijection between an overpartition and its second Frobenius representation, a bijection which we now describe. Beginning with an object ν counted by $p_{\mathcal{D},\mathcal{E}}(N)$, we want to map ν to an overpartition λ. The mapping we use is really just a dilation of the bijection presented in [5], so we shall not be too profuse with the details.

Step 1 - Expand. If an odd integer k less than the largest part on top does not occur overlined on top, we insert it, overlined and in its correct position. The integer $-k$ is placed (anywhere) in the bottom row.

Step 2 - Reorder. Reorder the entries on the bottom row, starting on the left with the odd integers in increasing order, followed by the even numbers in non-increasing order.

Step 3 - Add. Add column-by-column to obtain an overpartition. If the entry in the top row is non-overlined and the entry in the bottom row is odd, or if the top entry is overlined and the bottom entry is even, then the sum is overlined. Otherwise it is non-overlined.

For example, if

$$\nu = \begin{pmatrix} \overline{11} & 11 & 9 & 5 & 5 & 5 & \overline{3} & \overline{1} \\ 8 & 8 & 7 & 5 & 3 & 2 & 2 & 1 \end{pmatrix},$$

then after Step 1 we have

$$\nu = \begin{pmatrix} \overline{11} & 11 & \overline{9} & 9 & \overline{7} & \overline{5} & 5 & 5 & 5 & \overline{3} & \overline{1} \\ -9 & -7 & -5 & 8 & 8 & 7 & 5 & 3 & 2 & 2 & 1 \end{pmatrix},$$

after Step 2 we have

$$\nu = \left(\begin{array}{ccccccc|cccc} \overline{11} & 11 & \overline{9} & 9 & \overline{7} & \overline{5} & 5 & 5 & 5 & \overline{3} & \overline{1} \\ -9 & -7 & -5 & 1 & 3 & 5 & 7 & 8 & 8 & 2 & 2 \end{array} \right), \tag{2.4}$$

and the overpartition obtained after Step 3 is

$$\lambda = (13, 13, \overline{12}, \overline{10}, 10, 10, \overline{5}, \overline{4}, 4, \overline{3}, 2)$$

Just as in [5], the mapping is invertible, and a little consideration reveals that the equalities (i), (ii), and (iii) in the statement of the proposition do indeed hold. □

Before continuing we might point out that finding a bijective proof of the case $b, c = 1$ and $a = 0$ of (2.3) still masquerades as an open problem (*e.g.* [14, (4.4.2)]), even though we have just seen that such a bijection was already implied by the work in [5].

3. Generating functions

We are now ready to prove Theorem 1.2. Suppose that we have an overpartition λ whose second Frobenius representation is ν. In light of the work in the previous section, we need only describe the exponent of z in the series (1.1) entirely in terms of λ. Again, the term

$$\frac{(-1/a; q^2)_n (aq)^n}{(cq^2/z; q^2)_n}$$

is the generating function for the top row ν_1, and we deduce that the exponent of z is $\overline{n}_<(\nu_1) - (\ell(\nu_1) - 1)/2$. Since the number of parts in the overpartition λ is equal to the negative of this quantity plus the number of columns of ν, the exponent of z in the above term is $n(\nu_1) - n(\lambda)$.

Now the exponent of z in the term

$$\frac{(-q/b; q^2)_n (bc)^n}{(zq^2; q^2)_n},$$

which corresponds to the bottom row ν_2 of the F-partition, is

$$\left\lfloor \frac{\ell(\nu_2)}{2} \right\rfloor - o_<(\nu_2).$$

So far, then, we have that the exponent of z is equal to $\lfloor \frac{\ell(\nu_2)}{2} \rfloor - o_<(\nu_2) + n(\nu_1) - n(\lambda)$. To completely convert this expression into functions of λ, we need to consider four cases, depending on the parities of the largest part of λ and the largest part of ν_2. Conveniently, we'll obtain the M_2-rank in each case. We note, before continuing, that if $\ell(\lambda)$ is odd then $\ell(\nu_2)$ cannot be odd.

