FROBENIUS PARTITIONS AND THE COMBINATORICS OF RAMANUJAN'S ${}_1\psi_1$ SUMMATION

SYLVIE CORTEEL AND JEREMY LOVEJOY

Abstract. We examine the combinatorial significance of Ramanujan's famous summation. In particular, we prove bijectively a partition theoretic identity which implies the summation formula.

1. Introduction

One of the more remarkable identities in the theory of basic hypergeometric series is Ramanujan's product formula for the summation of the ${}_1\psi_1$ bilateral series. Namely, if $|q| < 1$ and $|\frac{b}{a}| < |z| < 1$ then

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(b/a;q)_\infty (q;q)_\infty (q/az;q)_\infty (az;q)_\infty}{(b;q)_\infty (b/az;q)_\infty (q/a;q)_\infty (z;q)_\infty} \tag{1}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n)$$

and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

The ${}_1\psi_1$ summation formula is a multi-parameter generalization of Jacobi's famous triple product identity:

$$\sum_{n=-\infty}^{\infty} q^{n(n+1)/2} z^n = (q;q)_\infty (-z^{-1};q)_\infty (-zq;q)_\infty \tag{2}$$

which can be obtained from (1) by replacing z with $-zq/a$ and letting $b \to 0$, $a \to \infty$. The summation of the ${}_1\psi_1$ has been proven in several ways [1, 2, 3, 5, 7, 8, 9, 10], typically by using some clever applications of other hypergeometric series identities. Most notable perhaps is Ismail's observation [8] that (1) is a corollary of the q - binomial theorem. Here we shall demonstrate how the ${}_1\psi_1$ summation formula is equivalent to a *combinatorial* statement about certain types of partitions.

2. Partitions

A partition π of $n := \sigma(\pi)$ is a nonincreasing sequence of natural numbers whose sum is n. We denote by $\mu(\pi)$ the number of parts in the partition π. Let $p_{A,B}(n)$ denote the number of generalized Frobenius partitions of n, that is, the number of two - rowed arrays

$$\begin{pmatrix} a_1 & a_2 & ... & a_m \\ b_1 & b_2 & ... & b_m \end{pmatrix}$$

where $\sum a_i$ is a partition of type A, $\sum b_i$ is a partition of type B, and $n = \sum (a_i + b_i + 1)$. If A denotes "distinct nonnegative parts," then Frobenius [6] observed that

$$p_{A,A}(n) = p(n) \tag{3}$$

where $p(n)$ is the number of ordinary partitions of n. In [4], Andrews discusses how this combinatorial identity is the essence of Jacobi's triple product formula. It turns out that (3) is just a specialization of a more general combinatorial identity which is essentially the ${}_1\psi_1$ summation.

Let C denote "distinct nonnegative parts and unrestricted nonnegative overlined parts, with $n > \bar{n}$." Let $f_{r,s}(n)$ be the number of generalized Frobenius partitions counted by $p_{C,C}(n)$ where there are r overlined parts in the top row and s overlined parts in the bottom row.

Now let $g_{r,s}(n)$ be the number of 4 - tuples of partitions $(\pi_1, \pi_2, \pi_3, \pi_4)$ where π_1 and π_2 are ordinary partitions, π_3 and π_4 are partitions into distinct parts, $\sigma(\pi_1) + \sigma_(\pi_2) + \sigma(\pi_3) + \sigma(\pi_4) = n$, $r = \mu(\pi_2) + \mu(\pi_3)$, and $s = \mu(\pi_2) + \mu(\pi_4)$. Notice that $f_{0,0}(n) = p_{A,A}(n)$ and $g_{0,0}(n) = p(n)$.

Theorem 1. *For all nonnegative integers n, r, s, $f_{r,s}(n) = g_{r,s}(n)$*

Proof: Given a Frobenius partition

$$\alpha = \begin{pmatrix} a_1 & a_2 & ... & a_m \\ b_1 & b_2 & ... & b_m \end{pmatrix}$$

counted by $f_{r,s}(n)$, transform α into another two-rowed array

$$\beta = \begin{pmatrix} c_1 & c_2 & ... & c_p \\ d_1 & d_2 & ... & d_p \end{pmatrix}$$

as follows:

- (i) Let $c_1, ..., c_p$ be $a_1, ..., a_m$ except k is inserted if $0 \leq k \leq a_1$ but k does not occur in row 1 of α.
- (ii) Let $d_1, ..., d_p$ be the $-k$'s from (i) written in increasing order, followed by the non-overlined parts of row 2 of α, incremented by 1 and written in increasing order, followed by the overlined parts from row 2 of α, incremented by 1 and written in non-increasing order.