Case 1: $\ell(\lambda)$ is even and $\ell(\nu_2)$ is odd. Then $\ell(\lambda)$ comes from the column just to the left of the bar in an array like the one in (2.4). We compute that

$$\begin{aligned} \ell(\lambda) &= \ell(\nu_2) + 2e(\nu_2) + 1 - 2n(\lambda_o) \\ &= \ell(\nu_2) + 2e(\nu_2) + 1 + 2o_<(\nu_2) - 2o_<(\nu_2) - 2n(\lambda_o) \\ &= 2\left\lfloor \frac{\ell(\nu_2)}{2} \right\rfloor + 1 + 2n(\nu_2) - 1 - 2o_<(\nu_2) - 2n(\lambda_o). \end{aligned}$$

Hence the exponent of z is equal to

$$\begin{aligned}\left\lfloor\frac{\ell(\nu_2)}{2}\right\rfloor - o_<(\nu_2) + n(\nu_1) - n(\lambda) &= \frac{1}{2}(\ell(\lambda) - 2n(\nu_2) + 2n(\lambda_o)) - (n(\lambda) - n(\nu_1)) \\ &= \left\lceil\frac{\ell(\lambda)}{2}\right\rceil - n(\lambda) + n(\lambda_o),\end{aligned}$$

which is the M_2-rank.

Case 2: $\ell(\lambda)$ is even and $\ell(\nu_2)$ is even. Observe that this can only happen if the largest odd part on the bottom is one less than $\ell(\nu_2)$. We compute that

$$\begin{aligned}\ell(\lambda) &= \ell(\nu_2) - 1 + 2e(\nu_2) + 1 - 2n(\lambda_o) \\ &= \ell(\nu_2) + 2e(\nu_2) + 2o_<(\nu_2) - 2o_<(\nu_2) - 2n(\lambda_o) \\ &= 2\left\lfloor\frac{\ell(\nu_2)}{2}\right\rfloor + 2n(\nu_2) - 2o_<(\nu_2) - 2n(\lambda_o).\end{aligned}$$

Then, as above, the exponent of z is equal to

$$\begin{aligned}\left\lfloor\frac{\ell(\nu_2)}{2}\right\rfloor - o_<(\nu_2) + n(\nu_1) - n(\lambda) &= \frac{1}{2}(\ell(\lambda) - 2n(\nu_2) + 2n(\lambda_o)) - (n(\lambda) - n(\nu_1)) \\ &= \left\lceil\frac{\ell(\lambda)}{2}\right\rceil - n(\lambda) + n(\lambda_o),\end{aligned}$$

which is again the M_2-rank.

Case 3: $\ell(\lambda)$ is odd and overlined, and $\ell(\nu_2)$ is even. Now $\ell(\lambda)$ comes from the column just to the right of the bar in a symbol like (2.4), and the top entry of this column is overlined. We compute that

$$\begin{aligned}\ell(\lambda) &= \ell(\nu_2) + 2e(\nu_2) - 1 - 2n(\lambda_o) \\ &= \ell(\nu_2) + 2e(\nu_2) - 1 + 2o_<(\nu_2) - 2o_<(\nu_2) - 2n(\lambda_o) \\ &= 2\left\lfloor\frac{\ell(\nu_2)}{2}\right\rfloor + 2n(\nu_2) - 1 - 2o_<(\nu_2) - 2n(\lambda_o).\end{aligned}$$

Then, the exponent of z is equal to

$$\begin{aligned}\left\lfloor\frac{\ell(\nu_2)}{2}\right\rfloor - o_<(\nu_2) + n(\nu_1) - n(\lambda) &= \frac{1}{2}(\ell(\lambda) + 1 - 2n(\nu_2) + 2n(\lambda_o)) - (n(\lambda) - n(\nu_1)) \\ &= \left\lceil\frac{\ell(\lambda)}{2}\right\rceil - n(\lambda) + n(\lambda_o),\end{aligned}$$

which is still the M_2-rank.

Case 4: $\ell(\lambda)$ is odd and non-overlined, and $\ell(\nu_2)$ is even. Here $\ell(\lambda)$ again corresponds to the column to the right of the bar, but now the top entry is non-overlined. We compute that

$$\begin{aligned}\ell(\lambda) &= \ell(\nu_2) + 2e(\nu_2) - 1 - 2n_<(\lambda_o) \\ &= \ell(\nu_2) + 2e(\nu_2) - 1 + 2o_<(\nu_2) - 2o_<(\nu_2) - 2n_<(\lambda_o) \\ &= 2\left\lfloor\frac{\ell(\nu_2)}{2}\right\rfloor + 2n(\nu_2) - 1 - 2o_<(\nu_2) - 2n_<(\lambda_o).\end{aligned}$$

Then, the exponent of z is equal to

$$\begin{aligned}\left\lfloor\frac{\ell(\nu_2)}{2}\right\rfloor - o_<(\nu_2) + n(\nu_1) - n(\lambda) &= \frac{1}{2}(\ell(\lambda) + 1 - 2n(\nu_2) + 2n_<(\lambda_o)) - (n(\lambda) - n(\nu_1)) \\ &= \left\lceil\frac{\ell(\lambda)}{2}\right\rceil - n(\lambda) + n_<(\lambda_o),\end{aligned}$$

which is one more time the M_2-rank, since here the largest part of λ is odd and non-overlined. □

We might remark that the definition of M_2-rank in terms of the second F-representation of an overpartition λ is contained in the first two paragraphs of above proof. We record this alternative definition here.

Definition 3.1. *In terms of its second F-representation ν, the M_2-rank of an overpartition is*

$$\overline{n}_<(\nu_1) - (\ell(\nu_1) - 1)/2 + \left\lfloor\frac{\ell(\nu_2)}{2}\right\rfloor - o_<(\nu_2).$$

Proof of Corollary 1.3: In (a limiting case of) Watson's transformation,

$$\sum_{n=0}^{\infty}\frac{(aq/bc,d,e)_n(\frac{aq}{de})^n}{(q,aq/b,aq/c)_n} = \frac{(aq/d,aq/e)_\infty}{(aq,aq/de)_\infty}\sum_{n=0}^{\infty}\frac{(a,\sqrt{a}q,-\sqrt{a}q,b,c,d,e)_n(aq)^{2n}(-1)^nq^{n(n-1)/2}}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e)_n(bcde)^n}, \tag{3.1}$$

take $q = q^2, a = 1, b = 1/z, c = z, d = -1$, and $e = -q$. The result is

$$\begin{aligned}\sum_{n\geq 0}\frac{(-1)_{2n}q^n}{(zq^2;q^2)_n(q^2/z;q^2)_n} &= \frac{(-q)_\infty}{(q)_\infty}\left(1 + 2\sum_{n\geq 1}\frac{(1/z,z;q^2)_n(-1)^nq^{n^2+2n}}{(zq^2,q^2/z;q^2)_n}\right) \\ &= \frac{(-q)_\infty}{(q)_\infty}\left(1 + 2\sum_{n\geq 1}\frac{(1-z)(1-1/z)(-1)^nq^{n^2+2n}}{(1-zq^{2n})(1-q^{2n}/z)}\right).\end{aligned}$$

Now it is easily verified that

$$\frac{(1-z)(1-1/z)q^{2n}}{(1-zq^{2n})(1-q^{2n}/z)} = 1 - \frac{(1-q^{2n})}{(1+q^{2n})}\sum_{m=0}^{\infty}z^mq^{2mn} - \frac{(1-q^{2n})}{(1+q^{2n})}\sum_{m=1}^{\infty}z^{-m}q^{2mn}.$$

Substituting this into the above equation and picking off the coefficient of z^m on both sides gives the desired result for $m \neq 0$. When $m = 0$, since $p_2(0,0) = 1$ we have

$$\sum_{n\geq 1}p_2(0,n)q^n = -1 + \frac{(-q)_\infty}{(q)_\infty}\left(1 + 2\sum_{n\geq 1}(-1)q^{n^2} + 2\sum_{n\geq 1}\frac{(-1)^{n+1}q^{n^2}(1-q^{2n})}{(1+q^{2n})}\right),$$

which gives the desired result upon recalling that

$$1 + 2\sum_{n\geq 1}(-1)^nq^{n^2} = \frac{(q)_\infty}{(-q)_\infty}.$$

□

4. The $2F$-Conjugation of an overpartition

We now describe the $2F$-conjugation of an overpartition. The idea is to swap the roles played by the partitions into even parts generated by $1/(cq^2/z;q^2)_n$ and $1/(zq^2;q^2)_n$ in (1.1).