For example, if

$$\alpha = \begin{pmatrix} \bar{5} & 3 & \bar{3} & \bar{3} & \bar{1} & 0 \\ 3 & \bar{3} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \end{pmatrix}$$

then

$$\beta = \begin{pmatrix} \bar{5} & 4 & 3 & \bar{3} & \bar{3} & 2 & 1 & \bar{1} & 0 \\ -4 & -2 & -1 & 4 & \bar{4} & \bar{1} & \bar{1} & \bar{1} & \bar{1} \end{pmatrix}.$$

Now map β to a 4 - tuple $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ by adding, for all i, a part of size $d_i + c_i$ to

$$\begin{cases} \pi_1 & \text{if neither } c_i \text{ nor } d_i \text{ is overlined} \\ \pi_2 & \text{if } c_i \text{ and } d_i \text{ are overlined} \\ \pi_3 & \text{if } c_i \text{ but not } d_i \text{ is overlined} \\ \pi_4 & \text{if } d_i \text{ but not } c_i \text{ is overlined} \end{cases}$$

In our example, we obtain

$$\pi = \begin{cases} \pi_1: & (2,2) \\ \pi_2: & (7,2) \\ \pi_3: & (1,7) \\ \pi_4: & (3,2,1) \end{cases}$$

It is easy to see that π_1 and π_2 are ordinary partitions and that π_3 and π_4 are partitions into distinct parts (where π_1 and π_3 are written in reverse order). By construction, $r = \mu(\pi_2) + \mu(\pi_3)$, $s = \mu(\pi_2) + \mu(\pi_4)$ and $\sum(c_i + d_i) = n$. In other words, the image of α is a 4 - tuple counted by $g_{r,s}(n)$.

This mapping is uniquely reversible. Given π, a 4-tuple of partitions counted by $g_{r,s}(n)$, first write π_2 and π_4 in reverse order. We use the notation $\pi_{i,1}$ for the first part of π_i and $\pi \backslash \pi_{i,1}$ for the 4-tuple of partitions π without the first part of π_i. We shall denote the empty partition by ϵ. With the following algorithm we reconstruct the two-rowed array β from π.

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a ← 0
β ← ε
While π2 ≠ ε or π4 ≠ ε do
  If π4 = ε or π2,1 ≤ π4,1
    1. β ← β ∪ ( ā / π2,1 − a ).
    2. π ← π\π2,1
  else
    1. β ← β ∪ ( a / π4,1 − a ).
    2. π4 ← π\π4,1
    3. a ← a + 1
While π1 ≠ ε or π3 ≠ ε
  If π1 = ε or π1,1 < π3,1
    1. β ← β ∪ ( ā / π3,1 − a ).
    2. π3 ← π\π3,1
  else
    1. β ← β ∪ ( a / π1,1 − a ).
    3. π1 ← π\π1,1
    3. a ← a + 1
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It is straightforward to recover the Frobenius partition α from β.

□

3. The summation formula

We first make the substitutions $z \to -zqa^{-1}$, $b \to -bq$, and $a \to -a^{-1}$ to obtain the equivalent form

$$\frac{(-aq;q)_\infty(-bq;q)_\infty}{(q;q)_\infty(abq;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-a^{-1};q)_n(zqa)^n}{(-bq;q)_n} = \frac{(-zq;q)_\infty(-z^{-1};q)_\infty}{(bz^{-1};q)_\infty(azq;q)_\infty} \tag{4}$$

where $|b| < |z| < |\frac{1}{aq}|$ and $|q| < 1$. Notice that the coefficient of z^0 on the left hand side of (4) is

$$\sum_{n,r,s\geq 0} g_{r,s}(n)a^r b^s q^n \tag{5}$$

while the coefficient of z^0 on the right hand side is

$$\sum_{n,r,s\geq 0} f_{r,s}(n)a^r b^s q^n \tag{6}$$

so that the truth of (4) implies Theorem 1. In fact, it is also true that Theorem 1 implies the ${}_1\psi_1$ summation formula.