Definition 4.1 (The $2F$-conjugation of an overpartition)**.** *Given an overpartition λ with second F-representation ν, the $2F$-conjugate λ' of λ is the overpartition whose second F-representation is ν', where ν' is obtained from ν in the following way: First, remove 1 from each of the n parts of the top row ν_1 to obtain $\tilde{\nu_1}$. Next, use the Joichi-Stanton map to decompose $\tilde{\nu_1}$ into a partition into n even non-negative parts λ_1 and a partition into distinct even non-negative parts less than $2n$, μ_1. Then use the Joichi-Stanton map to decompose the bottom row ν_2 into a partition into n even non-negative parts λ_2 and a partition into distinct odd parts less than $2n$, μ_2. Now, λ_2 and μ_1 are reassembled using the Joichi-Stanton map into an overpartition $\tilde{\nu_1}'$, to each part of which 1 is added to obtain the top row ν_1' of ν'. Finally, λ_1 and μ_2 are combined (using the Joichi-Stanton map, of course) to form the bottom row ν_2' of ν'.*

Two remarks are in order here. First, the $2F$-conjugation is clearly an involution. Second, since λ_1 is a partition generated by $1/(cq^2/z;q^2)_n$ and λ_2 is a partition generated by $1/(zq^2;q^2)_n$, the act of swapping them negates the exponent of z, which is the M_2-rank. In other words, the $2F$-conjugation reverses the sign of the M_2-rank.

For an example of this involution, consider the overpartition $\lambda = (13, 13, \overline{12}, \overline{10}, 10, 10, \overline{5}, \overline{4}, 4, \overline{3}, 2)$, whose second F-representation ν we saw in Section 3 is

$$\nu = \begin{pmatrix} \overline{11} & 11 & 9 & 5 & 5 & 5 & \overline{3} & \overline{1} \\ 8 & 8 & 7 & 5 & 3 & 2 & 2 & 1 \end{pmatrix}.$$

Using the Joichi-Stanton map, we decompose the top row ν_1 into a partition of ones, $(1,1,1,1,1,1,1,1)$, a partition into distinct non-negative even parts $(0,12,14)$, and a partition λ_1 into even non-negative parts $(6,6,4,0,0,0,0,0)$. The bottom row ν_2 is decomposed into a partition into distinct odd parts $(5,7,9,15)$ and a partition λ_2 into non-negative even parts $(0,0,0,0,0,0,0,0)$. Swapping λ_1 and λ_2 and reassembling using the Joichi-Stanton map gives a new overpartition

$$\nu' = \begin{pmatrix} \overline{5} & 5 & 5 & 5 & 5 & 5 & \overline{3} & \overline{1} \\ 14 & 14 & 11 & 5 & 3 & 2 & 2 & 1 \end{pmatrix}.$$

Following the bijection from the previous section, we see that $\lambda' = (19, 19, \overline{16}, \overline{10}, \overline{8}, 6, \overline{5}, \overline{3})$. Notice that the M_2-rank of λ is $\lceil 13/2 \rceil - 11 + 1 = -3$, while the M_2-rank of λ' is $\lceil 19/2 \rceil - 8 + 1 = 3$. The reader familiar with [13] may compute that the classical Ferrers conjugate as well as the F-conjugate of λ are not the same as λ'.

For another example, the reader may verify that the overpartition $(\overline{15}, 14, \overline{13}, 10, 8, 7, \overline{6}, \overline{3}, 3, 2)$ is $2F$-self-conjugate, its second Frobenius representation being

$$\begin{pmatrix} 13 & 13 & \overline{9} & \overline{5} & 3 & \overline{1} & 1 \\ 10 & 10 & 7 & 4 & 2 & 2 & 1 \end{pmatrix}.$$

5. Identities for $2F$-self-conjugate overpartitions

We now proceed to use elementary q-series identities to give a sample of five identities involving $2F$-self-conjugate overpartitions. The decomposition used to define $2F$-conjugation in the previous section can be used to give the generating function for $2F$-self-conjugate overpartitions. We simply count those overpartitions for which $\lambda_1 = \lambda_2$. So, the generating function for $2F$-self-conjugate overpartitions is

$$\sum_{n\geq 0} \frac{(-1)_{2n}q^n}{(q^4;q^4)_n} = 1 + 2q + 4q^2 + 4q^3 + 6q^4 + 12q^5 + \cdots . \tag{5.1}$$

Theorem 5.1. *Let $f(n)$ denote the number of overpartitions of n such that (i) there is at least one even non-overlined part, (ii) if $2k$ is the smallest such part, then $\overline{2k}$ does not occur, and (iii) all odd overlined parts are bigger than $2k$. Let $f^{\pm}(n)$ denote the number of overpartitions counted by $f(n)$ such that the number of even non-overlined parts plus the number of even overlined parts greater than $2k$ is even/odd. Then for $n \geq 1$ the number of $2F$-self-conjugate overpartitions of n is equal to*

$$\begin{cases} 4(f^-(n) - f^+(n)) + 2, & \text{if } n \text{ is a square,} \\ 4(f^-(n) - f^+(n)), & \text{otherwise.} \end{cases} \tag{5.2}$$

Proof. Starting with the second iteration of Heine's transformation [10],

$$\sum_{n=0}^{\infty} \frac{(a,b)_n z^n}{(c,q)_n} = \frac{(c/b, bz)_\infty}{(c,z)_\infty} \sum_{n=0}^{\infty} \frac{(abz/c, b)_n (c/b)^n}{(q, bz)_n}, \tag{5.3}$$

under the substitutions $q = q^2, a = -q, b = -1, c = -q^2$, and $z = q$, we have

$$\begin{aligned} \sum_{n\geq 0} \frac{(-1)_{2n}q^n}{(q^4;q^4)_n} &= \frac{(-q,q^2;q^2)_\infty}{(q,-q^2;q^2)_\infty} \sum_{n\geq 0} \frac{(-1;q^2)_n^2 q^{2n}}{(-q,q^2;q^2)_n} \\ &= \frac{(-q,q^2;q^2)_\infty}{(q,-q^2;q^2)_\infty} \left(1 + 4\sum_{n\geq 1} \frac{(-q^2;q^2)_{n-1}^2 q^{2n}}{(-q,q^2;q^2)_n} \right) \\ &= \sum_{n\in\mathbb{Z}} q^{n^2} + \frac{4}{(q;q^2)_\infty} \sum_{n\geq 1} \frac{(-q^2;q^2)_{n-1}(-q^{2n+1}, q^{2n+2};q^2)_\infty q^{2n}}{(-q^{2n};q^2)_\infty}. \end{aligned}$$

It is straightforward to see that the coefficient of q^n on the right hand side is the quantity in (5.2). □

Theorem 5.2. *Let $g^{\pm}(n)$ denote the number of overpartitions of n such that the M_2-rank of the sub-partition consisting only of even non-overlined parts and odd overlined parts is even/odd. Then the number of $2F$-self-conjugate overpartitions of n is equal to $g^+(n) - g^-(n)$.*

Proof. From Jackson's transformation,

$$\sum_{n=0}^{\infty} \frac{(a,b)_n z^n}{(c,q)_n} = \frac{(az)_\infty}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(a, c/b)_n (-bz)^n q^{n(n-1)/2}}{(c, az, q)_n}, \tag{5.4}$$

under the substitutions $q = q^2, a = -q, b = -1, c = -q^2$, and $z = q$, we have

$$\sum_{n\geq 0} \frac{(-1)_{2n}q^n}{(q^4;q^4)_n} = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n\geq 0} \frac{(-q;q^2)_n q^{n^2}}{(-q^2;q^2)_n^2}. \tag{5.5}$$

Letting $a = 0, b = 1, c = 1$, and $z = -1$ in Theorem 1.2 reveals that the right hand side is indeed the generating function for $g^+(n) - g^-(n)$.