Proof of ${}_1\psi_1$: If $\phi(z)$ denotes the right hand side of (4), then

$$\phi(zq) = \frac{(-zq^2;q)_\infty(-z^{-1}q^{-1};q)_\infty}{(bz^{-1}q^{-1};q)_\infty(azq^2;q)_\infty} \tag{7}$$

$$= \frac{(1+z^{-1}q^{-1})(1-azq)(-zq;q)_\infty(-z^{-1};q)_\infty}{(1+zq)(1+bz^{-1}q^{-1})(bz^{-1};q)_\infty(azq;q)_\infty} \tag{8}$$

$$= \frac{(1-azq)}{(zq-b)}\phi(z) \tag{9}$$

Since $\phi(z)$ is an analytic function of z in the annulus $|b| < |z| < |\frac{1}{aq}|$, it has a Laurent series

$$\phi(z) = \sum_{n=-\infty}^{\infty} A_n(a,b,q)z^n$$

First we assume that $|ab| < 1$ so that for all z with $|\frac{b}{q}| < |z| < |\frac{1}{aq}|$, applying (9) to this series yields

$$\sum_{n=-\infty}^{\infty} A_n(a,b,q)z^{n+1}q^{n+1} - \sum_{n=-\infty}^{\infty} A_n(a,b,q)bz^nq^n$$

$$= \sum_{n=-\infty}^{\infty} A_n(a,b,q)z^n - \sum_{n=-\infty}^{\infty} A_n(a,b,q)aqz^{n+1}$$

so that

$$\sum_{n=-\infty}^{\infty} A_{n-1}(a,b,q)(aq+q^n)z^n = \sum_{n=-\infty}^{\infty} A_n(a,b,q)(1+bq^n)z^n$$

and hence for all integers n we have that

$$A_n(a,b,q) = aq\left(\frac{1+a^{-1}q^{n-1}}{1+bq^n}\right)A_{n-1}$$

If $n > 0$, then this implies that

$$A_n = \frac{a^n q^n(-a^{-1};q)_n}{(-bq;q)_n}A_0$$

and if $n < 0$, say $n = -m$, then we have

$$\begin{aligned} A_{-m} &= \frac{a^{-m}q^{-m}(-bq^{-m+1};q)_m}{(-a^{-1}q^{-m};q)_m}A_0 \\ &= \frac{a^{-m}q^{-m}(-a^{-1};q)_{-m}}{(-bq;q)_{-m}}A_0 \end{aligned}$$

Therefore

$$\phi(z) = \sum_{n=-\infty}^{\infty} \frac{(-a^{-1};q)_n(zqa)^n}{(-bq;q)_n}A_0$$

But it follows from Theorem 1 and equations (4)-(6) that

$$A_0 = \frac{(-aq;q)_\infty(-bq;q)_\infty}{(q;q)_\infty(abq;q)_\infty}$$

By analytic continuation we can easily extend to $|b| < |z| < |\frac{1}{aq}|$. □

4. Concluding Remarks

It should be mentioned that for any integer n, it is indeed possible to bijectively prove the equality between coefficients of z^n on both sides of (4). The arguments are just more complicated variations on Theorem 1, which is the essential identity. We also wish to emphasize that a more careful consideration of the bijection presented in Theorem 1 yields an elegant proof of the $q-$analogue of a summation of Gauss,

$$\sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n(c/ab)^n}{(q;q)_n(c;q)_n} = \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}.$$

In fact, Frobenius partitions can be employed to give straightforward combinatorial proofs of several identities in the theory of basic hypergeometric series. This shall be demonstrated in a forthcoming paper.

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LRI, Bât 490, Université Paris-Sud, 91405 Orsay, France
E-mail address: Sylvie.Corteel@lri.fr

Dept. Of Math., University of Wisconsin, Madison, WI 53706
E-mail address: lovejoy@math.wisc.edu