□

Theorem 5.3. *Let $h^{\pm}(n)$ denote the number of $2F$-self-conjugate overpartitions of n whose second F-representation has an even/odd number of odd parts in the bottom row. Then for $n \geq 1$ we have*

$$h^+(n) - h^-(n) = \begin{cases} 2, & \text{if } n \text{ is a square,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This follows directly from the case $a = 1, b = -1$, and $c = -1$ of (2.3),

$$\begin{aligned} \sum_{n\geq 0} \frac{(-1, q; q^2)_n q^n}{(q^4; q^4)_n} &= \frac{(-q, q^2; q^2)_\infty}{(q, -q^2; q^2)_\infty} \\ &= \sum_{n\in\mathbb{Z}} q^{n^2}. \end{aligned}$$

□

Theorem 5.4. *Let $i(n)$ denote the number of divisors d of n, counted negatively if $d \equiv 0 \pmod 4$. Let $j^{\pm}(n)$ denote the number of $2F$-self-conjugate overpartitions of n such that the number of overlined parts less than the largest in the top row of the second F-representation is even/odd. Then for $n \geq 1$ we have $j^+(n) - j^-(n) = 2i(n)$.*

Proof. Take $b = 1$ and $c = -1$ in (2.3), differentiate with respect to a, set $a = -1$ and multiply both sides by -2. Using the fact that if $f = \prod_n f_n$, then

$$f' = f \sum_n \frac{f_n'}{f_n}, \tag{5.6}$$

as well as the fact that for $n \geq 1$,

$$\left.\frac{d}{dx}\right|_{x=1} (x)_n = -(q)_{n-1}, \tag{5.7}$$

one computes that

$$2 \sum_{n\geq 1} \frac{(-q; q^2)_n (q^2; q^2)_{n-1} q^n}{(q^4; q^4)_n} = 2 \left(\sum_{n\geq 1} \frac{q^{2n}}{1 + q^{2n}} + \sum_{n\geq 1} \frac{q^{2n-1}}{1 - q^{2n-1}} \right).$$

Expanding each term in the first sum on the right hand side as

$$q^{2n} - q^{2\cdot 2n} + q^{3\cdot 2n} - q^{4\cdot 2n} + \cdots,$$

and each term in the second sum on the right hand side as

$$q^{1\cdot(2n-1)} + q^{2\cdot(2n-1)} + q^{3\cdot(2n-1)} + q^{4\cdot(2n-1)} + \cdots$$

reveals the generating function for $2i(n)$. For the left hand side, it is sufficient to note that the proof of Proposition 2.1 shows that $2q^n(q^2; q^2)_{n-1}/(q^2; q^2)_n$ is the generating function for overpartitions λ into n odd parts weighted by $(-1)^{\overline{n}_<(\lambda)}$. □

Theorem 5.5. *Let $k(n)$ denote the number of odd divisors of n minus the number of even divisors of n plus $2(-1)^n$ times the number of odd divisors of n that exceed $\sqrt{2n}$. Let $l^{\pm}(n)$ denote the number of $2F$-self-conjugate overpartitions of n such that the number of overlined parts less than the largest on the top row of the second F-representation plus the number of odd parts on the bottom row of the second F-representation is even/odd. Then for $n \geq 1$ we have $l^{+}(n) - l^{-}(n) = 2k(n)$.*

Proof. Taking $q = q^2, a = 1, b = -1, c \to \infty$, and $d = q$ in (3.1), we get

$$\sum_{n\geq 0} \frac{(q;q^2)_n (e;q^2)_n (\frac{q}{e})^n}{(q^4;q^4)_n} = \frac{(q;q^2)_\infty (\frac{q^2}{e};q^2)_\infty}{(q^2;q^2)_\infty (\frac{q}{e};q^2)_\infty} \left(1 + 2\sum_{n\geq 1} \frac{(-1)^n q^{2n^2+n} (e;q^2)_n}{(\frac{q^2}{e};q^2)_n e^n}\right).$$

Using (5.6) and (5.7), differentiating with respect to e, setting $e = 1$ and multiplying both sides by -2 gives

$$2\sum_{n\geq 1} \frac{(q;q^2)_n (q^2;q^2)_{n-1} q^n}{(q^4;q^4)_n} = 2\sum_{n\geq 1} \frac{q^{2n-1}}{1-q^{2n-1}} - 2\sum_{n\geq 1} \frac{q^{2n}}{1-q^{2n}} + 4\sum_{n\geq 1} \frac{(-1)^n q^{2n^2+n}}{1-q^{2n}}.$$

The first two sums on the right combine to give (two times) the generating function for the number of odd divisors of m minus the number of even divisors of m. The last sum is the generating function for $(-1)^m$ times the number of odd divisors of m that exceed $\sqrt{2m}$, which can be seen by expanding it as

$$\sum_{n\geq 1} (-1)^{n(2n+1)} q^{n(2n+1)} + (-1)^{n(2n+3)} q^{n(2n+3)} + (-1)^{n(2n+5)} q^{n(2n+5)} \cdots$$

□

6. Concluding Remarks

Some ideas for future research naturally present themselves. First, continue to explore q-series related to the generating function (1.1), particularly the mock-theta functions that will arise. Which specializations of (1.1) are related to weak Maass forms [3, 4]? Second, generalizations of the D-rank were defined in terms of the first Frobenius representation of an overpartition and used to give combinatorial interpretations of certain multiple q-series identities [7, 8]. Are there generalizations of the M_2-rank that are similarly related to multiple series identities? Third, is there some kind of analogue of Dyson's adjoint for overpartitions like the one for partitions with no repeated odd parts in [2]? Finally, since the q-series identities employed in Section 5 are relatively elementary, one suspects that at least some of the results of this section have elegant bijective/combinatorial proofs.

7. Acknowledgement

Thanks are due to the referee for a careful reading of this paper and several key suggestions for its improvement.

References

[1] G.E. Andrews, Generalized Frobenius partitions, *Mem. Amer. Math. Soc.* **49** (1984), no. 301.

[2] A. Berkovich and F.G. Garvan, Some observations on Dyson's new symmetries of partitions, *J. Comb. Theory, Ser. A* **100** (2002), 61-93.

[3] K. Bringmann and K. Ono, The $f(q)$ mock theta function conjecture and partition ranks, *Invent. Math.* **165** (2006), 243-266.

[4] K. Bringmann and K. Ono, Dyson's rank and Maass forms, *Ann. Math.*, to appear.

[5] S. Corteel and J. Lovejoy, Frobenius partitions and the combinatorics of Ramanujan's ${}_1\psi_1$ summation, *J. Combin. Theory Ser. A* **97** (2002), 177-183.

[6] S. Corteel and J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.* **356** (2004), 1623-1635.

[7] S. Corteel, J. Lovejoy, and O. Mallet, An extension to overpartitions of the Rogers-Ramanujan identities for even moduli, *J. Number Theory*, to appear.

[8] S. Corteel and O. Mallet, Overpartitions, lattice paths, and Rogers-Ramanujan identities, *J. Combin. Theory Ser. A* **114** (2007), 1407-1437.

[9] F. J. Dyson, Some guesses in the theory of partitions, *Eureka (Cambridge)* **8** (1944), 10-15.

[10] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge Univ. Press, Cambridge, 1990.

[11] J. Joichi and D. Stanton, Bijective proofs of basic hypergeometric series identities, *Pacific J. Math.* **127** (1987), 103-120.

[12] J. Lovejoy, Overpartition theorems of the Rogers-Ramanujan type, *J. London Math. Soc.* **69** (2004), 562-574.

[13] J. Lovejoy, Rank and conjugation for the Frobenius representation of an overpartition, *Ann. Comb.* **9** (2005), 321-334.

[14] I. Pak, Partition bijections: a survey, *Ramanujan J.* **12** (2006), 5-76.

CNRS, LIAFA, Université Denis Diderot, 2, Place Jussieu, Case 7014, F-75251 Paris Cedex 05, FRANCE

E-mail address: `lovejoy@liafa.jussieu.fr`